A Singularity-Based Generalization of the Collatz Construction to Integer Bases

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Abstract

This paper develops a singularity-based extension of a methodological construction associated with the classical Collatz map. The starting point is a binary digit pattern previously considered in the study of the $3n+1$ sequence and its block generalization. We extend this pattern to positional representations in integer bases other than two. Let $p \geq 2$ be an integer that is not a proper perfect power, let $k \geq 1$, and put $d = p^k$. Writing $A_{p,k}$ for a block of k digits equal to $p-1$ and Z_k for a block of k zeros, a generalized singularity is represented canonically by $A_{p,k}(Z_k A_{p,k})^{\ell-1}, \qquad \ell \in \mathbb{N}^*$. Its numerical value is $a = (d-1)\sum_{j=0}^{\ell-1} d^{2j}$, which yields the identity $(1+d)a = d^{2\ell} - 1$. This identity motivates the correction function $S_{p,k}(n) = d - (n \bmod d)$ for $d \nmid n$, and the associated piecewise transformation that applies this corrected multiplication on nonmultiples of d and division by d on multiples of d. On every generalized singularity term, the correction equals one and the transformation produces the power $d^{2\ell}$. The construction also recovers the classical Collatz map when $p = 2$ and $k = 1$, and it has an explicit trivial cycle through 1 of length $2d-1$.

The arithmetic transformation itself is not claimed to be a new family: it is the specialization $b = d$, $m = 1$ of Bruschi's two-parameter family, [4], and it is closely related to Carnielli's congruence-based mapping, [3]. The focus here is the derivation and extension of the digit singularity that motivates the transformation. No universal boundedness or convergence theorem is proved. A systematic computational investigation of the generalized dynamics is reserved for future versions.

1 Introduction

The classical Collatz map on the positive integers is defined by

$$\mathrm{Col}(n) = \begin{cases} 3n+1, & \text{if } n \equiv 1 \pmod 2, \\ \dfrac{n}{2}, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

The Collatz conjecture asserts that every positive integer eventually reaches 1 under iteration of this map. Despite the simplicity of its definition, the conjecture remains a central open problem in number theory. Important partial progress is known. In particular, Tao proved that for every function f tending to infinity, the minimum value attained along the Collatz orbit is at most $f(n)$ for almost all starting values in the sense of logarithmic density [1]. This result gives a strong statement about almost all orbits, but it does not establish that every orbit reaches 1.

A substantial literature studies generalizations of the Collatz map. Bruschi introduced a two-parameter family of integer mappings, which in the notation used in this paper may be written as

$$F_{b,m}(n) = \begin{cases} \dfrac{n}{b}, & \text{if } b \mid n, \\ (b^m+1)n + b^m - (n \bmod b^m), & \text{if } b \nmid n. \end{cases}$$

where $b \geq 2$ and $m \geq 1$ [4]. Carnielli studied congruence-based generalizations, including the map

$$C_d(n) = \begin{cases} n/d, & d \mid n, \\ \dfrac{(d+1)n + d - r}{d}, & n \equiv r \pmod d, \quad 1 \leq r \leq d-1, \end{cases}$$

and discussed the occurrence of nontrivial cycles in such systems [3]. These works provide essential context for the arithmetic transformation considered below.

The present paper also continues a line of work in which a particular singularity pattern in the binary representation of terms associated with the classical $3n+1$ sequence is used as the starting point for a methodological generalization [2]. Rather than choosing new coefficients independently of the digit structure, that approach first identifies an alternating binary pattern, replaces its digits by blocks, and then derives a multiplicative factor and a correction term from the resulting representation. The purpose of the present

paper is to extend this singularity-centered approach from base two to integer bases other than two.

More precisely, let $p \geq 3$ be an integer that is not a proper perfect power, and let $k \geq 1$. The restriction on p is a normalization of the parameters: every integer $d \geq 2$ has a unique representation $d = p^k$ with p not a proper perfect power. The base-p digit alphabet is $0, 1, \ldots, p-1$. We consider blocks of k consecutive digits equal to $p-1$, separated by blocks of k zeros. If $A_{p,k}$ denotes the former block and Z_k the latter, the canonical representation of a generalized singularity term is $A_{p,k}(Z_k A_{p,k})^{\ell-1}, \qquad \ell \in \mathbb{N}^*$. Writing $d = p^k$, the numerical value (a) of such a term satisfies $a = (d-1)(1+d^2+\cdots+d^{2\ell-2})$, and therefore $(1+d)a = d^{2\ell} - 1$.

