Sequential Splitting of Smooth Real Functions with an Exact Truncation Error

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Abstract

A right triangle splits a signed scalar into two components whose squares sum to its square. Repeating the construction leaves a choice of angle at every step. We prescribe the angle as a smooth function of the current remainder and a fixed positive scale. The resulting sequence of smooth components defines a map into the real space of square-summable sequences. Its norm recovers the absolute value of the original function, and its first coordinate retains the sign, including across zero crossings. At every depth the remainder equals the truncation error in sequence norm. Convergence is uniform on compact subintervals and becomes at least quadratic once the remainder falls below the chosen scale. Pairs of coordinate idempotents record the extracted components. One example tracks the squared amplitude remaining after the first pair. A second illustrates how prescribed geometric coefficients determine the input magnitude.

Keywords: orthogonal decomposition, nonlinear recurrence, coordinate projections, truncation error, square-summable sequences

The choice of a splitting rule

Suppose a signed amplitude is represented by the hypotenuse of a right triangle. One leg is recorded and the other becomes the hypotenuse for the next step. Given the angles, repeated splitting is the familiar product-of-sines construction of spherical coordinates in successively higher dimensions [1]. The question here is how to choose those angles from the amplitude itself so that the construction passes smoothly through zero and the unrecorded length can be read off at any stage.

Fix a scale $s > 0$, in the same units as the amplitude. At a signed remainder r, choose the angle $\theta = \arctan(r/s)$. The recorded component is $r\cos\theta$, and the next remainder is $r\sin\theta$. This choice fixes a particular representation among many that preserve the sum of squares. An earlier exposition on the author's website introduced the construction in terms of paired projectors [3]. Here we specify the space in which the series converges, derive its error at arbitrary depth and examine the rate near zero.

Sequential splitting theorem

Let I be an open real interval, let $f : I \to \mathbb{R}$ be smooth, and fix $s > 0$. All operations below are pointwise in x. Starting from $r_0 = f$, compute one component and one new remainder at each step:

$$u_k = \frac{s r_k}{\sqrt{s^2 + r_k^2}}, \qquad r_{k+1} = \frac{r_k^2}{\sqrt{s^2 + r_k^2}}, \qquad k \geq 0. \tag{1}$$

Theorem. Each u_k and r_k is smooth. The sequence $F(x) = (u_0(x), u_1(x), ...)$ belongs to ℓ^2. If P_m retains its first m coordinates, then

$$\boxed{\|F\|_2^2 = f^2, \qquad \|F - P_m F\|_2 = |r_m|.} \tag{2}$$

The partial sums converge uniformly in sequence norm on every compact subinterval of I. For $m \geq 1$, the first component and the finite squared-length balance recover the signed input:

$$f = \operatorname{sgn}(u_0) \sqrt{\sum_{k=0}^{m-1} u_k^2 + r_m^2}, \qquad \operatorname{sgn}(0) = 0. \tag{3}$$

Proof. Since the denominator in (1) is strictly positive, both scalar maps in (1) are smooth on the whole real line. Induction gives the asserted regularity, also at a zero of f. Direct substitution yields

$$u_k^2 + r_{k+1}^2 = \frac{s^2 r_k^2 + r_k^4}{s^2 + r_k^2} = r_k^2, \qquad \sum_{k=0}^{m-1} u_k^2 + r_m^2 = f^2. \tag{4}$$

Thus the partial sums of squares are bounded. To determine their limit, take a compact subinterval K and a positive upper bound M for $|f|$ on K. Equation (4) gives $|r_k| \leq M$. The function $t/\sqrt{s^2 + t^2}$ increases on $[0, \infty)$, so iteration gives

$$|r_m| \leq M q^m, \qquad q = \frac{M}{\sqrt{s^2 + M^2}} < 1. \tag{5}$$

Consequently $r_m \to 0$ uniformly on K. Letting $m \to \infty$ in (4) proves the norm identity. Subtracting the first m squared coordinates gives the exact tail in (2), hence uniform convergence there as well. Finally u_0 has the sign of f, which proves (3). □

Rate after entry below the scale. Put $z_k = |r_k|/s$. Then $z_{k+1} = z_k^2/\sqrt{1 + z_k^2} \leq z_k^2$. If $z_j \leq \rho < 1$, induction gives

$$|r_{j+d}| \leq s\rho^{2^d}, \qquad d \geq 0. \tag{6}$$

Estimate (5) guarantees entry into this regime uniformly on each compact subinterval. Once the current remainder meets the requested absolute tolerance, (2) certifies the error directly.

