Helical Motion and the de Broglie Wavelength: A Phase Interpretation of Relativistic Energy

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Abstract. We examine the relativistic energy identity $E = \vec{p}\cdot\vec{v} + mc^2/\gamma$ and its connection to de Broglie's phase harmony and the free Dirac Hamiltonian. Dividing the two energy contributions by Planck's constant gives the spatial phase rate v/λ_B and the time-dilated Compton frequency f_C/γ, respectively. We propose a helical interpretation in which the center of charge moves at speed c and completes one revolution per Compton cycle. The resulting geometry reproduces the energy decomposition and gives $P\lambda_B = \lambda_C^2$, where P is the helical pitch and $\lambda_C = h/(mc)$. Under an additional assumption of uniform orbital phase at equal rest-frame time, the Lorentz-transformed orbital phase coincides with the de Broglie phase. This establishes a kinematic correspondence, while the dynamical origin of the matter wave remains unresolved.

1 Introduction

For a free particle of rest mass m, special relativity gives

$$E = \gamma mc^2 = \sqrt{m^2c^4 + p^2c^2} \tag{1}$$

where γ is the Lorentz factor and $\vec{p} = \gamma m\vec{v}$. The conventional separation into rest and kinetic energies is

$$E = mc^2 + (\gamma - 1)mc^2. \tag{2}$$

An alternative exact identity is

$$E = \vec{p}\cdot\vec{v} + \frac{mc^2}{\gamma} = \gamma mv^2 + \frac{mc^2}{\gamma}. \tag{3}$$

This paper examines the connection between the relativistic energy identity, de Broglie's phase harmony, and the free Dirac Hamiltonian. We then investigate a helical model in which the internal Compton frequency is identified with charge rotation. This interpretation relates the helical pitch to the de Broglie wavelength and clarifies the additional assumptions required to extend the orbital phase throughout space.

2 Relativistic Mechanics

2.1 Lagrangian

For a free particle of invariant mass m, the action between two events on its trajectory is proportional to the proper time elapsed between them,

$$S = -mc^2 \int d\tau = \int L\, dt. \tag{4}$$

The Lagrangian L gives the rate of change of the action along the trajectory with respect to laboratory time t. Since $d\tau = dt/\gamma$, we obtain

$$L = -mc^2\frac{d\tau}{dt} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} = -\frac{mc^2}{\gamma}. \tag{5}$$

The rest energy mc^2 is invariant. Comparing the expression for L with Eq. (3) gives

$$L = \vec{p}\cdot\vec{v} - E = -\frac{mc^2}{\gamma}. \tag{6}$$

2.2 Hamiltonian

Differentiating the free-particle Lagrangian in Eq. (5) with respect to velocity gives the canonical momentum,

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \gamma m\vec{v}. \tag{7}$$

The Hamiltonian is the Legendre transform of L with respect to $\vec{v}$,

$$H = \vec{p}\cdot\vec{v} - L. \tag{8}$$

Substituting the expressions for $\vec{p}$ and L reproduces the two terms in Eq. (3),

$$H = \gamma mv^2 + \frac{mc^2}{\gamma} = \gamma mc^2. \tag{9}$$

For convenience, we denote these contributions by E_t and E_r, anticipating their proposed translational and rotational interpretations within the helical model. At this stage, they are simply two terms of an exact relativistic identity.

$$E_t \equiv \vec{p}\cdot\vec{v} = pv, \qquad E_r \equiv -L = \frac{mc^2}{\gamma}. \tag{10}$$

Their sum gives the total energy,

$$E_T \equiv E = E_t + E_r = H. \tag{11}$$

2.3 Kinetic energy

Since $E_t = H + L$, its relation to kinetic energy can be examined by comparing the Newtonian and relativistic expressions for this sum.