Thus, multiplication by $1 + p^k$ turns the singularity term into a number whose base-p digits are all $p-1$; adding one produces a power of p^k. This identity motivates the correction $S_{p,k}(n) = p^k - (n \bmod p^k), \qquad p^k \nmid n,$

which is the least positive additive correction making the first branch divisible by p^k. The resulting transformation is

$$T_{p,k}(n) = \begin{cases} (1+p^k)n + S_{p,k}(n), & p^k \nmid n, \\ n/p^k, & p^k \mid n. \end{cases}$$

For a generalized singularity term a, one has $S_{p,k}(a) = 1$ and hence

$$T_{p,k}(a) = p^{2k\ell}.$$

The transformation recovers the classical Collatz map at $p = 2$, $k = 1$, and its orbit beginning at 1 contains an explicit generalized trivial cycle.

The relation with earlier generalized maps must be stated precisely. If $d = p^k$, then $T_{p,k}$ is exactly the specialization $F_{d,1}$ of Bruschi's family. Moreover, whenever $d \nmid n$, the value $T_{p,k}(n)$ is d times the corresponding value of Carnielli's map $C_d(n)$; the next iteration of $T_{p,k}$ divides by d and gives $C_d(n)$. Thus, the present paper does not claim to introduce a new arithmetic family of Collatz-type maps. Its focus is the singularity-first derivation, the explicit extension of the digit pattern to base p, and the algebraic relation between that pattern and the correction term. Whether this structural viewpoint yields further new results about orbit behavior or cycle structure remains a question for subsequent work.

The paper is organized as follows. Section 2 recalls the binary singularity and the methodological construction based on it. Section 3 extends the singularity pattern to integer bases other than two and establishes its numerical characterization. Section 4 derives the associated correction function and transformation, then examines the four design conditions. Section 5 places

the construction in the context of earlier generalizations, in particular those of Bruschi and Carnielli.

2 Reminder: The Methodological Generalization in the Binary Base

The classical Collatz transformation is defined by

$$T(n) = \begin{cases} 3n+1, & n \equiv 1 \pmod 2, \\ n/2, & n \equiv 0 \pmod 2, \end{cases} \qquad n \in \mathbb{N}^*.$$

The construction recalled in this section follows a structural approach: rather than replacing the coefficients in the classical transformation immediately, we first identify a singularity pattern in the binary representation and then use that pattern to determine a suitable arithmetic transformation.

2.1 The classical singularity and its binary generalization

The classical singularity considered in the preceding work is a binary word that begins and ends with the digit 1, with consecutive digits equal to 1 separated by a single digit equal to 0. Equivalently, its binary word has the form

$$1(01)^{\ell-1}, \qquad \ell \in \mathbb{N}^*,$$

where the word is read from the most significant digit to the least significant digit.

For a fixed integer $k \geq 1$, the generalized binary singularity is obtained by replacing each digit 1 in this alternating pattern by a block of k consecutive ones, and each separating digit 0 by a block of k consecutive zeros. Denote these two digit blocks by

$$\mathbf{I}_k = \underbrace{11\cdots1}_{k \text{ bits}}, \qquad \mathbf{Z}_k = \underbrace{00\cdots0}_{k \text{ bits}}.$$

Thus, a positive integer a is a generalized singularity term if its canonical binary representation has the form

$$A_{p,k}(Z_k A_{p,k})^{\ell-1}, \qquad \ell \in \mathbb{N}^*.$$

In particular, the representation begins and ends with a block of k ones, and every two consecutive such blocks are separated by exactly k zeros.

Writing $d = 2^k$, the numerical value of a term with this pattern is

$$a = (d-1)\left(1 + d^2 + d^4 + \cdots + d^{2\ell-2}\right).$$

Consequently,

$$\begin{aligned}(1+d)a &= (d^2-1)\left(1 + d^2 + \cdots + d^{2\ell-2}\right) &(1)\\ &= d^{2\ell} - 1. &(2)\end{aligned}$$

Thus, multiplying a generalized singularity term by $1 + 2^k$ produces a binary integer consisting entirely of ones, with a number of digits divisible by k. Adding one then produces a power of 2^k.

