What's the Area of the 4th Triangle? A Geometric Algebra Approach

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Abstract

Introductory-level Geometric Algebra provides a simple, structured solution to an intriguing problem that has become popular online: A rectangle of unknown dimensions and area is divided into four triangles. Given the areas of three of the triangles, find the area of the fourth.

1 Statement of the Problem

A rectangle of unknown dimensions and area is divided into four triangles as shown. Given A_1, A_2, and A_3, find A_4.

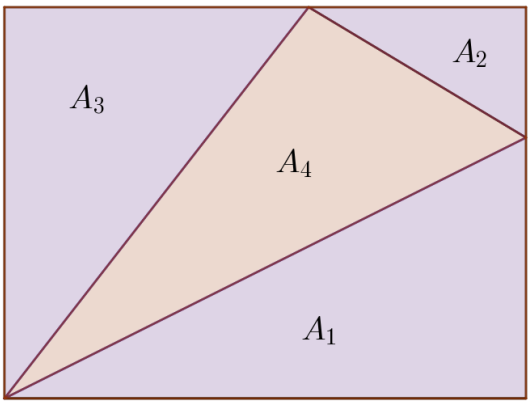
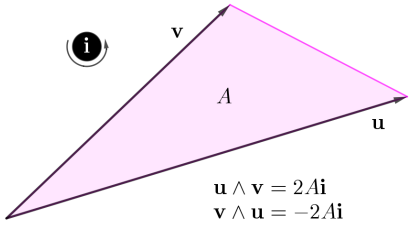


Figure 1: Given A_1, A_2, and A_3, find A_4.

2 Ideas that We Will Use

- The relationship between the outer product of two vectors and the oriented area of the triangle that they form:



- The definition of the equality of multivectors. Two multivectors $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal if and only if their respective parts are equal. That is, if the scalar part of $\mathcal{M}_1$ is equal to that of $\mathcal{M}_2$; the vector part of $\mathcal{M}_1$ is equal to that of $\mathcal{M}_2$; the bivector part of $\mathcal{M}_1$ is equal to that of $\mathcal{M}_2$; etc.

3 Formulation in GA Terms

As an initial formulation, we express the lengths of the sides of the "given" triangles in terms of their respective areas and the lengths of the sides of the rectangle (Fig. 2).

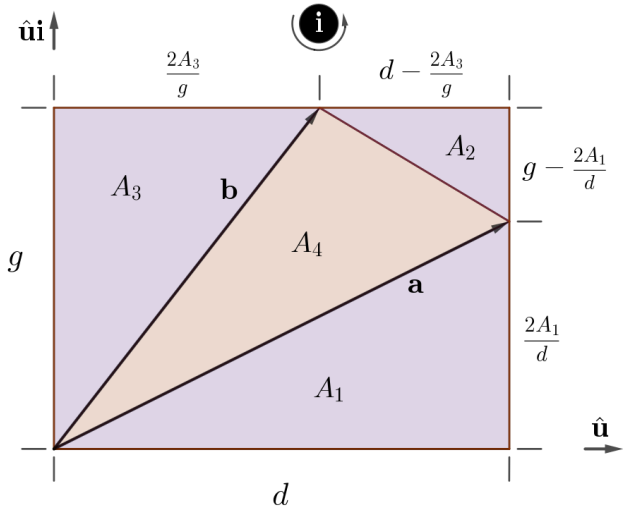


Figure 2: Initial formulation

Because $A_4 = \frac{1}{2}\|\mathbf{a} \wedge \mathbf{b}\|$, our final formulation (Fig. 3) expresses **a** and **b** in terms of the lengths that were identified in the initial formulation.