In Newtonian mechanics, the kinetic energy K_N, Lagrangian, and Hamiltonian of a particle moving in a velocity-independent potential $V(\vec{x})$ are

$$K_N = \frac{1}{2}mv^2, \qquad L = K_N - V, \qquad H = K_N + V. \tag{12}$$

For a velocity-independent potential, the Newtonian relations in Eq. (12) give $H + L = 2K_N = \vec{p} \cdot \vec{v}$. Since K_N is quadratic in velocity, this suggests examining the relativistic quantity

$$K_* \equiv \frac{H + L}{2} = \frac{E_t}{2} = \frac{pv}{2} = \frac{1}{2}\gamma mv^2. \tag{13}$$

The kinetic energy defined by the work required to accelerate the particle from rest is

$$K_{\rm rel} = H(v) - H(0) = (\gamma - 1)mc^2. \tag{14}$$

Both quantities approach $mv^2/2$ at low velocities, but differ relativistically. To determine whether K^* represents a distinct physical energy, consider the work required to accelerate a particle from rest.

$$W = \Delta K_{\rm rel} = \Delta E_t + \Delta E_r \tag{15}$$

$$W = pv + \left(\frac{mc^2}{\gamma} - mc^2\right) = (\gamma - 1)mc^2. \tag{16}$$

Work measurements determine the change in total kinetic energy, while momentum and velocity measurements allow $K^* = pv/2$ to be calculated independently. Neither measurement establishes whether K^* is a separately stored or exchanged physical energy. Such an interpretation would require additional evidence beyond these kinematic relations.

3 Matter-wave phase

3.1 De Broglie's phase harmony

De Broglie associated an internal periodic phenomenon with a material particle by combining Einstein's rest-energy relation with Planck's energy-frequency relation [1, 2],

$$E_0 = mc^2 = hf_C. \tag{17}$$

Here f_C is the Compton frequency of the particle in its rest frame. For a particle moving at velocity $\vec{v}$, special relativity leads to two different frequencies.

The total relativistic energy $E = \gamma mc^2$ corresponds to

$$f_T = \frac{E}{h} = \gamma f_C. \tag{18}$$

However, an internal periodic process accompanying the particle undergoes time dilation. Its frequency in laboratory time is therefore

$$f_r = \frac{f_C}{\gamma}. \tag{19}$$

De Broglie reconciled the total energy frequency $f_T = E/h$ with the time-dilated internal frequency $f_r = f_C/\gamma$ through phase harmony. The energy decomposition in Eq. (3) gives

$$\frac{E_T}{h} = \frac{E_r}{h} + \frac{E_t}{h}. \tag{20}$$

$$f_T = f_r + f_t, \qquad f_t = \frac{\vec{p} \cdot \vec{v}}{h}. \tag{21}$$

The following subsections show how this decomposition arises from the temporal and spatial terms of the matter-wave phase.

3.2 Temporal phase

De Broglie's phase harmony can be expressed through the relativistic transformation of the internal phase. In the particle's rest frame, an internal clock advances according to

$$\Phi' = \frac{mc^2 t'}{\hbar}. \tag{22}$$

To compare phases at different positions, we extend the internal phase beyond the particle's trajectory. We adopt a uniform phase on each rest-frame hypersurface of simultaneity. Treating this phase as a Lorentz scalar, its transformation to the laboratory frame gives

$$\Phi(\vec{x}, t) = \frac{mc^2}{\hbar}\gamma\left(t - \frac{\vec{v} \cdot \vec{x}}{c^2}\right) = \frac{E\,t - \vec{p} \cdot \vec{x}}{\hbar}. \tag{23}$$

For a free particle of definite energy and momentum, this phase can be represented by the plane wave

$$\psi(\vec{x}, t) = u\, e^{-{\rm i}\Phi(\vec{x},t)}. \tag{24}$$

Here u is a constant amplitude or spinor. This plane wave represents an idealized momentum eigenstate, rather than a spatially localized particle.

At a fixed laboratory position, the spatial phase remains constant. The temporal phase therefore advances at the frequency

$$f_T = \frac{E}{h} = \gamma f_C. \tag{25}$$

3.3 Phase along the trajectory

For a free particle with energy E and momentum $\vec{p}$, the associated classical velocity is $\vec{v} = \vec{p}c^2/E$.

