An Alternative Inequality To J. S. Bell's 1964 Inequality

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Abstract

This article derives an expression for an inequality relating the expectation values of the products of expectation values for pairs of three expectation values that gives theoretically correct results.

1 Introduction

Bell's 1964 inequality [2, equation (15)] has been shown to be incorrect both theoretically (including by Bell himself [2, equation (22)]) and, by inference, experimentally [1].

In the specific context of Einstein-Podolsky-Rosen-Bohm (EPRB) systems the standard scientific protocol for reconciling theory and experiment appears to have broken down.

For specific detector settings theoretical calculations and the corresponding empirical data appear to disagree with the bounds dictated by Bell's 1964 inequality.

Under the principles of the Scientific Method this systemic failure to predict verified outcomes indicates that Bell's inequality is incorrect for these domains. Consequently, the framework requires a formal mathematical modification rather than a metaphysical interpretation.

This article derives an alternative expression for an inequality relating the expectation values of the products of expectation values for pairs of three different expectation values that gives theoretically correct results.

2 Transformation

According to the Kolmogorov-type definition, the expected value $\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle$ over a domain Λ of the product of two functions $A(\mathbf{a},\lambda)$ and $B(\mathbf{b},\lambda)$ of the random variable λ with parameters $\mathbf{a}$ and $\mathbf{b}$ is given by [2, equation (2)]:

$$\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle_{\Lambda} = \int_{\Lambda}\rho(\lambda)\,A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\,d\lambda \tag{1}$$

Therefore:

$$\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle - \langle A(\mathbf{a},\lambda)\,B(\mathbf{c},\lambda)\rangle_{\Lambda} = \int_{\Lambda}\rho(\lambda)\left(A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda) - A(\mathbf{a},\lambda)\,B(\mathbf{c},\lambda)\right)d\lambda \tag{2}$$

Invoking the specified restriction [2, equation (13)] $B(\mathbf{b},\lambda) = -A(\mathbf{b},\lambda)$:

$$\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle = -\int\rho(\lambda)\,A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\,d\lambda \tag{3}$$

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Therefore, the difference of two different expectation values becomes:

$$\begin{aligned}
\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle - \langle A(\mathbf{a},\lambda)\,B(\mathbf{c},\lambda)\rangle &= \int \rho(\lambda)\left(-A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)+A(\mathbf{a},\lambda)\,A(\mathbf{c},\lambda)\right)d\lambda &(4)\\
&= \int \rho(\lambda)\left(A(\mathbf{a},\lambda)\,A(\mathbf{c},\lambda)-A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\right)d\lambda &(5)
\end{aligned}$$

With the aid of the transformation operator[1] $A(\mathbf{b},\lambda)\,A(\mathbf{b},\lambda)=1$:

$$\begin{aligned}
&\int \rho(\lambda)\left(A(\mathbf{a},\lambda)\,A(\mathbf{c},\lambda)-A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\right)d\lambda &(6)\\
=\ &\int \rho(\lambda)\,A(\mathbf{a},\lambda)\left(A(\mathbf{c},\lambda)-A(\mathbf{b},\lambda)\right)d\lambda &(7)\\
\Rightarrow\ &\int \rho(\lambda)\,A(\mathbf{b},\lambda)\,A(\mathbf{b},\lambda)\,A(\mathbf{a},\lambda)\left(A(\mathbf{c},\lambda)-A(\mathbf{b},\lambda)\right)d\lambda &(8)\\
=\ &\int \rho(\lambda)\,A(\mathbf{b},\lambda)\,A(\mathbf{a},\lambda)\left(A(\mathbf{b},\lambda)\,A(\mathbf{c},\lambda)-1\right)d\lambda &(9)\\
=\ &\langle A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\left(A(\mathbf{b},\lambda)\,A(\mathbf{c},\lambda)-1\right)\rangle &(10)\\
=\ &\langle A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\,A(\mathbf{b},\lambda)\,A(\mathbf{c},\lambda)-A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\rangle &(11)\\
=\ &\langle A(\mathbf{b},\lambda)\,A(\mathbf{c},\lambda)\rangle-\langle A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\rangle &(12)\\
=\ &\langle -A(\mathbf{b},\lambda)\,B(\mathbf{c},\lambda)\rangle+\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle &(13)\\
=\ &\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle-\langle A(\mathbf{b},\lambda)\,B(\mathbf{c},\lambda)\rangle &(14)
\end{aligned}$$

Thus, the effect of the transformation operator $A(\mathbf{b},\lambda)\,A(\mathbf{b},\lambda)$ is to transform the expression $\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle - \langle A(\mathbf{a},\lambda)\,B(\mathbf{c},\lambda)\rangle$ into the expression $\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle - \langle A(\mathbf{b},\lambda)\,B(\mathbf{c},\lambda)\rangle$.

