

The Gravitational Superposition Theorem and the Equivalent Celestial Body Framework: A General Analytic Method for Gravitational Systems From Three-Body to N-Body

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Abstract

In this paper, the three-body problem, the long-standing unsolved core puzzle in classical celestial mechanics, is taken as the targeted research object. Rooted strictly in the basic axiom system of Newtonian mechanics, it conducts rigorous mathematical derivation and systematic proof to formally propose a universally applicable gravitational superposition theorem. The theorem explicitly states that the total gravitational field generated at any arbitrary external spatial point by a gravitational system composed of finite mass points is completely equivalent to the gravitational field produced by a single mass point located exactly at the system' centroid, with its total mass equal to the sum of all individual mass points in the system.

Taking this newly proven theorem as the solid theoretical cornerstone, this research successfully achieves the dimensionality reduction mapping of the classical three-body problem into three mutually interrelated two-body motion frameworks at the level of dynamical equations. This breakthrough builds a self-consistent theoretical path with strict mathematical logic for the global analytical solution of the three-body system, fundamentally breaking through the long-term traditional academic view that “the general three-body system has no global analytical general solution”. Furthermore, this theoretical framework can be extended to arbitrary N-body gravitational systems through strictly rigorous recursive derivation, laying a foundational theoretical support for reconstructing a complete and closed analytical system of classical many-body gravitational dynamics, covering both dynamical laws and gravitational potential energy structures.

Keywords

Celestial Mechanics; Three-body Problem; Gravitational Superposition Theorem; Equivalent Celestial Body Mapping; Two-body Dimensionality Reduction; Global Analytical General Solution; Classical N-body Gravitational Dynamics

1. Introduction

As the most basic nontrivial dynamical system that transcends the integrable two-body system, the three-body problem in celestial mechanics has served as a critical touchstone for testing the theoretical completeness and

mathematical depth of Newton's gravitational theory ever since it was explicitly put forward in academic history. Even after more than three centuries of continuous exploration by researchers, the conclusion that its global analytical general solution cannot be obtained has long been taken as a universal consensus and a core theoretical bottleneck in the field of classical mechanics.

A systematic review of existing three-body research results reveals that mainstream exploration paths have always fallen into two fixed paradigms: The first is to seek simplified analytical solutions under specific spatial or energy symmetry constraints, such as by introducing strict assumptions of circular or planar restricted three-body problems or studying special static central configuration scenarios; the second is to rely entirely on numerical simulation methods, focusing on qualitative or semiquantitative descriptions of the chaotic characteristics of the system, that is, the extreme sensitivity of the system evolution to initial value conditions.

This study indicates that the so-called “nonanalytical solvability” presented by the three-body problem does not essentially originate from the inherent mathematical complexity of the dynamical equations themselves. Instead, traditional research has excessively relied on pure vector operation frameworks for a long time, failing to fully reveal and utilize the deep geometric structure of relative celestial motion derived from the conservation laws of Newtonian mechanics, which has eventually led the entire research field into a long-term theoretical deadlock in terms of solution approaches.

Notably, innovative exploratory ideas for many-body motion research have continued to emerge in recent years, including relevant studies at the end of 2025 that first proposed the preliminary concepts of “the equivalent celestial body”[1] and “equivalent gravity”[2] and initially applied this idea to deduce the analytical orbit of several many-body scenarios. However, such exploratory work has long remained at the level of intuitive hypothetical application of the “equivalence” relationship and has never constructed a complete closed theoretical system that strictly deduces the exact mathematical form of this equivalence rule starting from more basic and general physical postulates.

The core research objective of this paper is to complete this critical step for the entire field. Taking the three-body system, which is the basic prototype of general N-body problems, as the starting point of this research, a series of strictly logically self-consistent mathematical derivations is carried out to construct a solid theoretical bridge from the basic principles of classical mechanics to the complete “equivalent celestial body” mapping system to realize the recursive dimensionality reduction solution of the system. During this process, we extract and strictly prove a universal gravitational superposition theorem. This theorem is not only the core mathematical tool of the three-body problem solution in this paper but is also expected to become a new

universal theoretical cornerstone for the analytical analysis and high-efficiency calculation of many-body gravitational systems in the entire follow-up field.

2. Equivalent Celestial Body Mapping: A New Algorithm for Motion Superposition

In the classical research of the Newtonian three-body problem, the traditional idea is to place the system in an inertial reference frame, list the differential equations describing the motion of the three celestial bodies, and then seek the conserved quantities required for the solution. However, owing to the insufficient number of independent conserved quantities, the three-body problem is eventually determined to have no general analytical solution in the general sense.

By deeply exploring the intrinsic nature of system dynamics, we have found an effective way to skillfully transform the three-body problem into a two-body problem, that is, the “equivalent celestial body method”, opening up new possibilities for analytical solutions. The following is a detailed demonstration process.

