Algebraic Constants from a Discrete Mapping

Connections to Geometry, Physics, and Crystallography

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Abstract

Evaluating the rule $f(n) = \sqrt{n}/2 - n/2 + 2$ at the integers from 0 to 9 produces a small collection of familiar constants. Three caught my attention: the inverse golden ratio, the ideal tetrahedral interstitial radius ratio, and a coefficient in Gill's fourth-order Runge–Kutta method. This paper examines those exact matches and asks whether the remaining values have useful geometric or physical interpretations. I distinguish between an equality that can be verified directly and a proposed connection that still needs a construction or derivation. The tiling, quantum, and mechanical interpretations discussed below belong to the latter group.

The rule and its ten values

Between the integer inputs

Let $f(n) = \sqrt{n}/2 - n/2 + 2$ for $n \in \{0, 1, \ldots, 9\}$. The integer samples do not tell the whole story of the curve. If we allow real, nonnegative inputs, it rises briefly to a maximum of $f(x) = 2.125$ at $x = 0.25$, then falls as the linear term takes over. None of the integer inputs captures that initial rise. The curve crosses zero at $x = (9 + \sqrt{17})/2 \approx 6.5616$, between $n = 6$ and $n = 7$.

The discrete sequence

The perfect squares are a useful place to start: $f(0) = f(1) = 2$, $f(4) = 1$, and $f(9) = -1$. The six remaining outputs are irrational algebraic numbers.

After the repeated value 2, the sequence falls to $f(2) = 1 + \sqrt{2}/2 \approx 1.707107$ and $f(3) = (\sqrt{3} + 1)/2 \approx 1.366025$, then reaches unity at $n = 4$. At $n = 5$ we get $f(5) = (\sqrt{5} - 1)/2 \approx 0.618034$, the inverse golden ratio. The next output is $f(6) = (\sqrt{6} - 2)/2 \approx 0.224745$.

At $n = 7$ the output becomes negative: $f(7) = (\sqrt{7} - 3)/2 \approx -0.177124$. The last two values are $f(8) = \sqrt{2} - 2 \approx -0.585786$ and $f(9) = -1$. Apart from the equal opening pair, the integer sequence decreases throughout.

Geometric connections: equalities and constructions

The values at n = 2, 3, and 5 invite geometric comparisons, but there is a small trap here. Recognizing a term within an output is not the same as recognizing the entire output. That difference is easy to overlook when the radicals are familiar.

The n = 2 value: triangles and Gill's method

For n = 2, the rule gives f(2) = 1 + $\sqrt{2}/2$. We can recognize $\sqrt{2}/2$ as the leg of a right isosceles triangle with unit hypotenuse, or as sin(45°) = cos(45°). The same term occurs in 45° force and lattice-vector resolutions and in octagonal constructions. Those observations leave the added 1 unexplained, however. Gill's method offers a match to the whole output: $1 + \sqrt{2}/2$ is the coefficient multiplying k_3 in the fourth-stage argument [2]. Gill's original paper places the fourth-order method in the context of accumulated round-off error [3]. I also wonder whether the projection and inflation/deflation constructions of Ammann–Beenker tilings [1] contain a vertex-row distance equal to f(2). That would be a more substantial geometric connection, but I have not calculated such a distance here.

An altitude plus one-half

The n = 3 value, $(\sqrt{3} + 1)/2$, presents a similar difficulty. It is the altitude of a unit equilateral triangle, $\sqrt{3}/2$, plus one-half. Square–triangle tilings with dodecagonal symmetry, including the Stampfli tiling [4, 5], seem a reasonable place to look for that combined length. The secondary overview in [5] also lists Ni–Cr and V–Ni–Si alloy systems among dodecagonal quasicrystals. These are possible settings for an investigation; they do not yet explain why this particular sum should appear.

The golden-ratio identity

At n = 5 there is no extra term to account for: f(5) = $(\sqrt{5} - 1)/2 = \varphi^{-1}$. This exact identity is what draws me to comparisons with pentagonal and icosahedral proportions, logarithmic-spiral constructions [6], and golden-angle phyllotaxis models [7]. Penrose's nonperiodic tilings [9] and Shechtman's observation of an icosahedral phase in aluminum–manganese alloys in 1982, reported in 1984 [8], provide further context. But sharing the golden ratio does not mean that these structures follow from the rule. To make that case, I would first have to identify the lengths involved—whether in golden-ratio scaling or in the proportions of golden rhombohedra.