3 Derivation

Consider a further derivation using the equality of equation (14) with respect to equation (10):

$$\begin{aligned}
\langle A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)\rangle - \langle A(\mathbf{b},\lambda)\,B(\mathbf{c},\lambda)\rangle_{\Lambda} &= \int_{\Lambda} \rho(\lambda)\left(A(\mathbf{a},\lambda)\,B(\mathbf{b},\lambda)-A(\mathbf{b},\lambda)\,B(\mathbf{c},\lambda)\right)d\lambda &(15)\\
&= \int_{\Lambda} \rho(\lambda)\left(-A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)+A(\mathbf{b},\lambda)\,A(\mathbf{c},\lambda)\right)d\lambda &(16)\\
&= \int_{\Lambda} \rho(\lambda)\,A(\mathbf{b},\lambda)\left(A(\mathbf{c},\lambda)-A(\mathbf{a},\lambda)\right)d\lambda &(17)\\
&= \int_{\Lambda} \rho(\lambda)\,A(\mathbf{a},\lambda)\,A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\left(A(\mathbf{c},\lambda)-A(\mathbf{a},\lambda)\right)d\lambda &(18)\\
&= \int_{\Lambda} \rho(\lambda)\,A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\left(A(\mathbf{a},\lambda)\,A(\mathbf{c},\lambda)-1\right)d\lambda &(19)
\end{aligned}$$

As $|A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)|\leq 1$:

$$\begin{aligned}
\left|\int \rho(\lambda)\,A(\mathbf{a},\lambda)\,A(\mathbf{b},\lambda)\left(A(\mathbf{a},\lambda)\,A(\mathbf{c},\lambda)-1\right)d\lambda\right| &\leq \left|\int \rho(\lambda)\left(A(\mathbf{a},\lambda)\,A(\mathbf{c},\lambda)-1\right)d\lambda\right| &(20)\\
&\leq \left|\int \rho(\lambda)\left(1-A(\mathbf{a},\lambda)\,A(\mathbf{c},\lambda)\right)d\lambda\right| &(21)
\end{aligned}$$

[1] A transformation operation is indicated by the symbol $\Rightarrow$.

If $\mathbf{c} = \mathbf{a}$ then $A(\mathbf{a}, \lambda) A(\mathbf{a}, \lambda) = 1$ and $A(\mathbf{a}, \lambda) A(\mathbf{a}, \lambda) - 1 = 0$.
If $\mathbf{c} = \mathbf{a} + \pi$ then $A(\mathbf{a}, \lambda) A(\mathbf{a} + \pi, \lambda) = -1$ and $A(\mathbf{a}, \lambda) A(\mathbf{a} + \pi, \lambda) - 1 = -2$
Thus $-2 \leq A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) - 1 \leq 0$ so, $A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) - 1 \leq 0$
Therefore:

$$\int \rho(\lambda) (1 - A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda)) d\lambda \quad \geq \quad 0 \tag{22}$$

Therefore:

$$\left| \int \rho(\lambda) (1 - A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda)) d\lambda \right| \quad = \quad \int \rho(\lambda) (1 - A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda)) d\lambda \tag{23}$$

$$\begin{aligned} \left| \int \rho(\lambda) A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda) (A(\mathbf{b}, \lambda) A(\mathbf{c}, \lambda) - 1) d\lambda \right| &\leq \left| \int \rho(\lambda) (1 - A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda)) d\lambda \right| && (24) \\ &\leq \int \rho(\lambda) (1 - A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda)) d\lambda && (25) \end{aligned}$$

Therefore:

$$\begin{aligned} \left| \int \rho(\lambda) (A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) - A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda)) d\lambda \right| &\leq \int \rho(\lambda) (1) d\lambda - \int \rho(\lambda) A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) d\lambda && (26) \\ &\leq 1 - \int \rho(\lambda) A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) d\lambda && (27) \end{aligned}$$

Therefore:

$$1 - \int \rho(\lambda) A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) d\lambda \quad \geq \quad \left| \int \rho(\lambda) (A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) - A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda)) d\lambda \right| \tag{28}$$