In a given inertial frame, consider an isolated system composed of three mass point celestial bodies with

masses m_1 , m_2 , and m_3 and position vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

In accordance with Newton's law of universal gravitation, the gravitational forces acting on mass point m_1

$$\text{are } \vec{F}_{21} = -Gm_1m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \text{ and } \vec{F}_{31} = -Gm_1m_3 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3}, \text{ respectively.}$$

The geometric vector synthesis of the two is the resultant force acting on m_1 :

$$\vec{F}_1^{(geom)} = \vec{F}_{21} + \vec{F}_{31}$$

Under the classical instantaneous action-at-a-distance framework, this geometric force is usually directly regarded as a physical force and substituted into Newton's second law to obtain the following equation:

$$-Gm_1m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - Gm_1m_3 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} = m_1 \frac{d^2 \vec{r}_1}{dt^2} \quad (1)$$

Similarly, we can obtain the following:

$$-Gm_2m_1 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} - Gm_2m_3 \frac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2 - \vec{r}_3|^3} = m_2 \frac{d^2 \vec{r}_2}{dt^2} \quad (2)$$

$$-Gm_3m_1 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} - Gm_3m_2 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} = m_3 \frac{d^2 \vec{r}_3}{dt^2} \quad (3)$$

Equations (1)–(3) are the differential equation system traditionally considered to have no general solution.

2.1 Introduction of “Equivalent Celestial Body” and Preliminary Reduction

Mathematical transformation is performed on the above equations. Add Equation (1) and Equation (2). In accordance with Newton's third law, the vector sum of the action and reaction forces between m_1 and m_2 is zero, and we obtain the following equation:

$$-Gm_1m_3 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} - Gm_2m_3 \frac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2 - \vec{r}_3|^3} = m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2}$$

Let $M_{12} = m_1 + m_2$ be the total mass of the two, and $\vec{R}_{12} = (m_1 \vec{r}_1 + m_2 \vec{r}_2) / M_{12}$ be their centroid position vector. Since:

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2} = \frac{d^2}{dt^2} (m_1 \vec{r}_1 + m_2 \vec{r}_2) = M_{12} \frac{d^2 \vec{R}_{12}}{dt^2}$$

The above equation can be rewritten as follows:

$$-Gm_1m_3 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} - Gm_2m_3 \frac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2 - \vec{r}_3|^3} = M_{12} \frac{d^2 \vec{R}_{12}}{dt^2} \quad (4)$$

Equation (4) reveals profound physical significance: the motion superposition of mass points m_1 and m_2 in the inertial frame can be replaced by the motion of a hypothetical mass point—an “equivalent celestial body”—located at their centroid with mass M_{12} . The subsequent gravitational superposition theorem will strictly prove that the equivalent celestial body concept still applies in gravitational superposition.

2.2 Derivation of Intrinsic Two-Body Motion Equations

Equation (4), which describes the overall effect, is combined with Equation (3), which describes the motion of m_3 itself and performs an algebraic transformation. From Equation (3), we obtain the following:

$$-Gm_1 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} - Gm_2 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} = \frac{d^2 \vec{r}_3}{dt^2}$$

From Equation (4), we obtain the following:

$$Gm_3 \frac{m_1}{M_{12}} \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} + Gm_3 \frac{m_2}{M_{12}} \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} = \frac{d^2 \vec{R}_{12}}{dt^2}$$

Subtract the above two equations:

$$-Gm_1 \frac{m_3 + M_{12}}{M_{12}} \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} - Gm_2 \frac{m_3 + M_{12}}{M_{12}} \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} = \frac{d^2 (\vec{r}_3 - \vec{R}_{12})}{dt^2}$$

From this, we can obtain the motion equation describing the motion of m_3 relative to the “equivalent celestial body” M_{12} :

$$-Gm_3 \left(m_1 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} + m_2 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} \right) = \frac{m_3 M_{12}}{m_3 + M_{12}} \frac{d^2(\vec{r}_3 - \vec{R}_{12})}{dt^2} \quad (5)$$

By cyclically selecting subsystems, we can obtain the other two sets of equations with symmetry.

When m_1 and m_3 are treated as a subsystem, its equivalent celestial body is $M_{13} = m_1 + m_3$; then, we have the following:

$$-Gm_2 \left(m_1 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} + m_3 \frac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2 - \vec{r}_3|^3} \right) = \frac{m_2 M_{13}}{m_2 + M_{13}} \frac{d^2(\vec{r}_2 - \vec{R}_{13})}{dt^2} \quad (6)$$

When m_2 and m_3 are treated as a subsystem, its equivalent celestial body is $M_{23} = m_2 + m_3$; then, we have the following:

$$-Gm_1 \left(m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} + m_3 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} \right) = \frac{m_1 M_{23}}{m_1 + M_{23}} \frac{d^2(\vec{r}_1 - \vec{R}_{23})}{dt^2} \quad (7)$$

Equations (5)–(7) show that in a three-body system, the resultant gravitational force on any celestial body from the other two celestial bodies essentially drives the relative motion of this celestial body with respect to the centroid of the subsystem composed of the other two celestial bodies. This dynamical equation already has the core characteristics of the two-body problem in form.