2.2 From the singularity to the correction function

The preceding identity suggests beginning with the multiplicative expression $(1 + 2^k)n$ and adding one. However, a fixed addition of one does not, for an arbitrary integer n, ensure divisibility by 2^k. The correction term is therefore chosen so that the output of the non-divisible branch is always a multiple of 2^k.

For $n \in \mathbb{N}^*$ with $2^k \nmid n$, define the smallest no negative additive inverse of n modulo 2^k

$$S_k(n) = [2^k - (n \mod 2^k)],$$

where $(n \mod 2^k) \in \{1, 2, \ldots, 2^k - 1\}$. Equivalently, $S_k(n)$ is the unique integer in $\{1, 2, \ldots, 2^k - 1\}$ satisfying

$$(1 + 2^k)n + S_k(n) \equiv 0 \pmod{2^k}.$$

The equality follows from $1 + 2^k \equiv 1 \pmod{2^k}$. Hence the generalized Collatz transformation is defined by

$$T_k(n) = \begin{cases} (1 + 2^k)n + S_k(n), & 2^k \nmid n, \\ n/2^k, & 2^k \mid n. \end{cases} \tag{3}$$

The first branch always produces a positive multiple of 2^k, so the second branch is applicable at the next iteration.

2.3 The four design conditions

The correction function is selected in accordance with the four conditions stated in the methodological construction of the binary family:

1. **Recovery of the classical Collatz transformation.** For $k = 1$, every integer in the first branch is odd, and

$$S_1(n) = 2 - (n \bmod 2) = 1.$$

Therefore the first branch is $3n+1$, while the second branch is $n/2$. The classical $3n + 1$ Collatz transformation is thus recovered when $k = 1$.

2. **Preservation of the $+1$ action on generalized singularities.** If a is a generalized binary singularity term, then its last k binary digits are all ones. Hence

$$a \equiv 2^k - 1 \pmod{2^k}, \qquad S_k(a) = 1.$$

Equation (2) therefore gives

$$T_k(a) = (1 + 2^k)a + 1 = 2^{2k\ell},$$

which is a power of 2^k. Thus, on the generalized singularity, the correction function retains exactly the addition of one required by the original singularity argument.

3. **Existence of a generalized trivial cycle through 1.** Put $d = 2^k$. For every integer j with $1 \le j < d$, the correction rule gives

$$T_k(j) = (d + 1)j + (d - j) = d(j + 1),$$

and the following iteration divides by d, giving $j + 1$. Also, $T_k(d) = 1$. Consequently, the orbit beginning at 1 contains the cycle

$$1 \longrightarrow 2d \longrightarrow 2 \longrightarrow 3d \longrightarrow 3 \longrightarrow \cdots (d{-}1)d \longrightarrow d{-}1 \longrightarrow d^2 \longrightarrow d \longrightarrow 1$$

This is the generalized trivial cycle prescribed by the construction.

4. **Avoidance of divergence while retaining the possibility of convergence to 1.** The fourth condition is the dynamical objective motivating the correction: the generalized trajectories should not diverge to infinity, and convergence to 1 should remain possible. The definition of S_k guarantees divisibility by 2^k after every application of the first branch, but this divisibility property alone does not prove boundedness or convergence of all trajectories. The fourth condition must therefore be treated as a research objective, to be investigated separately by mathematical arguments and computation, rather than as a consequence already established by the definition. In particular, the computational report in the preceding preprint records nontrivial cycles for $k = 2$ and $k = 4$.

The purpose of this reminder is to retain the methodological order of the construction: identify the singularity of the classical sequence, generalize its digit pattern, infer the multiplicative factor from that pattern, and only then determine the correction term by the four design conditions.

3 Extending the Generalization of the Classical Collatz Singularity to Other Bases

We now extend the singularity pattern described in Section from the binary representation to positional representations in integer bases other than two. The aim of this section is to generalize the singularity itself. The convergence properties of a dynamical system defined in such a base are not assumed and will require a separate investigation.