Paired coordinate projectors

Use coordinatewise multiplication of real sequences. Write $\mathfrak{e}_n$ and $\bar{\mathfrak{e}}_n$ for the unit sequences supported at positions $2n$ and $2n + 1$, counting from zero. They are orthonormal in ℓ^2 and idempotent under multiplication. Let R_n be the bounded sequence equal to one at positions $2n, 2n + 1, ...$ and zero elsewhere. As diagonal multipliers on ℓ^2, these satisfy

$$R_n = \mathfrak{e}_n + \bar{\mathfrak{e}}_n + R_{n+1}, \qquad \mathfrak{e}_n \bar{\mathfrak{e}}_n = \mathfrak{e}_n R_{n+1} = \bar{\mathfrak{e}}_n R_{n+1} = 0. \tag{7}$$

Corollary (paired splitting). Let $h_n = r_{2n}$, so that $h_0 = f$. Denote the two scalar maps in (1) by

$$A(t) = \frac{st}{\sqrt{s^2 + t^2}}, \qquad B(t) = \frac{t^2}{\sqrt{s^2 + t^2}}.$$

Starting with h_n, the first step records a_n and passes $B(h_n)$ to the second step. The second step records b_n and supplies the input for the next pair. Thus

$$a_n = A(h_n), \qquad b_n = A(B(h_n)), \qquad h_{n+1} = B(B(h_n)).$$

These coefficients are u_{2n} and u_{2n+1}, respectively. Applying (4) twice gives $h_n^2 = a_n^2 + b_n^2 + h_{n+1}^2$. Grouping consecutive coordinates in (2) now gives

$$F = \sum_{n\geq 0} (a_n \mathfrak{e}_n + b_n \bar{\mathfrak{e}}_n), \qquad f^2 = \sum_{n<N} (a_n^2 + b_n^2) + h_N^2. \tag{8}$$

To see the splitting at a single level, apply (7) to F:

$$R_n F = a_n \mathfrak{e}_n + b_n \bar{\mathfrak{e}}_n + R_{n+1} F.$$

After N pairs the retained part is $P_{2N}F$. Since $R_N = \mathrm{Id} - P_{2N}$ as operators, (2) gives $\|R_N F\|_2 = |h_N|$. This tail tends to zero uniformly on each compact subinterval, which justifies the infinite projector sum in (8).

Multiplying by either idempotent selects its coordinate. At finite depth the compressed record belongs to $\mathbb{R}^{2N} \oplus \mathbb{R}$: the final scalar represents the length still to resolve. The multiplier R_N identifies its future positions. Counting an independent copy of h_N at each of these positions would give an infinite squared norm whenever $h_N \neq 0$.

A sign-changing example

For $f(x) = x$ and $s = 1$, two applications of (1) give

$$a_0 = \frac{x}{\sqrt{1+x^2}}, \quad b_0 = \frac{x^2}{\sqrt{1+x^2+x^4}}, \quad h_1 = \frac{x^4}{\sqrt{1+x^2}\sqrt{1+x^2+x^4}}. \tag{9}$$

At $x = -1$, the three entries are $-1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{6}$. Their squares sum to one. The error after keeping the first pair is exactly $1/\sqrt{6}$, and the first sign recovers the value -1. At $x = 1$ only the first coordinate changes sign. At zero every entry vanishes. For $f(x) = \sin x$, substitution of $\sin x$ for x gives the corresponding split at every zero crossing.

A geometric series in projector coordinates

Consider the reverse construction in [3]. Prescribe $a_n(x) = x^n$ and $b_n(x) = 0$ for $|x| < 1$, with $a_0 = 1$. Write G for the resulting sequence. Its squared norm is a geometric series, so its norm gives the positive function g:

$$G(x) = \sum_{n=0}^{\infty} x^n \mathfrak{e}_n, \qquad g(x) = \|G(x)\|_2 = \sqrt{\sum_{n=0}^{\infty} x^{2n}} = \frac{1}{\sqrt{1-x^2}}.$$

After N pairs the retained coordinates have indices $0 \leq n < N$. The omitted squared norm is the tail of the same series:

$$\|R_N G(x)\|_2^2 = \sum_{n=N}^{\infty} x^{2n} = \frac{x^{2N}}{1-x^2}.$$

For instance, the first two pairs give $\mathfrak{e}_0 + x\mathfrak{e}_1$. Their squared norm is $1+x^2$, and the remaining length is $x^2/\sqrt{1-x^2}$.

The choice of coefficients matters. If instead we use g as the input to (1) with $s = 1$, its first coordinate is $A(g(x)) = 1/\sqrt{2-x^2}$. The geometric construction starts with $a_0 = 1$. Both give the same norm, while the fixed splitting rule determines which coordinates carry it.

Interpretation and scope

In squared amplitudes the rule has a familiar allocation form. Write $w_k = r_k^2$ and $p_k = s^2/(s^2 + w_k)$. Then $u_k^2 = p_k w_k$ and $w_{k+1} = (1 - p_k)\, w_k$. In probability theory, stick-breaking constructions assign successive fractions of a remaining mass, with random fractions in models such as [2]. Here the fraction is determined by the current scalar and the fixed scale. This distinction accounts for the explicit error estimate (6).

Norm recovery alone imposes little structure: a single coordinate equal to the input already has the required length. For the present rule, even the first coordinate is invertible at fixed scale:

$$f = \frac{su_0}{\sqrt{s^2 - u_0^2}}, \qquad |u_0| < s. \tag{10}$$

The role of the extra coordinates is to resolve the squared amplitude into successive contributions and to track exactly how much of the norm remains. The construction acts on each value of the function separately. It makes no assertion about orthogonality of the component functions when integrated over the domain.

Data availability. All results follow from the displayed formulas. No external datasets were used.

References

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