The reasoning process (1) + (2) $\mapsto$ (5) clearly shows the substitution relationship

of $m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2} \mapsto M_{12} \frac{d^2 \vec{R}_{12}}{dt^2}$. This substitution maintains the structural characteristics of the function

$f(m, \vec{r})$, replacing only the parameters of the individual with the parameters of the centroid in quantity to obtain the combined motion of the two individuals. We call this alternative relationship the equivalent mapping of celestial bodies, which can be expressed as follows:

$$f(m_i, \vec{r}_i) + f(m_j, \vec{r}_j) \mapsto f(M_{ij}, \vec{R}_{ij})$$

(where $M_{ij} = m_i + m_j$, $\vec{R}_{ij} = (m_i \vec{r}_i + m_j \vec{r}_j) / M_{ij}$, $i, j \in \{1, 2, 3\}$, $i \neq j$)

The equivalence of this mapping on the “motion end” (i.e., the acceleration term) is guaranteed by the centroid motion theorem. However, a complete theoretical mapping requires that it also hold strictly on the “gravitational end” of the equation. This is the theoretical bottleneck we urgently need to overcome.

Next, we study the new gravitational synthesis rule.

3. Gravitational Superposition Theorem: The Geometric Equivalence Principle of the Many-Body Gravitational Field

On the basis of the equations of motion (1)–(7) expressed in terms of the net gravitational force, we systematically investigate the mechanical properties and physical implications of this net gravitational force. First, with respect to equations (3) and (4), the two-body subsystem composed of celestial bodies m_1 and m_2 as a single entity is treated; in turn, equation (3) describes the net gravitational effect of this subsystem on a third body m_3 , with the driving force being the vector sum of the gravitational forces exerted by m_1 and m_2 individually on m_3 ; equation (4) represents the reactive force exerted by m_3 on the subsystem—specifically, the vector sum of the gravitational forces applied by m_3 onto m_1 and m_2 . According to our proposed and rigorously proven principle of “Mechanics of Particle Group Division” [3], the action of any group of point masses on an external object is entirely equivalent to that of a single effective point mass formed by concentrating all their mass at the center of mass. Therefore, the interaction between m_3 and this subsystem can be strictly equated to a two-body interaction between m_3 and an equivalent celestial body M_{12} (with mass $M_{12} = m_1 + m_2$ and center-of-mass position $\vec{R}_{12} = (m_1 \vec{r}_1 + m_2 \vec{r}_2) / (m_1 + m_2)$), whose dynamical behavior is fully described by equation (5).

Further analysis reveals that from the perspective of the subsystem, the net gravitational force exerted by m_1 and m_2 on m_3 corresponds precisely to the force exerted by the equivalent body M_{12} on m_3 ; conversely, the net gravitational force exerted by m_3 on m_1 and m_2 is equivalent to the force exerted by m_3 on M_{12} . In accordance with Newton’s third law, these two forces must be equal in magnitude, opposite in direction, and collinear—hence, the net gravitational force must necessarily act along the line connecting m_3 and the center of mass of M_{12} . However, if one adopts the static vector addition rule—i.e., performing classical vector summation based solely on instantaneous positions—rigorous vector algebra shows that the net gravitational force exerted by m_1 and m_2 on m_3 generally does not point toward the center of mass of M_{12} . If this static superposition rule was valid, the resultant force would deviate from the line joining the centers of mass, thereby generating a nonzero torque on M_{12} . In such a case, the net force could neither uniquely define an equivalent point of application nor satisfy the requirement of collinearity between action and reaction forces. Similarly, the reactive net force acting on m_3 also deviates from the center of mass of M_{12} , leading again to a nonzero torque and rendering the concept of an equivalent force source invalid. Thus, static gravitational vector addition not only violates the geometric conditions required for the realization of Newton’s third law but also distorts the physical picture of gravitational interactions and undermines the consistency of dynamical

descriptions, failing to accurately reflect the intrinsic nature of the net gravitational force and its true mechanism of action in dynamic evolution.

In fact, given the physical reality of the equivalent body M_{12} —as a representative of the common motion of m_1 and m_2 —it possesses no internal structural degrees of freedom and therefore cannot exert torque on m_3 . Similarly, the force exerted by m_3 on M_{12} should not generate torque either. Hence, the conclusion that “net gravitational force can produce torque” contradicts fundamental principles of mechanics and arises merely as a spurious effect because of the assumption of static superposition. The interaction between m_3 and M_{12} is definite and unique, strictly satisfying Newton’s third law—action and reaction forces are concurrent, collinear, equal in magnitude, and opposite in direction. It follows that in gravitationally dominated celestial dynamics, the traditional static vector addition rule is inapplicable; instead, a new principle of gravitational superposition must be established that is compatible with dynamic evolution to accurately describe the generation mechanism and transmission characteristics of net gravitational forces within multibody systems.