3.1 Choice of the base and notation

Let $p \geq 2$ be an integer that is not a proper perfect power; that is, there do not exist integers $u \geq 2$ and $v \geq 2$ such that $p = u^v$. The integer p is not required to be prime. Let $k \geq 1$ and put

$$d = p^k.$$

The restriction on p is a normalization of the parameters: it avoids representing the same integer d using a perfect-power base and a smaller block parameter. Indeed, every integer $d \geq 2$ has a unique expression $d = p^k$ with p not a proper perfect power. To see this, write $d = \prod_{i=1}^{s} q_i^{e_i}$ as a prime factorization, set $k = \gcd(e_1, \ldots, e_s)$, and set $p = \prod_{i=1}^{s} q_i^{e_i/k}$. Uniqueness follows from the uniqueness of prime factorization. This convention does not impose any primality condition on p and is not needed for the algebraic identities below.

A base-p representation uses the digits

$$0, 1, \ldots, p-1.$$

We introduce two blocks of length k:

$$\mathbf{A}_{p,k} = \underbrace{(p-1)(p-1)\cdots(p-1)}_{k \text{ digits}}, \qquad \mathbf{Z}_k = \underbrace{00\cdots0}_{k \text{ digits}}.$$

Here the expressions denote digit strings, not ordinary powers.

3.2 Definition of the generalized singularity in base p

A positive integer a is called a *generalized singularity term in base p with block length k* if its canonical base-p representation, read from the most significant digit to the least significant digit, has the form

$$\boxed{\mathbf{A}_{p,k}\big(\mathbf{Z}_k\mathbf{A}_{p,k}\big)^{\ell-1}, \qquad \ell \in \mathbb{N}^*.} \tag{4}$$

In words, the representation begins and ends with a block of k digits equal to $p-1$, and any two consecutive such blocks are separated by exactly k zero digits. The number of blocks equal to $p-1$ is at least one.

For example, when $p = 3$ and $k = 2$, the word

$$(22)_3, (22\,00\,22)_3$$

is a generalized singularity. When $p = 6$ and $k = 2$, the corresponding pattern is

$$(55)_6, (55\,00\,55)_6.$$

These examples illustrate that primality of the base is not needed: only the usual positional representation in base p is used. Setting $p = 2$ recovers the binary singularity with blocks of k ones separated by blocks of k zeros.

3.3 Numerical characterization and the singularity identity

Let a have the representation in (4), with ℓ blocks equal to $\mathbf{A}_{p,k}$. Since each such block has numerical value $p^k - 1 = d - 1$ and successive blocks are separated by k zero digits, its value is

$$a = (d-1)\left(1 + d^2 + d^4 + \cdots + d^{2\ell-2}\right). \tag{5}$$

It follows that

$$\begin{aligned}(1+d)a &= (d^2-1)\left(1 + d^2 + d^4 + \cdots + d^{2\ell-2}\right) &(6)\\ &= d^{2\ell} - 1 &(7)\\ &= p^{2k\ell} - 1. &(8)\end{aligned}$$

Moreover, the last k base-p digits of a are all equal to $p-1$. Hence

$$a \equiv p^k - 1 = d - 1 \pmod{d}.$$

If the correction rule from the binary construction is carried over by setting

$$S_{p,k}(n) = p^k - (n \mod p^k), \qquad p^k \nmid n,$$

then, at every generalized singularity term a, it satisfies

$$S_{p,k}(a) = 1.$$

Combining this equality with (8) gives

$$\boxed{(1 + p^k)a + S_{p,k}(a) = p^{2k\ell} = (p^k)^{2\ell}.} \tag{9}$$

Thus the defining singularity pattern in base p has the same structural property as its binary counterpart: multiplication by $1 + p^k$, followed by the correction that is equal to one on the singularity, produces a power of p^k.

3.4 Reduction to the binary case and scope of the result

For $p = 2$, the blocks $\mathbf{A}_{p,k}$ and $\mathbf{Z}_k$ become respectively blocks of k ones and k zeros, so Definition (4) reduces exactly to the generalized binary singularity recalled in Section . For $p \geq 3$, it gives the corresponding singularity in a non-binary integer base, without requiring p to be prime.