Therefore:

$$\begin{aligned} 1 - \langle A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) \rangle &\geq |\langle A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda) \rangle - \langle A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda) \rangle| && (29) \\ 1 + \langle A(\mathbf{a}, \lambda) B(\mathbf{c}, \lambda) \rangle &\geq |- \langle A(\mathbf{a}, \lambda) B(\mathbf{c}, \lambda) \rangle + \langle A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \rangle| && (30) \end{aligned}$$

Finally:

$$1 + \langle A(\mathbf{a}, \lambda) B(\mathbf{c}, \lambda) \rangle_\Lambda \quad \geq \quad |\langle A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \rangle_\Lambda - \langle A(\mathbf{a}, \lambda) B(\mathbf{c}, \lambda) \rangle_\Lambda| \tag{31}$$

which can be compared with [2, equation (15)]:

$$1 + \langle A(\mathbf{b}, \lambda) B(\mathbf{c}, \lambda) \rangle \quad \geq \quad |\langle A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \rangle - \langle A(\mathbf{a}, \lambda) B(\mathbf{c}, \lambda) \rangle| \tag{32}$$

4 Confirmation

This confirmation of the inequality (31) utilizes the solution of equation (1) [3]:

$$\langle A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \rangle \quad = \quad \frac{2}{\pi} |\mathbf{b} - \mathbf{a}| - 1 \tag{33}$$

$\mathbf{a}$	$\mathbf{b}$	$\mathbf{c}$	$\lvert\mathbf{b}-\mathbf{a}\rvert$	$\lvert\mathbf{c}-\mathbf{a}\rvert$	$\lvert\mathbf{c}-\mathbf{b}\rvert$	$E_{\mathbf{ab}} = \frac{2}{\pi}\lvert\mathbf{b}-\mathbf{a}\rvert - 1$	$E_{\mathbf{ac}} = \frac{2}{\pi}\lvert\mathbf{c}-\mathbf{a}\rvert - 1$	$E_{\mathbf{bc}} = \frac{2}{\pi}\lvert\mathbf{c}-\mathbf{b}\rvert - 1$	$1 + E_{\mathbf{bc}}$	$\lvert E_{\mathbf{ab}} - E_{\mathbf{ac}}\rvert$	$1 + E_{\mathbf{bc}} \geq \lvert E_{\mathbf{ab}} - E_{\mathbf{ac}}\rvert$
0	0	0	0	0	0	-1	-1	-1	0	0	$=$
0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$=$
0	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$=$
0	0	$\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{4}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$=$
$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0	0	$=$
0	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{\pi}{8}$	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$=$
0	$\frac{\pi}{2}$	$\frac{\pi}{8}$	$\frac{\pi}{2}$	$\frac{\pi}{8}$	$\frac{3\pi}{8}$	0	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$=$
0	0	$\frac{\pi}{3}$	0	$\frac{\pi}{3}$	$\frac{\pi}{3}$	-1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$=$
0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{12}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$=$
0	$-\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{12}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$1\frac{1}{6}$	$\frac{1}{6}$	True
0	$\frac{\pi}{10}$	$-\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{\pi}{3}$	$\frac{13\pi}{30}$	$-\frac{4}{5}$	$-\frac{1}{3}$	$-\frac{2}{15}$	$\frac{13}{15}$	$\frac{7}{15}$	True

Table 1: Confirmation Of Inequalities

The following graphs depict the left (LS) and right (RS) sides of equation (31) for various parameter settings $\mathbf{a}$ and $\mathbf{b}$ over a $-\pi$ to π range of parameter $\mathbf{c}$ settings.[2]

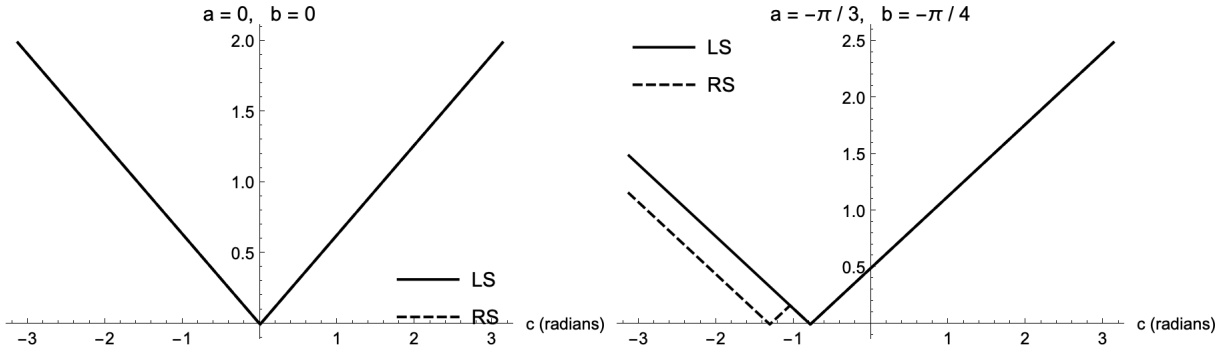


Figure 1: Inequality LS RS Comparisons, $a = 0$, $\mathbf{b} = 0$ and $a = -\frac{\pi}{3}$, $\mathbf{b} = -\frac{\pi}{4}$

