Wave Function of the Universe near the Big-Bang Singularity in Jackiw-Teitelboim Cosmology

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Abstract

Recently we investigated the wave function of the Universe near the Big-Bang Singularity in three-dimensional pure Einstein Gravity with a mini-superspace model, by extending the Hamiltonian constraint [1]. In this article, we apply this method to Jackiw-Teitelboim (JT) cosmology with a positive cosmological constant ($\Lambda > 0$). By extending the Hamiltonian constraint from $H = 0$ to $H \approx 0$ around the Planck Scale , $H\Psi = 0$ (Wheeler-DeWitt equation) may become $H\Psi = E\Psi$ (Schrodinger equation).Using the light-cone variables u and v, the Schrodinger-like equation is successfully mapped onto two independent equations associated with inverse harmonic oscillator. We show that the exact analytic solutions are described by parabolic cylinder functions, and the solution wave function is regular around the Big Bang Singularity($a = 0$).

1 Introduction

Recently we investigated the wave function of the universe near the Big Bang singularity in three dimensions by extending the Hamiltonian constraint [1]. In this report, we will apply this framework to the Jackiw-Teitelboim (JT) cosmology, which is a renormalizable quantum gravity model in two dimensions [2, 3] including a matter field ϕ.

In the standard JT cosmology, the universe is described by the wave function that satisfies the Wheeler-De Witt (WDW) equation. Of course the WDW equation is induced from the Hamiltonian constraint $H = 0$ [4, 5]. Therefore Iliesiu, Kruthoff, Turiaci, and Verlinde(IKTV) , Fanaras and Vilenkin(FV) , Maldacena , Turiaci , and Yang(MTY) used this Hamiltonian constraint , as a standard method , which often induces severe mathematical difficulties, [6–8].

We are much interested in the Planck scale physics where classical general relativity is no longer valid. Therefore, we lift the constraint from $H = 0$ to $H \approx 0$, or more precisely, $H + \delta H$, where δH includes the quantum gravitational information. This modification

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naturally converts the WDW equation into a Schrödinger-like evolution equation, $H\Psi = E\Psi$, The solutions of this Schrödinger-like equation are represented by the exact wave function of the parabolic cylinder functions,

The plan of this report is as follows. In Section 2 we begin by setting up the Lagrangian and Hamiltonian. In Section 3 we deal with the extended Hamiltonian constraint and the mapping onto two independent inverse harmonic oscillators. Section 4 covers the exact solution and the smooth regularization of the Big Bang singularity. Section 5 is devoted to conclusions.

2 Set-Up and Mini-superspace Hamiltonian

Our starting point is the Lorentzian JT action with a positive cosmological constant $\Lambda > 0$ defining a de Sitter (dS_2) background:

$$S = \frac{1}{16\pi G}\int dtdx\sqrt{-g}\,\phi(R - 2\Lambda) \tag{1}$$

We assume a homogeneous mini-superspace metric with the lapse function $N(t)$:

$$ds^2 = -N(t)^2dt^2 + a(t)^2dx^2 \quad (0 \leq x \leq 2\pi) \tag{2}$$

Integrating out the spatial section x, the Lagrangian L is exactly derived after integration by parts as:(see Appendix)

$$L = -\frac{1}{4G}\left(\frac{\dot{a}\dot{\phi}}{N} + N\Lambda a\phi\right) \tag{3}$$

where overdots denote derivatives with respect to t. The conjugate momenta to the scale factor $a(t)$ and the dilaton field $\phi(t)$ are represented respectively:

$$p_a = \frac{\partial L}{\partial \dot{a}} = -\frac{\dot{\phi}}{4GN}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = -\frac{\dot{a}}{4GN} \tag{4}$$

Performing the Legendre transformation, the Hamiltonian of the system is constructed as:

$$H = p_a\dot{a} + p_\phi\dot{\phi} - L = N(t)\left(-4Gp_ap_\phi + \frac{\Lambda}{4G}a\phi\right) \tag{5}$$

Varying the action with respect to N yields the Hamiltonian Constraint

$\partial L/\partial N = 0 \Longleftrightarrow \left(-4Gp_ap_\phi + \frac{\Lambda}{4G}a\phi\right) = 0.$

3 Schrödinger-like equation and Inverse Harmonic Oscillator

Following our fundamental concept, the extension of the Hamiltonian Constraint, we obtain the Schrödinger form $H\Psi = E\Psi$ by setting $N = 1$. Applying the canonical quantization rules $p_a \to -i\hbar\partial_a$ and $p_\Phi \to -i\hbar\partial_\Phi$, the partial differential equation(Schrödinger-like equation) is obtained:

$$\left(4G\hbar^2\frac{\partial^2}{\partial a\partial\phi} + \frac{\Lambda}{4G}a\phi\right)\Psi(a,\phi) = E\Psi(a,\phi) \tag{6}$$