The identity (9) establishes the algebraic compatibility between this generalized singularity and the correction value at singularity terms. It does not, by itself, establish that every orbit of a piecewise-defined base-p Collatz transformation is bounded or converges to 1. Those are separate dynamical questions and are not conclusions of the singularity generalization proved in this section.

4 Deduction of the Generalized Collatz Transformation from the Singularity in Base p

Throughout this section, let $p \geq 2$ be an integer that is not a proper perfect power, let $k \geq 1$, and put

$$d = p^k.$$

As explained in Section 3, the restriction on p is a normalization of the parameters: it gives a unique representation $d = p^k$ with p not a proper perfect power. The arguments below do not require p to be prime.

The purpose of this section is to derive a piecewise transformation from the generalized singularity in base p. We follow the same order as in the binary construction: first determine the multiplicative factor from the singularity, then define the correction term, and finally examine the four design conditions.

4.1 Derivation of the multiplicative factor

Let a be a generalized singularity term in base p with block length k, and let $\ell \in \mathbb{N}^*$ denote the number of blocks consisting of k digits equal to $p-1$. By the numerical characterization established in Section 3, the value of a is

$$a = (d-1)\left(1 + d^2 + d^4 + \cdots + d^{2\ell-2}\right) = (d-1)\sum_{j=0}^{\ell-1} d^{2j}. \tag{10}$$

Multiplying by $1+d$ gives

$$\begin{aligned}(1+d)a &= (1+d)(d-1)\sum_{j=0}^{\ell-1} d^{2j} \\ &= (d^2-1)\sum_{j=0}^{\ell-1} d^{2j} \\ &= d^{2\ell} - 1. \end{aligned} \tag{11}$$

Since $d = p^k$, this identity can also be written as

$$(1+p^k)a = p^{2k\ell} - 1. \tag{12}$$

In base p, the integer $p^{2k\ell}-1$ consists of exactly $2k\ell$ digits, all equal to $p-1$. Consequently, adding one produces a power of p^k:

$$(1+p^k)a + 1 = p^{2k\ell} = (p^k)^{2\ell} = d^{2\ell}. \tag{13}$$

This calculation determines the multiplicative factor $1+p^k$. It is not introduced arbitrarily: it is the factor for which the digit pattern of the generalized singularity yields an integer consisting entirely of digits equal to $p-1$, so that the addition of one produces a power of p^k.

4.2 Construction of the correction function $S_{p,k}$

The identity above explains why adding one is appropriate when the input is a generalized singularity. For an arbitrary positive integer, however, adding one to $(1+p^k)n$ does not necessarily make the result divisible by p^k. We therefore replace the fixed addition by the least positive correction that guarantees this divisibility.

Let

$$\mathcal{D}_{p,k} = \{n \in \mathbb{N}^* : p^k \nmid n\}.$$

For $n \in \mathcal{D}_{p,k}$, the residue $n \bmod p^k$ belongs to $\{1, 2, \ldots, p^k - 1\}$. Define

$$S_{p,k} : \mathcal{D}_{p,k} \longrightarrow \{1, 2, \ldots, p^k - 1\}, \qquad S_{p,k}(n) = p^k - (n \bmod p^k). \tag{14}$$

Because $1 + p^k \equiv 1 \pmod{p^k}$, we have

$$(1 + p^k)n + S_{p,k}(n) \equiv n + p^k - (n \bmod p^k) \equiv 0 \pmod{p^k}.$$

Thus the correction makes the output divisible by p^k. Moreover, since $n \bmod p^k$ is nonzero, the value $p^k - (n \bmod p^k)$ is the unique least positive integer in the required residue class. Hence $S_{p,k}(n)$ is precisely the smallest positive additive correction that must be added to $(1+p^k)n$ to obtain a multiple of p^k.

The correction also retains the required value on the generalized singularity. Indeed, the last k digits of a generalized singularity term are all equal to $p-1$, so

$$a \equiv p^k - 1 \pmod{p^k}, \qquad S_{p,k}(a) = 1. \tag{15}$$

4.3 Definition of the generalized transformation

We can now define the generalized Collatz transformation associated with the base p and block length k by

$$T_{p,k} : \mathbb{N}^* \longrightarrow \mathbb{N}^*, \qquad T_{p,k}(n) = \begin{cases} (1 + p^k)n + S_{p,k}(n), & p^k \nmid n, \\ \dfrac{n}{p^k}, & p^k \mid n. \end{cases} \tag{16}$$

By construction, the first branch always produces a positive multiple of p^k. Therefore, whenever the first branch is applied, the second branch is applicable at the next iteration.