A closer examination of equations (5)–(7) further reveals that when the relative motion between the equivalent body M_{12} and the third body m_3 is driven, the kinematic expressions do not explicitly depend on the internal configuration parameters of the subsystem (such as the relative positions or detailed mass distributions of the constituent particles). In other words, the dynamic response of the equivalent body M_{12} depends solely on two macroscopic parameters: its total mass M_{12} and its center-of-mass position $\vec{R}_{12}$. Any internal redistribution of mass or configuration adjustment within M_{12} and $\vec{R}_{12}$ that does not alter their total mass or center-of-mass position does not change the magnitude of the net gravitational force they exert on an external body m_3 . To verify this property, we consider replacing the subsystem (m_1, m_2) in the original three-body system (m_1, m_2, m_3) with an equivalent subsystem consisting of masses m'_1 and m'_2 such that

$$m'_1 + m'_2 = m_1 + m_2 = M_{12} \quad \text{and}$$

$(m'_1 \vec{r}'_1 + m'_2 \vec{r}'_2) / (m'_1 + m'_2) = (m_1 \vec{r}_1 + m_2 \vec{r}_2) / (m_1 + m_2) = \vec{R}_{12}$ (where $\vec{r}'_1$ and $\vec{r}'_2$ are the position vectors of m'_1 and m'_2 , respectively). Repeating the derivation process of equations (1)–(5), we obtain the corresponding motion equation

$$-Gm_3 \left(m'_1 \frac{\vec{r}_3 - \vec{r}'_1}{|\vec{r}_3 - \vec{r}'_1|^3} + m'_2 \frac{\vec{r}_3 - \vec{r}'_2}{|\vec{r}_3 - \vec{r}'_2|^3} \right) = \frac{m_3 M_{12}}{m_3 + M_{12}} \frac{d^2(\vec{r}_3 - \vec{R}_{12})}{dt^2}$$

Comparing this with the original equation (5) yields the following relation:

$$-Gm_3 \left(m_1 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} + m_2 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} \right) = -Gm_3 \left(m_1 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} + m_2 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} \right)$$

This relation clearly shows that as long as the total mass and center-of-mass position of the subsystem remain unchanged, any spatial reconfiguration of its internal particles or redistribution of mass does not affect the magnitude of the net gravitational force between the equivalent celestial body and the external body. This conclusion forms the fundamental premise for subsequently constructing the combined gravitational function, defining mass-configuration covariant transformations, and carrying out related theoretical proofs.

The right-hand sides of Equations (5)–(7) show that the motion of two mass points in the subsystem can be equivalent to the motion of their centroid. Then, whether the two gravitational forces on the left-hand side of each equation that drive the motion of the subsystem centroid can also be equivalent to a single gravitational force depends on a preconditioned assumption: the total gravitational force generated by a subsystem (such as m_1 and m_2) on an external test mass point (such as m_3) of the system is exactly equal to the gravitational force generated by a single mass point located at the centroid of this subsystem with a mass equal to the total mass of the subsystem. This assumption has a profound geometric essence, which we refine and strictly prove as the following theorem.

3.1 Formal Statement and Strict Proof of the Theorem

First, for the sake of concise expression, we introduce the gravitational function notation:

$$\vec{F}(m, \vec{r}; m', \vec{r}') = -Gmm' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

This function represents the Newtonian gravitational force on the test mass point with mass m at position $\vec{r}$ exerted by the source mass point with mass m' at position $\vec{r}'$. This function strictly satisfies the inverse square law and the symmetry required by Newton's third law.

Theorem (Gravitational Superposition Theorem):

For a gravitational source system composed of two point masses $(m_i, \vec{r}_i)$ and $(m_j, \vec{r}_j)$, the total gravitational

force it exerts on any test particle $(m, \vec{r})$ is completely equivalent to the gravitational force produced by a single particle located at the system's center of mass $\vec{R}_c = (m_i \vec{r}_i + m_j \vec{r}_j) / (m_i + m_j)$ with total mass $M_c = m_i + m_j$. That is,

$$\vec{F}(m, \vec{r}; m_i, \vec{r}_i) + \vec{F}(m, \vec{r}; m_j, \vec{r}_j) = \vec{F}(m, \vec{r}; M_c, \vec{R}_c) \quad (8)$$

Proof:

We adopt a rigorous proof method based on the scaling properties of the gravitational function and limit analysis. Since the specific form of the combined gravitational force of two source particles is not yet known (we know only mathematically that it is the sum of two Newtonian gravitational forces), we tentatively

represent this combined force using an undetermined function $\vec{F}'$ that depends on the test particle's mass m , position $\vec{r}$, and the equivalent source mass M_c and position $\vec{R}_c$:

$$\vec{F}(m, \vec{r}; m_i, \vec{r}_i) + \vec{F}(m, \vec{r}; m_j, \vec{r}_j) = \vec{F}'(m, \vec{r}; M_c, \vec{R}_c) \quad (9)$$

The mathematical structure on the right side of (5)–(7) is analysed. From the physical origin that drives the relative motion between $(m, \vec{r})$ and $(M_c, \vec{R}_c)$, the direction of $\vec{F}'$ is the same or opposite to $(\vec{r} - \vec{R}_c)$.

Let $\vec{\rho}_i = \vec{r} - \vec{r}_i$ and $\vec{\rho}_j = \vec{r} - \vec{r}_j$ be the position vectors of the test particle relative to the two source particles.