We consider the inertial reference trajectory $\vec{x} = \vec{v}t$, which also describes the mean motion of a sufficiently narrow free wave packet. Evaluating the matter-wave phase along this trajectory and using the relativistic energy decomposition gives

$$\Phi(\vec{v}t, t) = \frac{E - \vec{p} \cdot \vec{v}}{\hbar}\, t = \frac{mc^2}{\hbar}\, \tau, \tag{26}$$

where $\tau = t/\gamma$ is the proper time of the particle.

The corresponding phase frequency in laboratory time is

$$f_r = \frac{1}{2\pi}\frac{d\Phi}{dt} = \frac{mc^2}{\gamma h} = \frac{f_C}{\gamma}. \tag{27}$$

Thus, the matter-wave phase advances along the particle's trajectory at the time-dilated Compton frequency, reproducing de Broglie's phase harmony.

3.4 Spatial wavelength

The preceding results establish the temporal phase relation. We now examine its spatial dependence.

At fixed laboratory time, the phase difference between two points separated by Δx along the momentum direction is $\Delta\Phi = -p\Delta x/\hbar$. Requiring one complete phase cycle gives the de Broglie wavelength,

$$\lambda_B = \frac{h}{p} = \frac{h}{\gamma m v}. \tag{28}$$

Consequently, the spatial contribution to the phase rate along the particle trajectory is

$$f_t = \frac{v}{\lambda_B} = \frac{E_t}{h} \tag{29}$$

completing the frequency decomposition in Eq. (21).

3.5 Phase-front crossings

A surface of constant phase satisfies

$$Et - \vec{p}\cdot\vec{x} = \text{constant}. \tag{30}$$

For propagation along the positive x-axis, its velocity is

$$v_{\text{ph}} = \frac{E}{p} = \frac{c^2}{v}. \tag{31}$$

This superluminal phase velocity does not represent the transport of matter or information. A narrow wave packet centered at momentum p propagates at the group velocity

$$v_g = \frac{dE}{dp} = \frac{pc^2}{E} = v. \tag{32}$$

Successive phase fronts are separated by λ_B. Since the fronts move faster than the center of mass, they overtake it at a rate

$$\frac{v_{\text{ph}} - v}{\lambda_B} = \frac{E - pv}{h} = \frac{mc^2}{\gamma h} = f_r. \tag{33}$$

Thus, the rate at which successive phase fronts pass the center-of-mass trajectory equals the time-dilated Compton frequency. This provides a kinematic interpretation of de Broglie's phase harmony, without establishing a physical mechanism for generating the matter wave.

4 Breit's interpretation of the Dirac Hamiltonian

Breit [4] related the relativistic energy decomposition to the Dirac equation by associating the spatial Dirac matrices with velocity and the fourth matrix with $1/\gamma$. We examine this correspondence through expectation values for a free positive-energy electron state.

The free Dirac Hamiltonian [3] is

$$H_D = c\vec{\alpha}\cdot\vec{p} + \beta mc^2. \tag{34}$$

Here β denotes the Dirac matrix. The Heisenberg equation for position gives the velocity operator,

$$\hat{\vec{v}} = \frac{i}{\hbar}[H_D, \hat{\vec{x}}] = c\vec{\alpha}. \tag{35}$$

Breit's first correspondence thus follows directly from the equation of motion. To evaluate both energy terms, consider a normalized positive-energy spinor u of definite momentum $\vec{p}$, satisfying $H_D u = Eu$. The Dirac anticommutation relations give

$$\{H_D, \alpha_i\} = 2cp_i, \qquad \{H_D, \beta\} = 2mc^2. \tag{36}$$

In this state, $\langle\{H_D, A\}\rangle = 2E\langle A\rangle$. Applying this identity to α_i and β gives

$$\langle c\vec{\alpha}\rangle = \frac{c^2\vec{p}}{E} = \vec{v}, \qquad \langle\beta\rangle = \frac{mc^2}{E} = \frac{1}{\gamma}. \tag{37}$$

The first expectation value is the mean velocity of the electron. The second gives the factor relating proper time to laboratory time. Although β has eigenvalues ± 1, its expectation value in this state is $1/\gamma$.