Introducing the rescaled parameters $\Omega^2 = \frac{\Lambda}{16G^2\hbar^2} > 0$ and $\mathcal{E} = \frac{E}{4G\hbar^2}$, the above equation becomes:

$$\frac{\partial^2}{\partial a\partial\phi}\Psi + \Omega^2 a\phi\Psi = \mathcal{E}\Psi \tag{7}$$

To eliminate the cross-derivatives, we introduce the light-cone variables u and v:

$$u = \frac{1}{2}(a+\phi), \quad v = \frac{1}{2}(a-\phi) \tag{8}$$

Since $a\phi = u^2 - v^2$ and $\frac{\partial^2}{\partial a\partial\phi} = \frac{1}{4}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right)$, the system cleanly separates into u and v sectors:

$$\left[\left(\frac{\partial^2}{\partial u^2} + 4\Omega^2u^2\right) - \left(\frac{\partial^2}{\partial v^2} + 4\Omega^2v^2\right)\right]\Psi = 4\mathcal{E}\Psi \tag{9}$$

Using the separation ansatz $\Psi(u,v) = \psi_u(u)\psi_v(v)$ with $4\mathcal{E} = \varepsilon_u - \varepsilon_v$, the quantum cosmology of the JT model is perfectly mapped onto two completely independent inverse harmonic oscillators:

$$\left(\frac{d^2}{du^2} + 4\Omega^2u^2\right)\psi_u(u) = \varepsilon_u\psi_u(u) \tag{10}$$

$$\left(\frac{d^2}{dv^2} + 4\Omega^2v^2\right)\psi_v(v) = \varepsilon_v\psi_v(v) \tag{11}$$

Remark on the Dimensional Contrast and Potential Inversion:
It is highly instructive to compare this result with the three-dimensional pure Einstein gravity framework studied in our previous work [1]. In three dimensions, the effective minisuperspace Lagrangian reduces to a standard harmonic oscillator with a potential barrier ($U(a) \sim -a^2$) in some approximation, where a particle undergoes quantum tunneling at the full potential. [1]

On the other hand in JT cosmology the equations of the light-cone variables u and v become the inverted harmonic oscillator which is a unique and distinguishing feature of JT cosmology.

4 Big Bang Singularity Regularization and Parabolic Cylinder Functions

The separated differential equations in Eq (10),(11) are mathematically classified as the inverted harmonic oscillator.The solutions are given by linear combination of the parabolic cylinder functions.

$$\psi_u(u) = A_u D_{\nu_u}\left(\sqrt{2\Omega}(1+i)u\right) + B_u D_{\nu_u}\left(-\sqrt{2\Omega}(1+i)u\right), \tag{12}$$

$$\psi_v(v) = A_v D_{\nu_v}\left(\sqrt{2\Omega}(1+i)v\right) + B_v D_{\nu_v}\left(-\sqrt{2\Omega}(1+i)v\right), \tag{13}$$

where $A_{u,v}$ and $B_{u,v}$ are integration constants,D_{ν_u} and D_{ν_u} are the parabolic cylinder functions and the complex indices are determined by

$$\nu_u = -\frac{1}{2} + i\frac{\varepsilon_u}{4\Omega}, \qquad \nu_v = -\frac{1}{2} + i\frac{\varepsilon_v}{4\Omega}. \tag{14}$$

Using the separation relation $\varepsilon_u - \varepsilon_v = 4\mathcal{E} = E/(G\hbar^2)$ and $\Omega = \sqrt{\Lambda}/(4G\hbar)$, the extended energy eigenvalue E directly governs the index asymmetry between the two modes:

$$\nu_u - \nu_v = i\frac{E}{\hbar\sqrt{\Lambda}}. \tag{15}$$

Selecting the fundamental mode ($B_u = B_v = 0$) and transforming back to the original variables (a, ϕ), we obtain the explicit wave function of the Universe:

$$\Psi(a,\phi) = C\, D_{\nu_u}\left(\sqrt{\frac{\sqrt{\Lambda}}{2G\hbar}}(1+i)\frac{a+\phi}{2}\right) D_{\nu_v}\left(\sqrt{\frac{\sqrt{\Lambda}}{2G\hbar}}(1+i)\frac{a-\phi}{2}\right), \tag{16}$$

where $C = A_u A_v$ is an overall normalization constant.