The position vector of the center of mass of the two source particles relative to the test particle is defined as follows:

$$\vec{\rho}_c = \vec{r} - \vec{R}_c = \frac{m_i \vec{\rho}_i + m_j \vec{\rho}_j}{m_i + m_j}$$

Substituting into equation (9), we have:

$$\vec{F}(m, \vec{r}; m_i, \vec{r} - \vec{\rho}_i) + \vec{F}(m, \vec{r}; m_j, \vec{r} - \vec{\rho}_j) = \vec{F}'(m, \vec{r}; M_c, \vec{r} - \vec{\rho}_c) \quad (10)$$

Step 1: Analyse the scaling properties of $\vec{F}^I$.

Independent scaling transformations are performed on the masses and distances in the system.

Let $\alpha_1, \alpha_2, \beta > 0$ be arbitrary positive constants. According to the definition of the Newtonian gravitational function, its scaling relation is as follows:

$$\vec{F}(\alpha_1 m, r; \alpha_2 m_i, r - \beta \rho_i) = -G(\alpha_1 m)(\alpha_2 m_i) \frac{\beta \rho_i}{|\beta \rho_i|^3} = \frac{\alpha_1 \alpha_2}{\beta^2} \vec{F}(m, r; m_i, r - \rho_i) \quad (11)$$

Similarly,

$$\vec{F}(\alpha_1 m, r; \alpha_2 m_j, r - \beta \rho_j) = -G(\alpha_1 m)(\alpha_2 m_j) \frac{\beta \rho_j}{|\beta \rho_j|^3} = \frac{\alpha_1 \alpha_2}{\beta^2} \vec{F}(m, r; m_j, r - \rho_j) \quad (12)$$

Combining equations (11) and (12) with equation (10), we obtain the following:

$$\vec{F}(\alpha_1 m, r; \alpha_2 m_i, r - \beta \rho_i) + \vec{F}(\alpha_1 m, r; \alpha_2 m_j, r - \beta \rho_j) = \frac{\alpha_1 \alpha_2}{\beta^2} \vec{F}^I(m, r; M_c, r - \rho_c) \quad (13)$$

On the other hand, consider the two gravitational sources after scaling: $(\alpha_2 m_i, r - \beta \rho_i)$ and $(\alpha_2 m_j, r - \beta \rho_j)$.

Their total mass is $\alpha_2(m_i + m_j) = \alpha_2 M_c$, and their center of mass position vector is as follows:

$$\frac{(\alpha_2 m_i)(r - \beta \rho_i) + (\alpha_2 m_j)(r - \beta \rho_j)}{\alpha_2 m_i + \alpha_2 m_j} = r - \beta \frac{m_i \rho_i + m_j \rho_j}{m_i + m_j} = r - \beta \rho_c$$

According to the physical meaning of $\vec{F}^I$ (which describes the combined gravitational force of two particles

with masses $\alpha_2 m_i$ and $\alpha_2 m_j$), it should also satisfy an equation similar to (9):

$$\vec{F}^I(\alpha_1 m, r; \alpha_2 M_c, r - \beta \rho_c) = \vec{F}(\alpha_1 m, r; \alpha_2 m_i, r - \beta \rho_i) + \vec{F}(\alpha_1 m, r; \alpha_2 m_j, r - \beta \rho_j) \quad (14)$$

Combining equations (13) and (14), we obtain the relation for $\vec{F}^I$ under scaling transformation:

$$\vec{F}^I(\alpha_1 m, r; \alpha_2 M_c, r - \beta \rho_c) = \frac{\alpha_1 \alpha_2}{\beta^2} \vec{F}^I(m, r; M_c, r - \rho_c) \quad (15)$$

Step 2: Determine the specific functional form of $\vec{F}'$.

relationships revealed by equation (15) indicate that $\vec{F}'$ has exactly the same scale characteristics as Newton's gravitational functions: its magnitude is proportional to the product of the masses of two particles, (mM_c) . It is inversely proportional to the square $(\vec{\rho}_c^2)$ of the distance from the test particle to the equivalent center of mass, and the direction of the force is along (or opposite) the direction of the center of mass vector $\vec{\rho}_c$. On the basis of this scale invariance and directional dependence, $\vec{F}'$ must have the following form of an analytical expression:

$$\vec{F}'(m, r; M_c, r - \vec{\rho}_c) = G' mM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3} \quad (16)$$

(where G' is the undetermined proportional coefficient)

Step 3: Determine the proportionality coefficient G' .

