Closed-Form Evaluations of a Family of logarithmic Bessel integrals

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Abstract

We study the family of logarithmic Bessel integrals

$$I_n := \int_0^\infty J_0(x)^n \ln x \, dx,$$

where J_0 denotes the Bessel function of the first kind of order zero. Using Mellin transform methods and Mellin–Barnes contour representations, we derive explicit closed forms for several low-order cases. We also record a convolution-based recurrence for the Mellin transforms associated with this family and discuss the asymptotic behavior of I_n for large n. The methods illustrate how increasing powers of J_0 lead naturally to higher special-function complexity.

Keywords: Definite integrals, Bessel functions, Mellin transforms, hypergeometric functions, Meijer G-function, closed-form evaluation.

MSC 2020: 33C10, 33C20, 33C60, 44A15.

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1 Introduction

1.1 Background

In this paper, we study the family of logarithmic Bessel integrals

$$I_n := \int_0^\infty J_0(x)^n \ln x \, dx,$$

where $J_0(x)$ is the Bessel function of the first kind of order zero and n is a positive integer. Integrals involving powers and products of Bessel functions occur frequently in mathematical physics, harmonic analysis, and probability theory. They are often difficult to evaluate in closed form because of the oscillatory nature of Bessel functions on the unbounded interval $(0, \infty)$.

A natural way to analyze such integrals is through Mellin transforms; standard background on the Bessel function and its Mellin transform may be found in Watson [1]. The logarithmic factor $\ln x$ is especially well suited to this approach, since it can be obtained by differentiating with respect to the Mellin parameter. This makes the family I_n a useful testing ground for Mellin-transform methods and Mellin–Barnes contour techniques.

1.2 Motivation

The integrals

$$I_n = \int_0^\infty J_0(x)^n \ln x \, dx$$

combine two sources of analytic complexity. First, the factor $J_0(x)^n$ produces an increasingly intricate oscillatory structure as n grows. Second, the logarithmic weight $\ln x$ introduces sensitivity to the behavior of the Mellin transform near its central value and naturally leads to differentiation of parameter-dependent integral representations.

This combination makes the family suitable for studying how special-function representations evolve with increasing powers of J_0. In particular, it illustrates the transition from elementary formulas to Meijer G-functions and then to multivariable hypergeometric structures.

1.3 Main Contributions

The main results of the paper are as follows:

- We derive the elementary evaluation of the case $n = 1$
- We show that the case $n = 2$ diverges.
- We express the cases $n = 3$ and $n = 4$ in terms of Meijer G-functions via Mellin–Barnes integrals.
- We derive Kampé de Fériet representations for the cases $n = 5$ and $n = 6$.
- We formulate a Mellin-convolution recurrence for the associated Mellin transforms and discuss the large-n asymptotic behavior of I_n.

These formulas show that the analytic complexity of the family grows in a controlled way with the exponent n.

1.4 Organization of the Paper

Section 2 collects notation and standard facts about the Gamma function, the Mellin transform, the Meijer G-function, and the Kampé de Fériet function. Section 3 states the main results. Section 4 gives the derivations of the closed forms for $n = 1, \dots, 6$. Section 5 discusses the Mellin-convolution recurrence, the large-n asymptotic behavior, and analytic continuation. Section 6 concludes the paper.

2 Preliminaries and Notation

2.1 Gamma and Beta Functions

Recall the definitions of the Gamma and Beta functions:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

2.2 Pochhammer Symbol

For $a \in \mathbb{C}$ and $n \in \mathbb{N}_0$, the Pochhammer symbol is

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

2.3 The Kampé de Fériet Function

We use the two-variable Kampé de Fériet function in the form (see also Luke [2] and Srivastava–Karlsson [3])

$$F^{m:n;k}_{p:q;r}\left[\begin{array}{c}(a_1,\dots,a_p):(b_1,\dots,b_q);(c_1,\dots,c_r)\\(\alpha_1,\dots,\alpha_m):(\beta_1,\dots,\beta_n);(\gamma_1,\dots,\gamma_k)\end{array};x,y\right] = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{u+v}\prod_{j=1}^{q}(b_j)_u\prod_{j=1}^{r}(c_j)_v}{\prod_{j=1}^{m}(\alpha_j)_{u+v}\prod_{j=1}^{n}(\beta_j)_u\prod_{j=1}^{k}(\gamma_j)_v}\frac{x^u}{u!}\frac{y^v}{v!}.$$