A useful equivalent expression follows by writing $n = qd + r$, where $q = \lfloor n/d \rfloor$ and $1 \le r \le d - 1$. In this case,

$$\begin{aligned} T_{p,k}(n) &= (1 + d)n + (d - r) \\ &= d(n + q + 1). \end{aligned} \tag{17}$$

Applying the division branch at the next iteration gives the two-step identity

$$T_{p,k}^2(n) = T_{p,k}\big(d(n + q + 1)\big) = n + q + 1 = n + \left\lfloor \frac{n}{p^k} \right\rfloor + 1, \qquad p^k \nmid n. \tag{18}$$

Here $T_{p,k}^2$ denotes two successive applications of $T_{p,k}$.

4.4 Examination of the four design conditions

The correction function is introduced in accordance with the four design conditions used in the binary construction. We examine each condition for the transformation (16).

4.4.1 Recovery of the classical Collatz transformation

For $p = 2$ and $k = 1$, we have $p^k = 2$. Every positive integer in the first branch is odd, and therefore

$$S_{2,1}(n) = 2 - (n \bmod 2) = 1.$$

Consequently,

$$T_{2,1}(n) = \begin{cases} 3n+1, & n \equiv 1 \pmod 2, \\ n/2, & n \equiv 0 \pmod 2. \end{cases}$$

Thus the classical $3n + 1$ Collatz transformation is recovered exactly when $p = 2$ and $k = 1$. The first design condition is satisfied.

4.4.2 Preservation of the $+1$ action on generalized singularities

Let a be any generalized singularity term in base p with block length k. By (15), its correction is exactly one. Combining this with (13), we obtain

$$T_{p,k}(a) = (1 + p^k)a + S_{p,k}(a) = p^{2k\ell}.$$

The multiplication-correction branch therefore transforms every generalized singularity term into a power of p^k in one iteration. The subsequent iterations apply the division branch:

$$p^{2k\ell} \longmapsto p^{k(2\ell-1)} \longmapsto \cdots \longmapsto p^k \longmapsto 1.$$

Hence the correction retains exactly the addition of one on the generalized singularity, as required by the singularity construction. The second design condition is satisfied.

4.4.3 Existence of a generalized trivial cycle through 1

Put $d = p^k$. For every integer j with $1 \le j < d$, we have $j \bmod d = j$, so

$$S_{p,k}(j) = d - j.$$

It follows that

$$\begin{aligned} T_{p,k}(j) &= (1+d)j + (d-j) \\ &= d(j+1). \end{aligned}$$

Since $d(j+1)$ is divisible by d, the next iteration gives

$$T_{p,k}\big(d(j+1)\big) = j+1.$$

In particular, as j runs from 1 to $d-1$, these transitions yield the cycle [4]

$$1 \longrightarrow 2d \longrightarrow 2 \longrightarrow 3d \longrightarrow 3 \longrightarrow \cdots \ (d-1)d \longrightarrow d-1 \longrightarrow d^2 \longrightarrow d \longrightarrow 1. \tag{19}$$

The final transition follows from $T_{p,k}(d) = d/d = 1$. The displayed cycle has period $2d-1 = 2p^k - 1$. Thus a generalized trivial cycle containing 1 exists for every admissible p and every $k \geq 1$. The third design condition is satisfied.

4.4.4 Avoidance of divergence and the possibility of convergence to 1

The fourth design condition asks for the generalized trajectories to avoid divergence to infinity while retaining the possibility of convergence to 1. The construction establishes the latter possibility: the cycle (19) consists of terms whose trajectories reach 1 and then repeat periodically.

However, the divisibility property proved for the first branch does not establish that every trajectory is bounded, nor does it rule out divergence or other nontrivial cycles. In particular, the two-step identity (18) shows that, whenever $p^k \nmid n$, the value after the multiplication-correction step and the immediately following division step is strictly greater than n. Further divisions may subsequently occur, but this identity alone gives no global bound on an orbit.