The momentum and mass terms of H_D therefore reproduce the two contributions in Eq. (3) separately,

$$\langle c\vec{\alpha}\cdot\vec{p}\rangle = \vec{p}\cdot\vec{v} = E_t, \tag{38}$$

$$\langle\beta mc^2\rangle = \frac{mc^2}{\gamma} = E_r. \tag{39}$$

Breit also emphasized that the components of the Dirac velocity operator do not commute [4].

The analysis of the Dirac velocity operators discussed by Breit was followed by Schrödinger's identification, in 1930, of Zitterbewegung: a rapid oscillatory component of the electron's motion, with characteristic angular frequency $2mc^2/\hbar$ and an amplitude on the Compton scale [5]. In his 1933 Nobel lecture, Dirac explicitly acknowledged this oscillatory motion as a consequence of his electron equation, noting that the electron's instantaneous speed in this description equals the speed of light [6]. Subsequent studies by Huang [7], Barut and Zanghi [8], Hestenes [9], Rivas [10], and Consa [11] developed different interpretations of this internal motion, including circular and helical models in which the center of charge moves at the speed of light around the

center of mass. This historical development motivates the helical kinematic model examined in the following section, without requiring a unique classical charge trajectory to follow from the Dirac equation.

5 Helical interpretation

5.1 Charge motion and energy

We model the electron by a timelike center of mass (CM) and a lightlike center of charge (CC). In the CM rest frame, the charge moves uniformly at speed c on a circle of radius R.

Consider a laboratory frame in which the CM moves at speed v along the normal to the orbital plane. With the z-axis along this direction, the charge follows a helical trajectory,

$$\vec{r}_{CC}(t) = \big(R\cos\omega t,\ R\sin\omega t,\ vt\big). \tag{40}$$

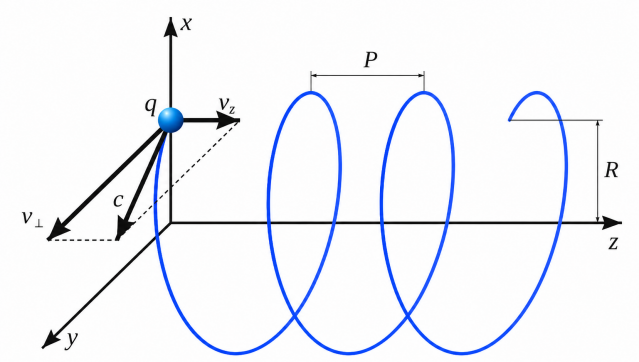

Figure 1: Helical trajectory of the center of charge along the z-axis, showing the radius R and pitch P.

The charge velocity has two perpendicular components: the transverse orbital velocity $v_\perp$ and the longitudinal velocity v. By the Pythagorean theorem, the squared speed of the charge is the sum of the squares of these components. Since the charge moves at speed c, we obtain

$$c^2 = v_\perp^2 + v^2. \tag{41}$$

An axial Lorentz boost preserves the transverse radius and dilates the orbital period. Consequently,

$$v_\perp = R\omega = \frac{c}{\gamma}, \qquad \omega = \frac{\omega_0}{\gamma}, \qquad \omega_0 = \frac{c}{R}. \tag{42}$$

The proper time used here belongs to the CM, while the charge follows a lightlike worldline.

The Pythagorean velocity relation provides a geometric counterpart to the relativistic energy decomposition. Multiplying Eq. (41) by γm gives

$$E = \underbrace{\gamma m v_\perp^2}_{E_r} + \underbrace{\gamma m v^2}_{E_t} = \frac{mc^2}{\gamma} + \vec{p}\cdot\vec{v}, \tag{43}$$

which reproduces Eq. (3).