2.4 Meijer G-Function

Define the Meijer G-function.

$$G_{p,q}^{m,n}\left(\left.\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right|z\right)=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+u)\prod_{j=1}^{n}\Gamma(1-a_j-u)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-u)\prod_{j=n+1}^{p}\Gamma(a_j+u)}z^{-u}du$$

2.5 Mellin Transform

Define the Mellin transform

$$\mathcal{M}\{f\}(s)=\int_0^\infty x^{s-1}f(x)\,dx.$$

3 Statement of Main Results

3.1 Main Closed-Form Formula

Theorem 3.1 (Case $n=1$)**.**

$$\int_0^\infty J_0(x)\ln x\,dx=-\gamma-\ln 2.$$

Theorem 3.2 (Case $n=2$)**.** *The integral diverges,*

$$\int_0^\infty J_0(x)^2\ln x\,dx=+\infty.$$

Theorem 3.3 (Case $n=3$)**.**

$$\int_0^\infty J_0(x)^3\ln x\,dx=\frac{1}{2\sqrt{\pi}}\frac{d}{ds}\left[G_{3,3}^{2,1}\left(\left.\frac{1}{4}\right|\begin{matrix}1-\frac{s}{2},1-\frac{s}{2},1-\frac{s}{2}\\ 0,\frac{1}{2}-\frac{s}{2},0\end{matrix}\right)\right]_{s=1}$$

Theorem 3.4 (Case $n=4$)**.**

$$\int_0^\infty J_0(x)^4\ln x\,dx=\frac{1}{2\pi}\frac{d}{ds}\left[G_{4,4}^{2,2}\left(1\left|\begin{matrix}\frac{1}{2},\ 1-\frac{s}{2},\ 1-\frac{s}{2},\ 1-\frac{s}{2}\\ 0,\ \frac{1}{2}-\frac{s}{2},\ \ 0,\ \ \ 0\end{matrix}\right.\right)\right]_{s=1}$$

Theorem 3.5 (Case $n=5$)**.**

$$\int_0^\infty J_0(x)^5\ln x\,dx=\frac{d}{ds}\left[\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{2\sqrt{\pi}\Gamma\left(1-\frac{s}{2}\right)^2}F_{1:2;1}^{3:1;0}\left[\begin{matrix}\frac{s}{2},\frac{s}{2},\frac{s}{2}:\frac{1}{2};-\\ \frac{s}{2}+\frac{1}{2}:1,1;1\end{matrix};\frac{1}{4},\frac{1}{4}\right]\right]\Bigg|_{s=1}.$$

Theorem 3.6 (Case $n=6$)**.**

$$\int_0^\infty J_0(x)^6\ln x\,dx=\frac{d}{ds}\left[\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{8\pi^{5/2}\Gamma\left(1-\frac{s}{2}\right)^2}F_{1:2;2}^{3:1;1}\left(\begin{matrix}\frac{s}{2},\frac{s}{2},\frac{s}{2}:\ \frac{1}{2}\ ;\ \frac{1}{2}\\ \frac{s}{2}+\frac{1}{2}\ :1,1\ ;\ 1,1\end{matrix};1,1\right)\right]\Bigg|_{s=1}$$

4 Derivation of the Closed Forms

4.1 Case $n = 1$

Proof.

$$\int_0^\infty J_0(x) \ln x \, dx$$

Let

$$F_1(s) := \int_0^\infty x^{s-1} J_0(x) \, dx.$$

Then

$$I_1 = \int_0^\infty J_0(x) \ln x \, dx = F_1'(1).$$

Use the Mellin transform

$$F_1(s) = \int_0^\infty x^{s-1} J_0(x) \, dx = 2^{s-1} \frac{\Gamma(s/2)}{\Gamma(1 - s/2)}.$$