Accordingly, the fourth condition is a dynamical objective of the construction, not a property proved by the definition of $S_{p,k}$. Its validity for all starting values and all admissible parameters requires a separate analysis. Computational experiments can provide evidence and detect particular cycles, but cannot by themselves establish universal convergence. The fourth condition therefore remains open at this stage.

The construction therefore satisfies the first three design conditions, while the fourth remains a separate dynamical problem to be investigated.

5 Relation to Previous Generalizations and Methodological Positioning

The transformation defined in Section belongs to a broader literature on generalized Collatz mappings. It is therefore important to distinguish the arithmetic map itself from the methodological route used in the present work. In particular, the present paper does not claim that the piecewise mapping $T_{p,k}$, considered solely as an integer-valued transformation, is introduced here for the first time. The purpose of this section is to identify the relevant antecedents and to state precisely the scope of the contribution developed from the singularity pattern.

5.1 Relation to Bruschi's two-parameter family

Bruschi introduced a two-parameter family of generalized Collatz mappings. In the notation used here, his map can be written as

$$F_{b,m}(n) = \begin{cases} \dfrac{n}{b}, & b \mid n, \\ (b^m+1)n + b^m - (n \bmod b^m), & b \nmid n, \end{cases} \qquad b \geq 2, \quad m \geq 1. \tag{20}$$

Bruschi studies the convergence behavior of these mappings and gives both a limited convergence result and examples of nontrivial cycles for some parameter choices [4].

The relation with the transformation of Section is exact. Set

$$b = p^k, \qquad m = 1.$$

Then $b^m = b = p^k$, and the condition $b \nmid n$ is exactly $p^k \nmid n$. Consequently, Equation (20) becomes

$$F_{p^k,1}(n) = \begin{cases} \dfrac{n}{p^k}, & p^k \mid n, \\ (1+p^k)n + p^k - (n \bmod p^k), & p^k \nmid n, \end{cases} = T_{p,k}(n). \tag{21}$$

Thus, the transformation proposed in this paper is precisely the $m = 1$ member of Bruschi's family, with $b = p^k$. Moreover, the convention that p is not a proper perfect power is a canonical parametrization of the integer b: every integer $b \geq 2$ has a unique representation $b = p^k$ of this form. It does not select a new set of integer maps beyond Bruschi's $m = 1$ subfamily.

This equivalence has a direct consequence for the claims made in the present paper. The piecewise arithmetic rule in Equation (16) should not be presented

as a new mapping independent of Bruschi's construction. In addition, convergence and cycle observations for this rule must be discussed in light of the results already reported for the corresponding parameters in [4].

5.2 Relation to Carnielli's work

Carnielli's paper, *Some Natural Generalizations of the Collatz Problem*, is another relevant antecedent in the study of generalized Collatz mappings [3]. The earlier binary study that motivates the present manuscript already acknowledged a relationship between its family, with divisor parameter $d = 2^k$, and generalized mappings discussed by Carnielli. That relationship should remain explicit in the present work: the binary case $p = 2$ is not being presented as unrelated to the existing theory of generalized Collatz maps.

The purpose of citing Carnielli here is to place the construction within that broader context, rather than to claim that every parameter choice in the present family is distinct from every mapping considered there. A complete term-by-term comparison with all of Carnielli's formulations would require a separate analysis. Accordingly, no stronger claim of disjointness or priority is made in this paper.

5.3 The methodological focus of the present work

Although the arithmetic transformation in Equation (16) coincides with a known member of Bruschi's family, the organizing objective of the present paper is different. The construction is developed by taking a specific singularity pattern of the classical $3n + 1$ sequence as its starting point and following the structure of that pattern through successive generalizations:

1. The classical binary singularity is identified from its alternating digit pattern.

2. The binary pattern is generalized by replacing each digit 1 and each separating digit 0 with blocks of k digits.

3. The resulting block structure is extended from base two to an arbitrary integer base $p \geq 2$, using blocks of k digits equal to $p - 1$, separated by blocks of k zeros.

4. The numerical value of this singularity is used to derive the factor $1+p^k$. The correction function is then selected so that it is equal to one on every term having the prescribed singularity and so that the non-divisible branch produces a multiple of p^k.