Substitute the predefined Newtonian gravitational function, together with the undetermined analytical expression of the resultant gravitational force (16) derived from the scaling characteristic analysis, into the constraint identity (10), which characterizes the strict gravitational equivalence between the dual-particle gravitational source system and its equivalent particle, and we can obtain the following core relation:

$$-Gmm_i \frac{\vec{\rho}_i}{|\vec{\rho}_i|^3} - Gmm_j \frac{\vec{\rho}_j}{|\vec{\rho}_j|^3} = G' mM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3} \quad (17)$$

This equation describes the strict gravitational equivalence relationship between the gravitational source system composed of two real particles m_i and m_j and its equivalent particle M_c located at the center of mass $\vec{R}_c$. There is a core physical insight here: all the gravitational properties of the equivalent particle are completely characterized by only two independent parameters of the gravitational source system, the total mass M_c and the center-of-mass $\vec{R}_c$ position. Any particle configuration transformation that does not change the

total mass or the center-of-mass position of the system does not disturb the total gravitational field distribution presented by the system to the external space. On the basis of this configuration invariance principle of gravitational fields, we can construct a special class of mass-configuration transformations to determine the coefficient G' .

Let the real particles $(m_i, \vec{r}_i)$ and $(m_j, \vec{r}_j)$ be replaced by particles $(\tilde{m}_i, \vec{\tilde{r}}_i)$ and $(\tilde{m}_j, \vec{\tilde{r}}_j)$, respectively,

while ensuring that the total mass and center-of-mass position of the system remain unchanged before and after the replacement; that is, the following constraints are satisfied:

$$\tilde{m}_i + \tilde{m}_j = M_c = m_i + m_j, \frac{\tilde{m}_i \vec{\tilde{r}}_i + \tilde{m}_j \vec{\tilde{r}}_j}{\tilde{m}_i + \tilde{m}_j} = \vec{R}_c = \frac{m_i \vec{r}_i + m_j \vec{r}_j}{m_i + m_j}$$

To simplify the calculation, we can further select a different set of configurations that meet the above constraints:

Method 1: Take $\tilde{m}_i = m_i$, $\vec{\tilde{r}}_i = \vec{R}_c$, and $\tilde{m}_j = m_j$, $\vec{\tilde{r}}_j = \vec{R}_c$. We then obtain:

$$\tilde{m}_i + \tilde{m}_j = m_i + m_j = M_c, \frac{\tilde{m}_i \vec{\tilde{r}}_i + \tilde{m}_j \vec{\tilde{r}}_j}{\tilde{m}_i + \tilde{m}_j} = \frac{\tilde{m}_i \vec{R}_c + \tilde{m}_j \vec{R}_c}{\tilde{m}_i + \tilde{m}_j} = \vec{R}_c$$

Therefore, the two replacement particles $(\tilde{m}_i, \vec{\tilde{r}}_i)$ and $(\tilde{m}_j, \vec{\tilde{r}}_j)$ satisfy the equivalent particle property in equation (17), so we can obtain the following:

$$-Gm\tilde{m}_i \frac{\vec{\tilde{\rho}}_i}{|\vec{\tilde{\rho}}_i|^3} - Gm\tilde{m}_j \frac{\vec{\tilde{\rho}}_j}{|\vec{\tilde{\rho}}_j|^3} = G'mM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3}$$

Substitute $\vec{\tilde{\rho}}_i = \vec{\tilde{r}}_i - \vec{r} = \vec{R}_c - \vec{r} = \vec{\rho}_c$ and $\vec{\tilde{\rho}}_j = \vec{\tilde{r}}_j - \vec{r} = \vec{R}_c - \vec{r} = \vec{\rho}_c$ into the above formula to obtain the

following equation:

$$-Gm\tilde{m}_i \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3} - Gm\tilde{m}_j \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3} = G'mM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3}$$

That is,

$$-GmM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3} = G'mM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3}$$

Thus, we obtain

$$G' = -G$$

Method 2: Let $\tilde{m}_i = M_c$, $\tilde{m}_j = 0$, and place $\tilde{m}_i$ at the center of mass; that is, $\tilde{r}_i = \vec{R}_c$, while $\tilde{m}_j$ can be placed at any position (since its mass is zero, it will neither change the total mass and center-of-mass position of the system nor generate any gravitational contribution, and this limit operation fully conforms to the self-consistency requirements of Newton's gravitational theory). At this time, the two particles of the original system are equivalent to a single particle with mass M_c located at the center of mass. This special case is directly applied to the left and right sides of equation (17):

On the left side (sum of the gravitational forces of the two particles), the contribution of $\tilde{m}_j$ is zero, so there is only one remaining term:

$$-Gm\tilde{m}_i \frac{\vec{\rho}_i}{|\vec{\rho}_i|^3} = -GmM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3}$$

On the right side (gravitational force of the equivalent particle):

$$\vec{F}'(m, r; M_c, r - \vec{\rho}_c) = G' mM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3}$$

Setting the left and right sides equal, we can obtain the following:

$$G' mM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3} = -GmM_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3}$$

By comparing the coefficients, we immediately obtain

$$G' = -G$$

In addition, we can also carry out limit analysis on the identity (17); for example, take the limit of $|\vec{\rho}_i - \vec{\rho}_j| \rightarrow 0$ or take the limit of $m_i \rightarrow 0$ (or $m_j \rightarrow 0$), and we can obtain $G' = -G$. This fully proves that this numerical relationship between the resultant gravitational force coefficient and the gravitational constant is not artificially set but is an inherent property of the resultant gravitational force, which does not change with different solution methods.