There is also a tempting dynamical question. The continued fraction $\varphi^{-1} = [0; 1, 1, 1, \ldots]$ underlies the description of the golden ratio as "most irrational" [6]. It suggests looking at quasiperiodicity and phase locking in nonlinear oscillators. I stop short of making that connection here, because the rule gives me a number, not an oscillator model.

Sphere packing and possible physical interpretations

A tetrahedral gap between touching spheres

For $f(6) = (\sqrt{6} - 2)/2$, the geometric interpretation is unusually clear. The value equals the limiting radius ratio $r/R \approx 0.225$ for a small sphere that just fits into a tetrahedral gap in ideal close packing [10]. Here the surrounding spheres touch, and the smaller sphere fills the available clearance. Those assumptions do much of the work. Tetrahedral bonding in methane or diamond is a different problem, despite the shared geometry. Any comparison with coordination polyhedra or zinc-blende structures would also have to address the gap between the touching-sphere model and the physical structure.

Interpreting the negative values

I find the negative outputs less straightforward. At $n = 7$, $f(7) = (\sqrt{7} - 3)/2 \approx -0.177124$. A negative number could represent a signed translation in projected heptagonal geometry, a phase inversion, or an energy below a chosen reference level in a quantum potential well. These interpretations use the sign in quite different ways. The rule alone gives no basis for choosing among them.

The value $f(8) = \sqrt{2} - 2$ suggests octagonal tilings because it shares $\sqrt{2}$ with the silver ratio, $\delta_s = 1 + \sqrt{2}$. Tile deflation [1] is one avenue I would like to examine; destructive interference is another possible comparison. The presence of $\sqrt{2}$ is not enough, though. The real question is whether either setting produces the specific quantity $\sqrt{2} - 2$.

A proposed cantilever connection

The cantilever proposal is more concrete: it assigns $f(6)$, $f(7)$, and $f(8)$ to normalized cantilever-deflection residuals under non-uniform, parabolic loading. It is also a claim that can be checked. One must calculate the deflection produced by the load, choose a normalization, and show that the resulting residuals equal the stated values. Theory of Elasticity [11] provides a framework for investigating this possibility. I have not carried out that derivation in this paper.

How far do the connections go?

Of the connections considered here, three are exact matches to identified quantities: $f(5)$ is the inverse golden ratio [6], $f(2)$ is Gill's fourth-stage coefficient [2, 3], and $f(6)$ is the ideal tetrahedral clearance ratio [10]. These are the strongest results in the paper. They establish equalities, not a governing role for the rule in the systems where those constants occur.

The triangle examples show why I hesitate to go further. At $n = 2$, the triangle leg explains $\sqrt{2}/2$ but not the added 1; at $n = 3$, the equilateral altitude explains $\sqrt{3}/2$ but not the added one-half. The sphere-packing match is exact, yet it depends on touching-sphere assumptions. Moving from that ideal geometry to a molecule or crystal would require a separate argument.

There are several ways to take the geometric questions forward. One could specify and calculate vertex-row distances in Ammann–Beenker tilings [1], examine the proposed relations to Stampfli tilings and FCC/HCP geometry [4, 5], or identify lengths and ratios in Penrose and icosahedral structures that exhibit golden-ratio scaling or golden-rhombohedron proportions [8, 9]. Quantum and interference proposals would start elsewhere, with a defined reference energy or phase relation. In each case, I would want a calculation that could fail as well as succeed. Otherwise, the comparison remains an association between familiar numbers.

What would make the proposals testable?

What keeps my interest is the economy of the rule: ten integer samples produce several recognizable constants. I think that is worth examining. Whether it amounts to more than a collection of identities depends on what we can actually derive from it.

My next step would be to test one proposal at a time. For a tiling, that means specifying a geometric construction; for a quantum interpretation, it means defining a model and its reference levels. The cantilever idea offers a particularly clear test: calculate the deflection under the stated parabolic load, define the normalization and residuals, and compare the results with f(6), f(7), and f(8). Until that work is done, I regard the exact identities as results and the physical connections as questions.

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