

# What's the Area of the 4th Triangle? A Geometric Algebra Approach

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## Abstract

Introductory-level Geometric Algebra provides a simple, structured solution to an intriguing problem that has become popular online: A rectangle of unknown dimensions and area is divided into four triangles. Given the areas of three of the triangles, find the area of the fourth.

## 1 Statement of the Problem

A rectangle of unknown dimensions and area is divided into four triangles as shown. Given  $A_1$ ,  $A_2$ , and  $A_3$ , find  $A_4$ .

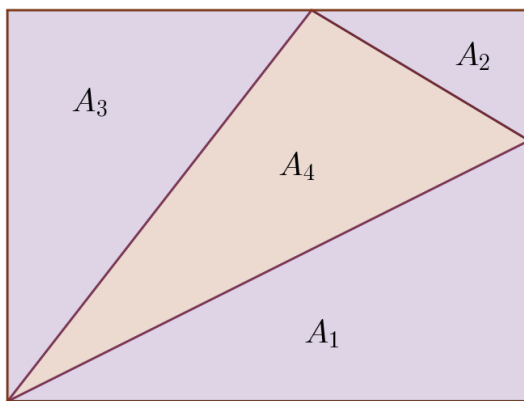
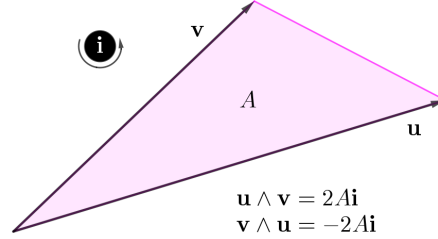


Figure 1: Given  $A_1$ ,  $A_2$ , and  $A_3$ , find  $A_4$ .

## 2 Ideas that We Will Use

The only “GA” idea that we need is the relationship between the outer product of two vectors and the oriented area of the triangle that they form:



## 3 Formulation in GA Terms

As an initial formulation, we express the lengths of the sides of the “given” triangles in terms of the sides of the rectangle and the areas of those same triangles (Fig. 2).

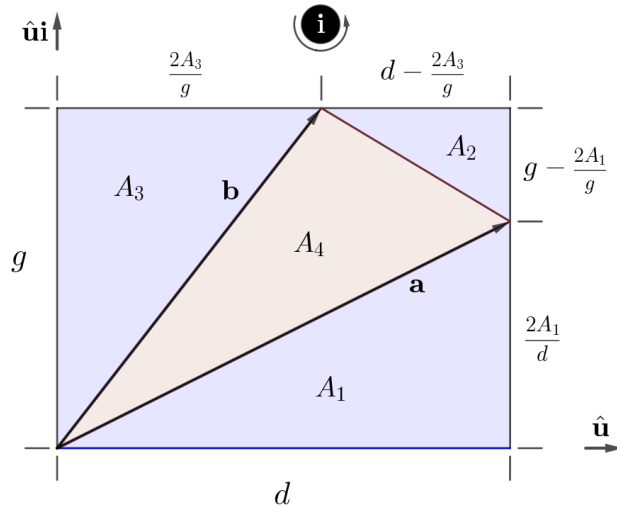


Figure 2: Initial formulation

Because  $A_4 = \frac{1}{2}\|\mathbf{a} \wedge \mathbf{b}\|$ , our final formulation (Fig. 3) expresses  $\mathbf{a}$  and  $\mathbf{b}$  in terms of the lengths that were identified in the initial formulation.