The present construction is purely kinematic. It does not specify the internal momentum distribution or derive the electron's spin and magnetic moment from the proposed motion. These questions require a dynamical model beyond the scope of this paper.

5.2 Orbital and matter-wave phases

We identify one revolution of the charge in the CM rest frame with one cycle of de Broglie's internal clock. This fixes the orbital frequency, radius, and circumference:

$$f_0 = f_C = \frac{mc^2}{h}, \tag{44}$$

$$R = \frac{c}{2\pi f_C} = \frac{\hbar}{mc}, \qquad 2\pi R = \lambda_C. \tag{45}$$

The orbital angle advances at the Compton frequency per unit CM proper time. In laboratory time,

$$\theta(t) - \theta(0) = \frac{mc^2}{\gamma\hbar}t = \frac{E - \vec{p}\cdot\vec{v}}{\hbar}t. \tag{46}$$

This is the same phase increment obtained for the matter wave along the CM reference trajectory. With a common initial phase, the orbital angle and the matter-wave phase therefore remain synchronized along that trajectory.

To extend this correspondence beyond the trajectory, we assign the orbital angle uniformly to each hypersurface of CM rest-frame simultaneity and treat the resulting phase as a Lorentz scalar. We denote this extended orbital phase by $\Phi_{\rm orb}(z,t)$ and the de Broglie phase introduced in Sec. 3.2, restricted to motion along the z-axis, by $\Phi_{\rm B}(z,t)$. Choosing a common initial phase, the Lorentz transformation gives

$$\Phi_{\rm orb}(z,t) = \frac{E\,t - pz}{\hbar} = \Phi_{\rm B}(z,t). \tag{47}$$

Since both phases have been assigned the same rest-frame frequency and the same Lorentz-scalar extension, their equality follows from these assumptions. This establishes the kinematic compatibility of the proposed orbital motion with the de Broglie phase, but does not independently derive the matter wave from charge circulation. In particular, the existence and physical origin of the assumed spatial phase distribution remain to be established.

5.3 Pitch and wavelength

The helical pitch P is the axial displacement of the center of mass during one charge revolution. Since the laboratory rotation period is γ/f_C, the pitch is

$$P = \frac{\gamma v}{f_C} = \gamma\frac{v}{c}\lambda_C. \tag{48}$$

By contrast, the de Broglie wavelength λ_B is the spatial period of the de Broglie phase at fixed laboratory time. Using the previously established relation,

$$\lambda_B = \frac{\lambda_C}{\gamma(v/c)}. \tag{49}$$

For $v > 0$ and the assumed orbital frequency $f_0 = f_C$, the two lengths satisfy

$$P\lambda_B = \lambda_C^2, \tag{50}$$

$$\frac{P}{\lambda_B} = \gamma^2 - 1. \tag{51}$$

As momentum increases, the helical pitch grows while the de Broglie wavelength decreases, keeping their product constant. Both lengths equal λ_C at $p = mc$, corresponding to $v = c/\sqrt{2}$.

6 Conclusion

The relativistic identity $E = \vec{p} \cdot \vec{v} + mc^2/\gamma$ connects de Broglie's phase harmony with the expectation values of the free Dirac Hamiltonian. Dividing its two contributions by Planck's constant gives the spatial phase rate v/λ_B and the time-dilated Compton frequency f_C/γ.

A helical model with charge motion at speed c provides a geometric interpretation of this decomposition. Identifying one charge revolution with one Compton cycle leads to the relation $P\lambda_B = \lambda_C^2$. Under an additional assumption of uniform rest-frame phase extension, the orbital phase coincides with the de Broglie phase.

This correspondence is kinematic and depends on the assumed orbital frequency and spatial phase extension. Establishing a dynamical origin for the circulating charge, the associated matter-wave phase, and the resulting diffraction amplitudes remains necessary for a physical interpretation of the model.

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