Taking logarithmic derivatives

$$\frac{F_1'(s)}{F_1(s)} = \ln 2 + \frac{1}{2}\psi\left(\frac{s}{2}\right) + \frac{1}{2}\psi\left(1 - \frac{s}{2}\right).$$

At $s = 1$,

$$F_1(1) = 1, \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2.$$

Therefore

$$F_1'(1) = \ln 2 + \psi\left(\frac{1}{2}\right) = \ln 2 - \gamma - 2\ln 2 = -\gamma - \ln 2.$$

Thus

$$\boxed{\int_0^\infty J_0(x) \ln x \, dx = -\gamma - \ln 2.}$$

□

4.2 Case $n = 2$

Proof. We show that the integral diverges. The classical asymptotic expansion (see Watson [1])

$$J_0(x) = \sqrt{\frac{2}{\pi x}} \left(\cos\Big(x - \frac{\pi}{4}\Big) + O\Big(\frac{1}{x}\Big) \right), \qquad x \to \infty,$$

implies

$$J_0(x)^2 = \frac{1}{\pi x}\big(1 + \sin 2x\big) + O\left(\frac{1}{x^2}\right).$$

Consequently,

$$\int_1^R J_0(x)^2 \ln x \, dx = \frac{1}{\pi}\int_1^R \frac{\ln x}{x}\, dx + \frac{1}{\pi}\int_1^R \frac{\ln x}{x}\sin 2x \, dx + \int_1^R O\left(\frac{\ln x}{x^2}\right) dx$$

The first term equals $\frac{1}{2\pi}(\ln R)^2$ and diverges as $R \to \infty$. The oscillatory term converges by Dirichlet's test, and the error term is absolutely convergent. Hence the integral does not converge. □

4.3 Case $n = 3$

Proof.

$$\int_0^\infty J_0(x)^3 \ln x \, dx$$

Let

$$F_3(s) := \int_0^\infty x^{s-1} J_0(x)^3 \, dx,$$
$$I_3 = \int_0^\infty J_0(x)^3 \ln x \, dx = F_3'(1).$$

Use the Mellin transform of J_0,

$$A(s) := \int_0^\infty x^{s-1} J_0(x)\, dx = 2^{s-1}\frac{\Gamma(s/2)}{\Gamma(1 - s/2)},$$

and

$$C(s) := \int_0^\infty x^{s-1} J_0(x)^2 \, dx = \frac{\Gamma(s/2)\Gamma((1-s)/2)}{2\sqrt{\pi}\Gamma(1 - s/2)^2}$$

Since $J_0(x)^3 = J_0(x)J_0(x)^2$, the Mellin convolution gives,

$$F_3(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A(z)C(s - z)\, dz.$$

Substitute A and C:

$$F_3(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{z-1}\frac{\Gamma(z/2)}{\Gamma(1 - z/2)} \cdot \frac{\Gamma((s-z)/2)\Gamma((1-s+z)/2)}{2\sqrt{\pi}\Gamma(1 - (s-z)/2)^2}\, dz.$$

Let $z = 2u$. Then

$$F_3(s) = \frac{1}{4\pi^{3/2} i}\int_L 4^u \frac{\Gamma(u)\Gamma(\frac{s}{2} - u)\Gamma(\frac{1}{2} - \frac{s}{2} + u)}{\Gamma(1 - u)\Gamma(1 - \frac{s}{2} + u)^2}\, du.$$

Use the Meijer G-function

$$F_3(s) = \frac{1}{2\sqrt{\pi}} G^{2,1}_{3,3}\left(\frac{1}{4}\middle| \begin{matrix} 1 - \frac{s}{2}, & 1 - \frac{s}{2}, & 1 - \frac{s}{2} \\ 0, & \frac{1}{2} - \frac{s}{2}, & 0 \end{matrix}\right).$$