Fortunately this explicit wave function smoothly regularizes the Big-Bang singularity. Taking the limit $a \to 0$, the wave function reduces to a completely analytic function of the dilaton field ϕ:

$$\lim_{a\to 0}\Psi(a,\phi) = C\, D_{\nu_u}\left(\sqrt{\frac{\sqrt{\Lambda}}{8G\hbar}}(1+i)\phi\right) D_{\nu_v}\left(-\sqrt{\frac{\sqrt{\Lambda}}{8G\hbar}}(1+i)\phi\right). \tag{17}$$

At the exact origin $(a, \phi) \to (0, 0)$, using the identity $D_\nu(0) = 2^{\nu/2}\sqrt{\pi}/\Gamma\left(\frac{1-\nu}{2}\right)$, we arrive at the finite, non-divergent amplitude:

$$\Psi(0,0) = C\,\frac{2^{(\nu_u+\nu_v)/2}\,\pi}{\Gamma\left(\frac{3}{4} - i\frac{\varepsilon_u}{8\Omega}\right)\Gamma\left(\frac{3}{4} - i\frac{\varepsilon_v}{8\Omega}\right)}. \tag{18}$$

Because the Gamma function has no zeros in the complex plane, the denominator never vanishes. Thus, the extended constraint $\hat{H}\Psi = E\Psi$ ($E \neq 0$) naturally prevents the wave function from blowing up at the singularity.

In the $H = 0$ case discussed by FV [7], IKTV [6] and MY [8], the wave functions around both the Big-Bang Singularity and the cosmological horizon are divergent. On the other hand in our case the wave functions are not divergent there.

5 Conclusions

We have investigated the wave function of the universe near the Big-Bang singularity in JT cosmology by extending the Hamiltonian constraint from $H = 0$ to $H \approx 0$, leading to the Schrodinger-like equation $H\Psi = E\Psi$. The solution wave functions are described as an exact parabolic cylinder function which has convergent wave functions around the Big Bang Singularity. In the near future, we will apply this method to Unimodular JT cosmology and wormhole .

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Appendix

We explain how to obtain the Lagrangian. The metric is

$$ds^2 = -N^2dt^2 + a(t)^2dx^2 = g_{\mu\nu}dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 & 0 \\ 0 & a^2 \end{pmatrix}, g^{\mu\nu} = \begin{pmatrix} -N^{-2} & 0 \\ 0 & a^{-2} \end{pmatrix}$$

Non-vanishing terms of the Chlistoffel Symbols:$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$

$$\Gamma^0_{11} = \frac{a\dot{a}}{N^2}, \Gamma^1_{10} = \Gamma^1_{01} = \frac{\dot{a}}{a}$$

Next , non vanishing terms of the Ricci tensor $R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\nu\rho}\Gamma^\rho_{\mu\sigma}$ iare

$$R_{00} = -\frac{\ddot{a}}{a}, R_{11} = \frac{a\ddot{a}}{N^2} \tag{19}$$

Then,

the scalar curvature is

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} = -N^{-2}(-\frac{\ddot{a}}{a}) + a^{-2}\frac{a\ddot{a}}{N^2} = \frac{2\ddot{a}}{aN^2}$$

$$g = det(g_{\mu\nu}) = -N^2a^2, \sqrt{-g} = \sqrt{N^2a^2} = Na$$

Therefore

$$
\begin{aligned}
S &= \frac{1}{16\pi G}\int dtdx\sqrt{-g}\,\phi(R-2\Lambda) && (20)\\
&= \frac{1}{16\pi G}\int dtdx(Na\phi R - 2Na\phi\Lambda) && (21)\\
&= -\frac{2}{16\pi G}\int dtdx(\frac{-\phi\ddot{a}}{N} + Na\phi\Lambda) && (22)\\
&= -\frac{2}{16\pi G}\int_0^{2\pi} dx\int dt(\frac{-\phi\ddot{a}}{N} + Na\phi\Lambda) && (23)\\
&= -\frac{1}{4G}\int dt(\frac{-\phi\ddot{a}}{N} + Na\phi\Lambda) && (24)\\
&= -\frac{1}{4G}\int dt(\frac{\dot{a}\dot{\phi}}{N} + NA\phi\Lambda) && (25)\\
&= \int dtL(t) && (26)
\end{aligned}
$$

where we used the integration of part;

$$\int dt(-\frac{\ddot{a}\phi}{N}) = -\frac{\dot{a}\phi}{N}|_i^f - \int dt(-\frac{\dot{a}\dot{\phi}}{N}) = \int(\frac{\dot{a}\dot{\phi}}{N})dt$$

Consequently

$$L = -\frac{1}{4G}\left(\frac{\dot{a}\dot{\phi}}{N} + N\Lambda a\phi\right) \quad (27)$$

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