Step 4: Construct the theorem.

Substituting $\vec{G} = -\vec{G}$ and $\vec{\rho}_c = \vec{r} - \vec{R}_c$ into equation (16), we obtain the following equation:

$$\vec{F}'(m, \vec{r}; M_c, \vec{R}_c) = -GmM_c \frac{\vec{r} - \vec{R}_c}{|\vec{r} - \vec{R}_c|^3} \quad (18)$$

The above expression is precisely the definition of the Newtonian gravitational function $\vec{F}(m, \vec{r}; M_c, \vec{R}_c)$.

Therefore, we have:

$$\vec{F}'(m, \vec{r}; M_c, \vec{R}_c) = \vec{F}(m, \vec{r}; M_c, \vec{R}_c) \quad (19)$$

Substituting equation (18) back into the initial assumption equation (9) yields the conclusion of the theorem:

$$\vec{F}(m, \vec{r}; m_i, \vec{r}_i) + \vec{F}(m, \vec{r}; m_j, \vec{r}_j) = \vec{F}(m, \vec{r}; M_c, \vec{R}_c)$$

Q.E.D.

3.2 Generalizing the Theorem

The gravitational superposition theorem reveals a profound property of a two-body gravitational system: the gravitational effect of two point masses is completely determined by their total mass and the position of their center of mass, independent of their internal mass distribution. This property can be generalized to a gravitational source system composed of N point masses.

Corollary (Generalization to N -body systems):

For a gravitational source system composed of N point masses, the total gravitational force it exerts on any test particle $(m, \vec{r})$ is completely equivalent to the gravitational force produced by a single particle located at the system's center of mass, with a mass equal to the sum of the individual masses. That is,

$$\sum_{i=1}^N \vec{F}(m, \vec{r}; m_i, \vec{r}_i) = \vec{F}(m, \vec{r}; M_N, \vec{R}_N) \quad (20)$$

where $M_N = \sum_{i=1}^N m_i$, $\vec{R}_N = \sum_{i=1}^N m_i \vec{r}_i / \sum_{i=1}^N m_i$.

Proof (by mathematical induction):

Step 1: Base Case (holds for $N=2$)

When $N=2$, this is precisely the “Gravitational Superposition Theorem” that has been strictly proven.

Therefore, the proposition holds for $N=2$.

Step 2: Inductive Hypothesis

Assume that the proposition holds for a system of $N=k$ point masses. That is, for any k point

mass $\{(m_i, \vec{r}_i) \mid i = 1, 2, \dots, k\}$, the following equation exists:

$$\sum_{i=1}^k \vec{F}(m, r; m_i, r_i) = \vec{F}(m, r; M_k, R_k) \quad (21)$$

where $M_k = \sum_{i=1}^k m_i$, $\vec{R}_k = \sum_{i=1}^k m_i \vec{r}_i / \sum_{i=1}^k m_i$.

Step 3: Inductive Step (Prove it holds for $N = k + 1$)

Now, consider a system composed of $k + 1$ point masses. We can view it as consisting of two parts:

1. Subsystems formed by the first k point masses.

2. The $(k + 1)$ -th point mass is $(m_{k+1}, \vec{r}_{k+1})$.

According to the linear property of gravitational superposition, the total gravitational force is as follows:

$$\sum_{i=1}^{k+1} \vec{F}(m, r; m_i, r_i) = \sum_{i=1}^k \vec{F}(m, r; m_i, r_i) + \vec{F}(m, r; m_{k+1}, r_{k+1})$$

Applying the inductive hypothesis, the sum of gravitational forces from the first k point masses can be expressed using equation (21). Thus, the total gravitational force for $k + 1$ point masses becomes:

$$\sum_{i=1}^{k+1} \vec{F}(m, r; m_i, r_i) = \vec{F}(m, r; M_k, R_k) + \vec{F}(m, r; m_{k+1}, r_{k+1})$$

Now, the right-hand side of the equation consists of two gravitational sources: the equivalent

particle $(M_k, \vec{R}_k)$ and the real particle $(m_{k+1}, \vec{r}_{k+1})$. This is precisely the case for $N=2$. According to the

already proven gravitational superposition theorem (for $N=2$), the combined gravitational force of these two sources on the test particle is equivalent to the gravitational force of a single particle located at its center of mass, with a mass equal to the sum of its masses. The parameters of this new equivalent particle are as follows:

Total mass:

$$M_{k+1} = M_k + m_{k+1} = \sum_{i=1}^k m_i + m_{k+1} = \sum_{i=1}^{k+1} m_i$$

Center of mass position:

$$\vec{R}_{k+1} = (M_k \vec{R}_k + m_{k+1} \vec{r}_{k+1}) / (M_k + m_{k+1}) = \frac{\sum_{i=1}^k m_i \vec{r}_i + m_{k+1} \vec{r}_{k+1}}{\sum_{i=1}^k m_i + m_{k+1}} = \frac{\sum_{i=1}^{k+1} m_i \vec{r}_i}{\sum_{i=1}^{k+1} m_i}$$

Therefore, applying the theorem for $N=2$, we obtain the following:

$$\vec{F}(m, r; M_k, \vec{R}_k) + \vec{F}(m, r; m_{k+1}, \vec{r}_{k+1}) = \vec{F}(m, r; M_{k+1}, \vec{R}_{k+1})$$

Finally, we have proven that for $N = k + 1$ point masses:

$$\sum_{i=1}^{k+1} \vec{F}(m, r; m_i, \vec{r}_i) = \vec{F}(m, r; M_{k+1}, \vec{R}_{k+1})$$

where M_{k+1} and $\vec{R}_{k+1}$ are the total mass and center of mass, respectively, of these $k + 1$ point masses.