Therefore

$$\boxed{\int_0^\infty J_0(x)^3 \ln x \, dx = \frac{1}{2\sqrt{\pi}}\frac{d}{ds}\left[G^{2,1}_{3,3}\left(\frac{1}{4}\middle| \begin{matrix} 1 - \frac{s}{2}, 1 - \frac{s}{2}, 1 - \frac{s}{2} \\ 0, \frac{1}{2} - \frac{s}{2}, 0 \end{matrix}\right)\right]_{s=1}}$$

□

4.4 Case $n = 4$

Proof.

$$I = \int_0^\infty J_0(x)^4 \ln x \, dx$$

Let

$$F(s) = \int_0^\infty x^{s-1} J_0(x)^4 \, dx$$
$$I = F'(1)$$

Use $J_0(x)^2 = \frac{1}{\pi}\int_0^\pi J_0\left(2x\sin\left(\frac{\theta}{2}\right)\right)\, d\theta$ and The Mellin transform,

$$C(s) = \int_0^\infty x^{s-1} J_0(x)^2 \, dx = \frac{1}{\pi}\int_0^\pi \left(\int_0^\infty x^{s-1} J_0\left(2x\sin\left(\frac{\theta}{2}\right)\right)\, dx\right) d\theta$$

Use, $\int_0^\infty x^{s-1} J_0(cx)\, dx = c^{-s}2^{s-1}\frac{\Gamma(s/2)}{\Gamma(1-s/2)}$

$$C(s) = \frac{\Gamma(s/2)}{2\pi\Gamma(1-s/2)}\int_0^\pi \sin^{-s}\left(\frac{\theta}{2}\right)\, d\theta$$
$$C(s) = \frac{\Gamma(s/2)}{\pi\Gamma(1-s/2)}\left[\frac{1}{2}B\left(\frac{1-s}{2},\frac{1}{2}\right)\right]$$

Where $B(x,y)$ is the beta function

$$C(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\Gamma\left(1-\frac{s}{2}\right)^2}$$

We have $J_0(x)^4 = (J_0(x)^2)^2$. The Mellin transform of the product gives,

$$F(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} C(z)C(s-z)\, dz$$
$$F(s) = \frac{1}{8\pi^2 i}\int_L \frac{\Gamma(\frac{z}{2})\Gamma(\frac{1-z}{2})\Gamma(\frac{s-z}{2})\Gamma(\frac{1-s+z}{2})}{\Gamma(1-\frac{z}{2})^2\Gamma(1-\frac{s-z}{2})^2}\, dz$$

Let $z = 2u$

$$F(s) = \frac{1}{4\pi^2 i}\int_L \frac{\Gamma(u)\Gamma(\frac{1}{2}-u)\Gamma(\frac{s}{2}-u)\Gamma(\frac{1}{2}-\frac{s}{2}+u)}{\Gamma(1-u)^2\Gamma(1-\frac{s}{2}+u)^2}\, du$$

Use the Meijer G function

$$F(s) = \frac{1}{2\pi}G^{2,2}_{4,4}\left(1\left|\begin{matrix}\frac{1}{2}, & 1-\frac{s}{2}, & 1-\frac{s}{2}, & 1-\frac{s}{2}\\ 0, & \frac{1}{2}-\frac{s}{2}, & 0, & 0\end{matrix}\right.\right)$$
$$I = F'(1)$$

$$\boxed{\int_0^\infty J_0(x)^4 \ln x \, dx = \frac{1}{2\pi}\frac{d}{ds}\left[G^{2,2}_{4,4}\left(1\left|\begin{matrix}\frac{1}{2}, & 1-\frac{s}{2}, & 1-\frac{s}{2}, & 1-\frac{s}{2}\\ 0, & \frac{1}{2}-\frac{s}{2}, & 0, & 0\end{matrix}\right.\right)\right]_{s=1}}$$

□

4.5 Case $n = 5$

Proof.