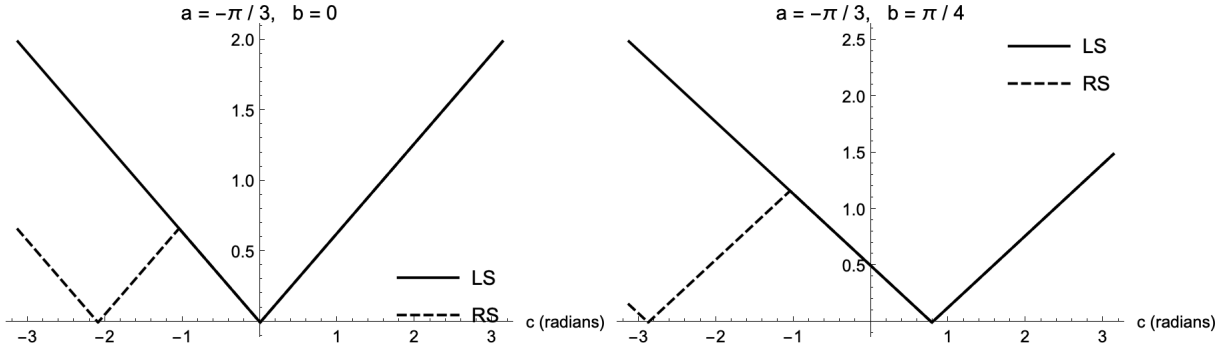


Figure 2: Inequality LS RS Comparisons, $a = -\frac{\pi}{3}$, $\mathbf{b} = 0$ and $a = -\frac{\pi}{3}$, $\mathbf{b} = \frac{\pi}{4}$

[2]The right side plot is hidden whenever the left and right sides are equal to each other.

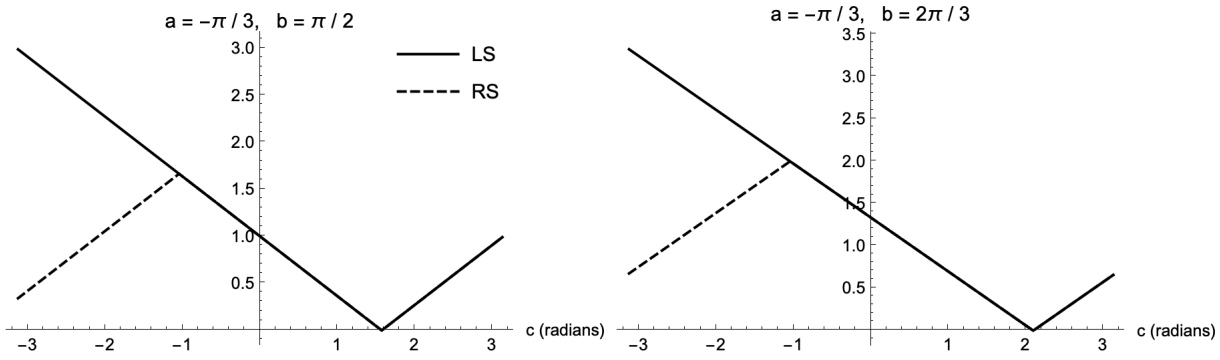


Figure 3: Inequality LS RS Comparisons, $a = -\frac{\pi}{3}$, $\mathbf{b} = \frac{\pi}{2}$ and $a = -\frac{\pi}{3}$, $\mathbf{b} = \frac{2\pi}{3}$