In particular, the central identity established in Section 3 is

$$(1 + p^k)a = p^{2k\ell} - 1,$$

for a generalized singularity term a with ℓ blocks of k digits equal to $p - 1$. Since the last k base-p digits of a are all equal to $p-1$, the correction satisfies $S_{p,k}(a) = 1$, and hence

$$T_{p,k}(a) = (1 + p^k)a + 1 = p^{2k\ell}.$$

The emphasis is therefore on the singularity as the structural object from which the multiplicative factor and the value of the correction on that singularity are derived.

This methodological distinction should be stated with precision. Bruschi's paper also uses base-b digit expansions in part of its analysis; consequently, the present work does not claim to be the first use of positional representations in the study of generalized Collatz mappings. Rather, its focus is the explicit formulation and extension of the particular singularity pattern inherited from the classical binary problem, and the systematic derivation of the associated correction rule from that pattern.

5.4 Scope of the contribution

The contribution claimed here is thus a singularity-centered structural presentation and extension, not the introduction of a new piecewise integer map. The identity between $T_{p,k}$ and the $m = 1$ specialization of Bruschi's family is part of the mathematical context of the paper, not a point to be concealed. Likewise, the relation with Carnielli's work is acknowledged. The results established in the present manuscript concern the generalized digit singularity and its algebraic compatibility with the correction function. They do not establish universal boundedness or convergence of all trajectories. Any further claim of originality should therefore be based on additional theorems about the singularity structure or its consequences for the dynamics, rather than on the formula for $T_{p,k}$ alone.

6 Conclusion

This paper has presented a singularity-based extension of the methodological construction associated with the classical Collatz map. Starting from the alternating binary pattern and its block generalization, we extended the digit

structure to integer bases $p \geq 3$, with a block length $k \geq 1$. The base p was not required to be prime; the condition that it is not a proper perfect power serves only to give a canonical parametrization of the divisor $d = p^k$.

For a generalized singularity term with ℓ blocks of k digits equal to $p-1$, we obtained

$$a = (p^k - 1) \sum_{j=0}^{\ell-1} p^{2kj}, \qquad (1 + p^k)a = p^{2k\ell} - 1.$$

The last k base-p digits of such a term are all equal to $p-1$, so the correction function satisfies $S_{p,k}(a) = 1$. Consequently, the corrected multiplicative branch maps every term of the prescribed singularity form to the power $p^{2k\ell}$. We also showed that the construction recovers the classical $3n+1$ map when $p = 2$ and $k = 1$, and that the generalized transformation has a trivial cycle through 1 of length $2p^k - 1$.

The scope of these results must nevertheless be kept precise. The first three design conditions are verified by algebraic arguments. The fourth condition, concerning the prevention of divergence and the global dynamical behavior of all trajectories, is not established here. In addition, the piecewise transformation is not claimed as a new arithmetic family: it coincides with the $m = 1$ specialization of Bruschi's two-parameter family, while its relation to Carnielli's congruence-based map is described in Section 5. The contribution of the present work is therefore the singularity-centered derivation and its extension to other integer bases, not a proof of universal convergence or the discovery of a new map family.

In future versions, we plan to carry out a systematic computational verification of the generalized dynamics for a range of admissible bases p and block lengths k. Such an investigation will use exact integer arithmetic, record the tested ranges and stopping criteria, identify and verify any nontrivial cycles encountered, and compare the results with the cycles and parameter cases already reported in the literature. These computations will be presented as reproducible experimental evidence only; the absence of a cycle in a finite search will not be treated as a proof that no such cycle exists, nor will numerical convergence establish convergence for all positive integers. A general proof of boundedness or convergence, if true, remains a separate mathematical problem.

References

[1] Terence Tao, *Almost all orbits of the Collatz map attain almost bounded values*, DOI: 10.48550/arXiv.1909.03562 [math.NT], 2026.

[2] Ammar Hamdous, *Methodological Generalization of the Collatz Sequences*, Zenodo, 2026, doi.org/10.5281/zenodo.22864107.

[3] W. Carnielli, *Some Natural Generalizations of the Collatz Problem*, Applied Mathematics E-Notes, 15 (2015), 207–215.

[4] M. Bruschi, "A generalization of the Collatz problem and conjecture," arXiv:0810.5169 [math.NT], 2008. `https://arxiv.org/abs/0810.5169`.