$$I = \int_0^\infty J_0(x)^5 \ln x \, dx$$

$$F_5(s) := \int_0^\infty x^{s-1} J_0(x)^5 \, dx, \quad I_5 = \int_0^\infty J_0(x)^5 \ln x \, dx = F_5'(1).$$

Using, $J_0(x)^5 = J_0(x)J_0(x)^4$, and the Mellin transform

$$A(s) := \int_0^\infty x^{s-1} J_0(x) \, dx = 2^{s-1} \frac{\Gamma(s/2)}{\Gamma(1-s/2)},$$

we write

$$F_5(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A(z) F_4(s-z) \, dz.$$

From the $n = 4$ case,

$$F_4(t) = \frac{1}{4\pi^2 i} \int_L \frac{4^u \Gamma(u)\Gamma(\frac{1}{2}-u)\Gamma(\frac{t}{2}-u)\Gamma(\frac{1}{2}-\frac{t}{2}+u)}{\Gamma(1-u)^2\Gamma(1-\frac{t}{2}+u)^2} \, du.$$

Substituting $t = s - z$ gives

$$F_5(s) = \frac{1}{2\pi i} \int 2^{z-1} \frac{\Gamma(z/2)}{\Gamma(1-z/2)} \left[\frac{1}{4\pi^2 i} \int \frac{4^u \Gamma(u)\Gamma(\frac{1}{2}-u)\Gamma(\frac{s-z}{2}-u)\Gamma(\frac{1}{2}-\frac{s-z}{2}+u)}{\Gamma(1-u)^2\Gamma(1-\frac{s-z}{2}+u)^2} \, du \right] dz.$$

Now set $z = 2v$

$$F_5(s) = \frac{1}{8\pi^3 i^2} \iint 4^{u+v} \frac{\Gamma(u)\Gamma(\frac{1}{2}-u)\Gamma(v)\Gamma(\frac{s}{2}-u-v)\Gamma(\frac{1}{2}-\frac{s}{2}+u+v)}{\Gamma(1-u)^2\Gamma(1-v)\Gamma(1-\frac{s}{2}+u+v)^2} \, du \, dv.$$

Closing both contours to the left and taking residues at $u = -m, v = -n, m, n \geq 0$,

$$\mathrm{Res}_{u=-m} \Gamma(u) = \frac{(-1)^m}{m!}, \quad \mathrm{Res}_{v=-n} \Gamma(v) = \frac{(-1)^n}{n!}.$$

Hence, by the double residue theorem,

$$F_5(s) = \frac{1}{8\pi^3 i^2} (2\pi i)^2 \sum_{m,n\geq 0} \left[4^{u+v} \frac{\Gamma(\frac{1}{2}-u)\Gamma(\frac{s}{2}-u-v)\Gamma(\frac{1}{2}-\frac{s}{2}+u+v)}{\Gamma(1-u)^2\Gamma(1-v)\Gamma(1-\frac{s}{2}+u+v)^2} \cdot \frac{(-1)^{m+n}}{m!n!} \right]_{u=-m,v=-n}$$

Now evaluating each factor at $u = -m, v = -n$ and substituting them into $F_5(s)$ yields:

$$F_5(s) = \frac{\Gamma(\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})}{2\sqrt{\pi}\Gamma(1-\frac{s}{2})^2} \sum_{m,n\geq 0} \frac{(\frac{s}{2})_{m+n}^3 (\frac{1}{2})_m}{(\frac{s}{2}+\frac{1}{2})_{m+n} (m!)^3 (n!)^2} \left(\frac{1}{4}\right)^{m+n}.$$