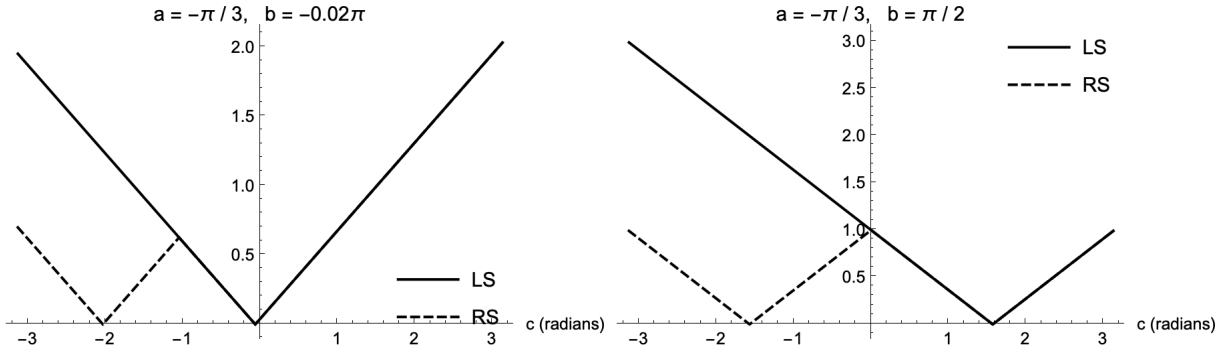


Figure 4: Inequality LS RS Comparisons, $a = -\frac{\pi}{3}$, $\mathbf{b} = 0.02\pi$ and $a = -\frac{\pi}{3}$, $\mathbf{b} = \frac{\pi}{2}$

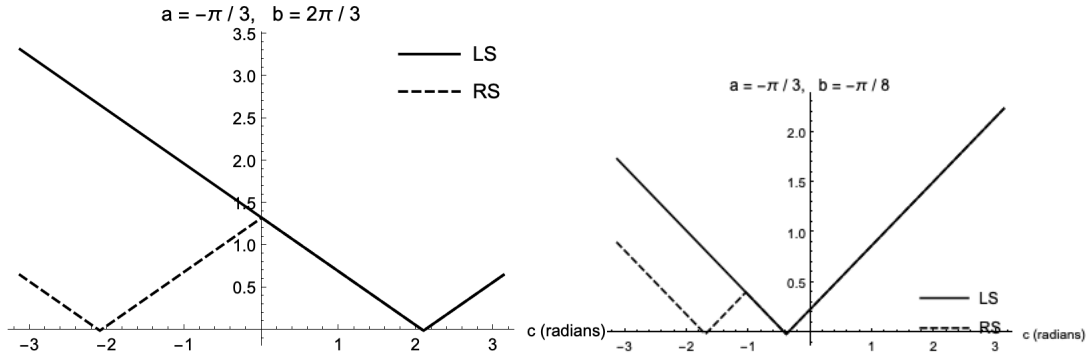


Figure 5: Inequality LS RS Comparisons, $a = -\frac{\pi}{3}$, $\mathbf{b} = -\frac{2\pi}{3}$ and $a = -\frac{\pi}{3}$, $-\frac{\pi}{8}$

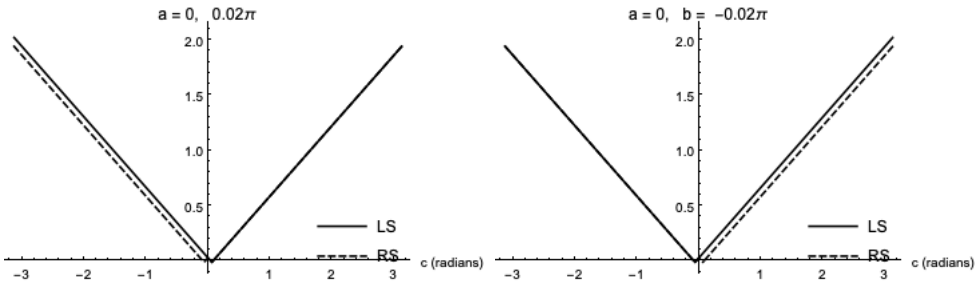


Figure 6: Inequality LS RS Comparisons, $a = 0$, $\mathbf{b} = 0.02\pi$ and $a = 0$, $\mathbf{b} = 0.02\pi$

5 Conclusion

An expression for an inequality relating the expectation values of the products of three expectation values in pairs has been derived. The resulting expression supports the hypothesis that the inequality (31) is correct.

Experiments are not necessarily expected to produce this result exactly as each individual experiment can only process data from a necessarily small subdomain of an infinite population. However, proper statistical analysis of experimental results is expected to support the hypothesis that inequality (31) is correct.

The hypothesis that inequality(31) is valid does not preclude the possibility of hidden variable theories.

6 Acknowledgements

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References

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[2] J. S. Bell. On the einstein-podolsky-rosen paradox. *Physics*, 1:195–200, 1964.

[3] R. L. Mc Murtrie. J. s. bell's inequality postulate reviewed by direct comparison formulae. *Current Research In Statistics And Mathematics*, 5(1):1 – 8, April 2026.