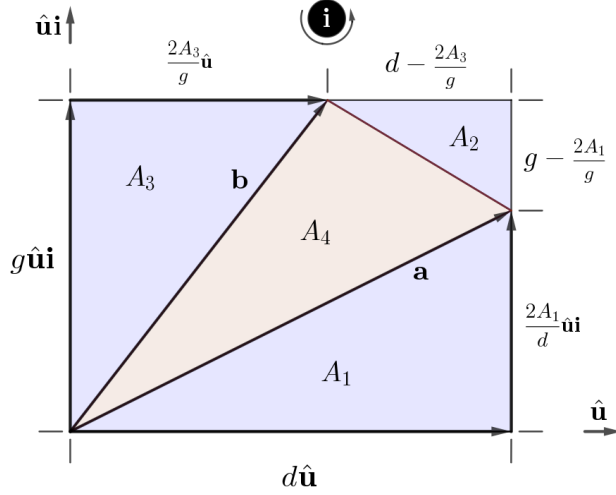


Figure 3: Final formulation:  $\mathbf{a} = d\hat{\mathbf{u}} + \left(\frac{2A_1}{d}\right)\hat{\mathbf{u}}\mathbf{i}$ ;  $\mathbf{b} = \left(\frac{2A_3}{g}\right)\hat{\mathbf{u}} + g\hat{\mathbf{u}}\mathbf{i}$ ;  $2A_4\mathbf{i} = \mathbf{a} \wedge \mathbf{b}$ .

Specifically,

$$\mathbf{a} = d\hat{\mathbf{u}} + \left(\frac{2A_1}{d}\right)\hat{\mathbf{u}}\mathbf{i} \quad (1)$$

and

$$\mathbf{b} = \left(\frac{2A_3}{g}\right)\hat{\mathbf{u}} + g\hat{\mathbf{u}}\mathbf{i}. \quad (2)$$

Thus, we can express  $A_4$  in terms of  $d$ ,  $g$ , and the given areas  $A_1$  and  $A_3$ :

$$\begin{aligned} 2A_4\mathbf{i} &= \mathbf{a} \wedge \mathbf{b} \\ &= \left[d\hat{\mathbf{u}} + \left(\frac{2A_1}{d}\right)\hat{\mathbf{u}}\mathbf{i}\right] \wedge \left[\left(\frac{2A_3}{g}\right)\hat{\mathbf{u}} + g\hat{\mathbf{u}}\mathbf{i}\right] \\ &= dg\mathbf{i} - \left[\frac{4A_1A_3}{dg}\right]\mathbf{i}. \end{aligned} \quad (3)$$

Although we cannot find either  $d$  or  $g$  alone, we can find their product  $dg$  by solving a quadratic equation for  $A_2$ :

$$2A_2 = \left[g - \frac{2A_1}{d}\right] \left[d - \frac{2A_3}{g}\right], \quad (4)$$

from which

$$\begin{aligned} dg + \frac{4A_1A_3}{dg} - 2(A_1 + A_2 + A_3) &= 0; \\ (dg)^2 - 2(A_1 + A_2 + A_3)(dg) + 4A_1A_3 &= 0 \\ dg &= \frac{2(A_1 + A_2 + A_3) \pm \sqrt{4(A_1 + A_2 + A_3)^2 - 16A_1A_3}}{2} \\ dg &= A_1 + A_2 + A_3 + \sqrt{(A_1 + A_2 + A_3)^2 - 4A_1A_3}. \end{aligned} \quad (5)$$

Only the ”+” solution of the quadratic is valid, because  $dg$  is the area of the entire rectangle, and must therefore be greater than  $A_1 + A_2 + A_3$ . From our result for  $dg$ ,

$$\begin{aligned}\frac{1}{dg} &= \frac{1}{A_1 + A_2 + A_3 + \sqrt{(A_1 + A_2 + A_3)^2 - 4A_1A_3}} \\ &= \frac{A_1 + A_2 + A_3 - \sqrt{(A_1 + A_2 + A_3)^2 - 4A_1A_3}}{4A_1A_3} .\end{aligned}\tag{6}$$

Substituting the expressions from Eqs. (5) and (6) in Eq. (3), then simplifying, we obtain

$$A_4 = \sqrt{(A_1 + A_2 + A_3)^2 - 4A_1A_3} .\tag{7}$$