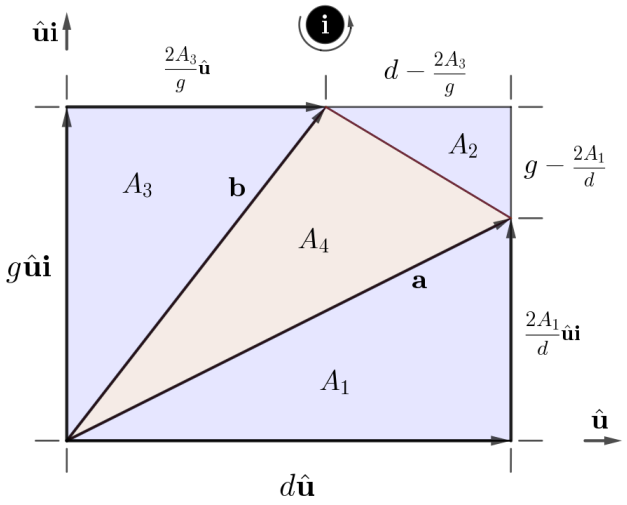


Figure 3: Final formulation: $\mathbf{a} = d\hat{\mathbf{u}} + \left(\frac{2A_1}{d}\right)\hat{\mathbf{u}}\hat{\mathbf{i}}$; $\mathbf{b} = \left(\frac{2A_3}{g}\right)\hat{\mathbf{u}} + g\hat{\mathbf{u}}\hat{\mathbf{i}}$; $2A_4\mathbf{i} = \mathbf{a} \wedge \mathbf{b}$.

Specifically,

$$\mathbf{a} = d\hat{\mathbf{u}} + \left(\frac{2A_1}{d}\right)\hat{\mathbf{u}}\hat{\mathbf{i}} \tag{1}$$

and

$$\mathbf{b} = \left(\frac{2A_3}{g}\right)\hat{\mathbf{u}} + g\hat{\mathbf{u}}\hat{\mathbf{i}}\ . \tag{2}$$

Thus, we can express A_4 in terms of d, g, and the given areas A_1 and A_3:

$$\begin{aligned} 2A_4\mathbf{i} &= \mathbf{a} \wedge \mathbf{b} \\ &= \left[d\hat{\mathbf{u}} + \left(\frac{2A_1}{d}\right)\hat{\mathbf{u}}\hat{\mathbf{i}}\right] \wedge \left[\left(\frac{2A_3}{g}\right)\hat{\mathbf{u}} + g\hat{\mathbf{u}}\hat{\mathbf{i}}\right] \\ &= \left[dg - \frac{4A_1A_3}{dg}\right]\mathbf{i}\ . \end{aligned}$$

By the definition of the equality of multivectors, we now write

$$2A_4 = dg - \frac{4A_1A_3}{dg}\ . \tag{3}$$

Although we cannot find either d or g alone, we can find their product dg by solving a quadratic equation for A_2:

$$2A_2 = \left[g - \frac{2A_1}{d}\right]\left[d - \frac{2A_3}{g}\right]\ , \tag{4}$$

from which

$$\begin{aligned} dg + \tfrac{4A_1A_3}{dg} - 2\left(A_1 + A_2 + A_3\right) &= 0; \\ \left(dg\right)^2 - 2\left(A_1 + A_2 + A_3\right)\left(dg\right) + 4A_1A_3 &= 0 \\ dg = \frac{2\left(A_1 + A_2 + A_3\right) \pm \sqrt{4\left(A_1 + A_2 + A_3\right)^2 - 16A_1A_3}}{2} & \\ dg = A_1 + A_2 + A_3 + \sqrt{\left(A_1 + A_2 + A_3\right)^2 - 4A_1A_3}\,. & \end{aligned} \tag{5}$$

Only the "+" solution of the quadratic is valid, because dg is the area of the entire rectangle, and must therefore be greater than $A_1 + A_2 + A_3$. From our result for dg,

$$\begin{aligned} \frac{1}{dg} &= \frac{1}{A_1 + A_2 + A_3 + \sqrt{\left(A_1 + A_2 + A_3\right)^2 - 4A_1A_3}} \\ &= \frac{A_1 + A_2 + A_3 - \sqrt{\left(A_1 + A_2 + A_3\right)^2 - 4A_1A_3}}{4A_1A_3}\,. \end{aligned} \tag{6}$$

Substituting the expressions from Eqs. (5) and (6) in Eq. (3), then simplifying, we obtain

$$A_4 = \sqrt{\left(A_1 + A_2 + A_3\right)^2 - 4A_1A_3}\,. \tag{7}$$

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).