Using the Kampé de Fériet Function

$$F_5(s) = \frac{\Gamma(\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})}{2\sqrt{\pi}\Gamma(1-\frac{s}{2})^2} F^{3:1;0}_{1:2;1} \left(\begin{matrix} \frac{s}{2}, \frac{s}{2}, \frac{s}{2} : \frac{1}{2}; - \\ \frac{s}{2}+\frac{1}{2} : 1, 1; 1 \end{matrix}; \frac{1}{4}, \frac{1}{4} \right),$$

and hence

$$\boxed{\int_0^\infty J_0(x)^5 \ln x \, dx = \frac{d}{ds} \left[\frac{\Gamma(\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})}{2\sqrt{\pi}\Gamma(1-\frac{s}{2})^2} F^{3:1;0}_{1:2;1} \left(\begin{matrix} \frac{s}{2}, \frac{s}{2}, \frac{s}{2} : \frac{1}{2}; - \\ \frac{s}{2}+\frac{1}{2} : 1, 1; 1 \end{matrix}; \frac{1}{4}, \frac{1}{4} \right) \right] \Big|_{s=1}}.$$

□

4.6 Case $n = 6$

Proof.

$$I = \int_0^\infty J_0(x)^6 \ln x \, dx$$

$$F_6(s) := \int_0^\infty x^{s-1} J_0(x)^6 \, dx, \quad I_6 = \int_0^\infty J_0(x)^6 \ln x \, dx = F_6'(1).$$

Using $J_0(x)^6 = \left(J_0(x)^2\right)^3$ and the Mellin transform,

$$A(s) := \int_0^\infty x^{s-1} J_0(x)^2 \, dx = \frac{\Gamma(s/2)\Gamma((1-s)/2)}{2\sqrt{\pi}\Gamma(1-s/2)^2}.$$

Then by Mellin convolution,

$$F_6(s) = \frac{1}{(2\pi i)^2} \iint A(z)A(w)A(s-z-w) \, dz \, dw.$$

Substituting the formula for A and setting $z = 2u, w = 2v$, we obtain

$$F_6(s) = \frac{1}{8\pi^{7/2} i^2} \iint \frac{\Gamma(u)\Gamma\left(\frac{1}{2}-u\right)\Gamma(v)\Gamma\left(\frac{1}{2}-v\right)\Gamma\left(\frac{s}{2}-u-v\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}+u+v\right)}{\Gamma(1-u)^2\Gamma(1-v)^2\Gamma\left(1-\frac{s}{2}+u+v\right)^2} \, du \, dv.$$

Now close both contours to the left and take residues at $u = -m, \quad v = -n, \quad m, n \geq 0$

$$\mathrm{Res}_{u=-m}\Gamma(u) = \frac{(-1)^m}{m!}, \quad \mathrm{Res}_{v=-n}\Gamma(v) = \frac{(-1)^n}{n!},$$

we get

$$F_6(s) = \frac{1}{8\pi^{7/2}} \sum_{m,n\geq 0} \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+m+n\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}-m-n\right)}{(m!)^3(n!)^3\Gamma\left(1-\frac{s}{2}-m-n\right)^2}.$$

$$F_6(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{8\pi^{5/2}\Gamma\left(1-\frac{s}{2}\right)^2} \sum_{m,n\geq 0} \frac{\left(\frac{s}{2}\right)_{m+n}^3 \left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n}{\left(\frac{s}{2}+\frac{1}{2}\right)_{m+n} (m!)^3(n!)^3}.$$

Using the Kampé de Fériet Function

$$F_6(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{8\pi^{5/2}\Gamma\left(1-\frac{s}{2}\right)^2} F^{3:1;1}_{1:2;2}\left(\begin{matrix}\frac{s}{2},\frac{s}{2},\frac{s}{2} : \frac{1}{2} \; ; \; \frac{1}{2} \\ \frac{s}{2}+\frac{1}{2} : 1,1 \; ; \; 1,1\end{matrix}; 1, 1\right),$$

and therefore

$$\boxed{\int_0^\infty J_0(x)^6 \ln x \, dx = \frac{d}{ds}\left[\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{8\pi^{5/2}\Gamma\left(1-\frac{s}{2}\right)^2} F^{3:1;1}_{1:2;2}\left(\begin{matrix}\frac{s}{2},\frac{s}{2},\frac{s}{2} : \frac{1}{2} \; ; \; \frac{1}{2} \\ \frac{s}{2}+\frac{1}{2} : 1,1 \; ; \; 1,1\end{matrix}; 1, 1\right)\right]\Bigg|_{s=1}}.$$

□

4.7 General pattern

The cases $n = 1, \ldots, 6$ exhibit a clear structural progression. The case $n = 1$ is elementary. The case $n = 2$ diverges because

$$J_0(x)^2 \sim \frac{1}{\pi x} \qquad (x \to \infty).$$

For $n = 3$ and $n = 4$, repeated Mellin convolution reduces the problem to a single Barnes integral and hence to a Meijer G-function. For $n = 5$ and $n = 6$, the natural closed forms are two-variable Kampé de Fériet functions.

Each additional factor of J_0 increases the depth of the Mellin convolution by one, and the resulting contour integral becomes correspondingly more complex. The special-function class therefore grows in a controlled and predictable way.

5 Further Properties

5.1 Recurrence Relations

Proposition 5.1 (Mellin-family relations for powers of J_0)**.** *Let $n \geq 3$, and define*

$$F_n(s) = \int_0^\infty x^{s-1} J_0(x)^n \, dx, \qquad A_n(s) = \int_0^\infty x^{s-1} J_0(x)^{n-1} J_1(x) \, dx,$$

$$C_n(s) = \int_0^\infty x^{s-1} J_0(x)^{n-2} J_1(x)^2 \, dx,$$

for $0 < \Re(s) < \frac{n}{2}$, where the integrals converge absolutely.

Then the following identities hold:

$$A_n(s) = \frac{s-1}{n} F_n(s-1), \tag{1}$$

$$F_n(s+1) = (n-1)\, C_n(s+1) - \frac{(s-1)^2}{n} F_n(s-1), \tag{2}$$

$$(n-1)\, C_n(s) = F_n(s) + \frac{(s-2)^2}{n} F_n(s-2). \tag{3}$$

The logarithmic moment

$$I_n = \int_0^\infty J_0(x)^n \ln x \, dx$$

is given by

$$I_n = F_n'(1).$$

Proof. We begin with the Bessel identity

$$J_0'(x) = -J_1(x),$$

which implies

$$\big(J_0(x)^n\big)' = -n\, J_0(x)^{n-1} J_1(x).$$

Hence, by integration by parts,

$$A_n(s) = -\frac{1}{n} \int_0^\infty x^{s-1} \big(J_0(x)^n\big)' \, dx = \frac{s-1}{n} \int_0^\infty x^{s-2} J_0(x)^n \, dx = \frac{s-1}{n} F_n(s-1),$$

which proves (1).

Next, differentiate the product $x^s J_0(x)^{n-1} J_1(x)$. we obtain

$$\begin{aligned}\frac{d}{dx}\Big(x^s J_0(x)^{n-1} J_1(x)\Big) &= s\, x^{s-1} J_0(x)^{n-1} J_1(x) - (n-1)\, x^s J_0(x)^{n-2} J_1(x)^2 \\ &\quad + x^s J_0(x)^n - x^{s-1} J_0(x)^{n-1} J_1(x) \\ &= (s-1) x^{s-1} J_0(x)^{n-1} J_1(x) - (n-1) x^s J_0(x)^{n-2} J_1(x)^2 \\ &\quad + x^s J_0(x)^n.\end{aligned}$$

Integrating over $(0, \infty)$ and using the vanishing of the boundary terms gives

$$0 = (s-1) A_n(s) - (n-1) C_n(s+1) + F_n(s+1),$$

hence

$$F_n(s+1) = (n-1) C_n(s+1) - (s-1) A_n(s).$$

Substituting (1) yields (2).

Finally, shifting $s \mapsto s-1$ in (2) and rearranging gives

$$(n-1) C_n(s) = F_n(s) + \frac{(s-2)^2}{n} F_n(s-2),$$

which is (3).

For the logarithmic moment, differentiate the Mellin transform under the integral sign:

$$F_n'(s) = \int_0^\infty x^{s-1} (\ln x)\, J_0(x)^n \, dx,$$

so in particular

$$I_n = F_n'(1).$$

□

Remark 5.2. The identities above provide the natural Mellin-family framework for I_n. The logarithmic moment is obtained by differentiation at $s = 1$, while the auxiliary transforms A_n and C_n encode the shift structure generated by the Bessel differential equations.

5.2 Asymptotic Behavior

For large n, the dominant contribution comes from the neighborhood of $x = 0$. Since

$$J_0(x) = 1 - \frac{x^2}{4} + O(x^4) \qquad (x \to 0),$$

we have

$$J_0(x)^n = \exp\left(-\frac{n x^2}{4} + O(n x^4)\right).$$

After the scaling $x = 2u/\sqrt{n}$, we get,

$$I_n = 2\sqrt{n} \int_0^\infty \exp(-u^2) \left(\ln(2u) - \frac{1}{2} \ln n\right) du + O\Big(n^{-3/2} \ln n\Big).$$

we obtain the asymptotic estimate

$$I_n = -\frac{\sqrt{\pi}}{2\sqrt{n}} \big(\ln n + \gamma\big) + O\Big(n^{-3/2} \ln n\Big), \qquad n \to \infty.$$

5.3 Analytic Continuation

The Mellin transforms appearing above are initially derived in strips of absolute convergence. The Barnes-type integral representations then provide analytic continuations in the Mellin parameter s. In particular, the logarithmic moments are recovered by differentiation at $s = 1$ after continuation:

$$I_n = F_n'(1), \qquad F_n(s) := \int_0^\infty x^{s-1} J_0(x)^n \, dx,$$

whenever the continuation is regular at $s = 1$ or can be differentiated in the usual meromorphic sense. This is the mechanism underlying the $n = 1, 3, 4, 5, 6$ evaluations.

6 Conclusion

We have rigorously derived closed-form expressions for a complex family of definite integrals utilizing Mellin transform techniques and Mellin-Barnes contour integrals. The generated formulas elegantly span generalized hypergeometric functions and the Meijer G-function, mapping escalation in analytic complexity as the exponent n increases.

Potential avenues for future research include exploring multivariate generalizations, establishing representations involving the Fox H-function, and pursuing a universally generalized closed-form framework applicable to arbitrary values of n.

A Auxiliary Results and Technical Details

A.1 Mellin transforms used in the derivations

The following standard Mellin transforms are used repeatedly:

$$\int_0^\infty x^{s-1} J_0(x) \, dx = 2^{\,s-1} \frac{\Gamma(s/2)}{\Gamma(1 - s/2)},$$

and

$$\int_0^\infty x^{s-1} J_0(x)^2 \, dx = \frac{\Gamma(s/2)\Gamma((1-s)/2)}{2\sqrt{\pi}\,\Gamma(1 - s/2)^2}.$$

A.2 Useful Gamma-function identities

We repeatedly use the shift relation

$$\Gamma(z + m) = \Gamma(z)(z)_m, \qquad m \in \mathbb{N}_0,$$

as well as its backward version

$$\Gamma(z - m) = (-1)^m \frac{\Gamma(z)}{(1 - z)_m}.$$

In particular,

$$\Gamma\left(\frac{1}{2} + m\right) = \sqrt{\pi} \left(\frac{1}{2}\right)_m.$$

A.3 Residue expansions for $n = 5$ and $n = 6$

For $n = 5$ and $n = 6$, the Mellin-convolution formulas lead to double Barnes integrals. Closing the contours to the left and summing residues at the poles of $\Gamma(u)$ and $\Gamma(v)$ gives the Kampé de Fériet series recorded in the main text.

A.4 Differentiation at $s = 1$

If a Mellin transform

$$F_n(s) = \int_0^\infty x^{s-1} J_0(x)^n \, dx$$

admits an analytic continuation to a neighborhood of $s = 1$, then

$$\frac{d}{ds} F_n(s)\Big|_{s=1} = \int_0^\infty J_0(x)^n \ln x \, dx,$$

provided the differentiation is justified under the continuation. This is the standard Mellin-parameter differentiation rule.

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