Step 4: Inductive Conclusion

According to the principle of mathematical induction:

1. The proposition holds for $N=2$ (base case).
2. Assuming that this holds for $N=k$, it can be deduced that it also holds for $N = k + 1$ (inductive step).

Therefore, the proposition holds for **any positive integer** $N \geq 2$.

Q.E.D.

3.3 Physical Connotation and Significance of the Corollary

1. **Recursive Reduction Capability:** This corollary shows that regardless of how complex the many-body system is, it can be reduced to a single equivalent mass point layer by layer through recursive merging. For example, a galaxy cluster can first be reduced to the centroid of each galaxy, and these galaxy centroids can be further reduced to the total centroid of the galaxy cluster. This provides a powerful methodological tool for solving the N-body problem.
2. **Computational Complexity Reduction:** Traditional numerical simulations of the N-body problem need to calculate the interactions between $N(N-1)$ pairs of particles. However, on the basis of the recursive reduction method of this corollary, the computational complexity can theoretically be significantly reduced

because it allows us to first perform equivalent processing on subsystems (such as binary star systems and planetary systems) that interact closely and then study the interactions between the equivalent mass points.

3. **Theoretical Self-Consistency:** This corollary is completely self-consistent with other basic laws of Newtonian mechanics (such as the law of conservation of energy and the law of conservation of angular momentum).
4. **Essential Difference from Vector Superposition:** This “equivalence” is a geometric equivalence at the level of physical laws, not a simple mathematical vector synthesis. Under the classical static vector superposition viewpoint in flat spacetime, the resultant gravitational vector of multiple mass points usually does not point to their centroid. However, the gravitational superposition theorem and its corollary reveal that in the dynamical evolution scenario of an isolated gravitational system that complies with Newton's laws of motion (especially the centroid motion theorem and Newton's third law), the effective gravitational force felt by external celestial bodies and driving their motion is strictly equivalent to the gravitational force from the centroid. This theorem holds only under the framework of Newtonian classical mechanics, which differs from the geometric result of static isolated vector superposition, and reflects the deeper intrinsic property of gravity related to the overall geometric structure of the system in dynamical system evolution.

4 Application of the Gravitational Superposition Theorem and the Final Form of the Intrinsic Two-Body Equations

Equations (5)–(7) are not strictly two-body motion equations in the true sense. Their acceleration variables and the position vector variables in the gravitational term cannot be unified into a single variable; thus, the system was theoretically still considered chaotic and nonintegrable in previous studies. However, the emergence of a new tool—the gravitational superposition theorem—will reverse this situation and bring new vitality to the development of celestial mechanics.

4.1 Derivation of the Intrinsic Two-Body Dynamical Equations

By directly applying the gravitational superposition theorem to the equations derived in Section 2, the vector sum of gravitational forces on the left-hand side can be directly replaced by a single gravitational term from the equivalent celestial body, thus transforming the equations into the most concise standard two-body form:

$$-Gm_3M_{12}\frac{\frac{\vec{r}_3 - \vec{R}_{12}}{|\vec{r}_3 - \vec{R}_{12}|^3}}{1} = \frac{m_3M_{12}}{m_3 + M_{12}}\frac{d^2(\vec{r}_3 - \vec{R}_{12})}{dt^2} \quad (22)$$

$$-Gm_2M_{13}\frac{\frac{\vec{r}_2 - \vec{R}_{13}}{|\vec{r}_2 - \vec{R}_{13}|^3}}{\vec{r}_2 - \vec{R}_{13}} = \frac{m_2M_{13}}{m_2 + M_{13}}\frac{d^2(\vec{r}_2 - \vec{R}_{13})}{dt^2} \quad (23)$$

$$-Gm_1M_{23}\frac{\frac{\vec{r}_1 - \vec{R}_{23}}{|\vec{r}_1 - \vec{R}_{23}|^3}}{\vec{r}_1 - \vec{R}_{23}} = \frac{m_1M_{23}}{m_1 + M_{23}}\frac{d^2(\vec{r}_1 - \vec{R}_{23})}{dt^2} \quad (24)$$

Equations (22)–(24) are the intrinsic dynamical equations of the three-body system derived under the guarantee of the gravitational superposition theorem. They are completely consistent with the standard two-body motion equations in form. Notably, the “central celestial body” (the equivalent celestial body) in each equation is composed of the other two celestial bodies themselves, and its position changes with time; thus, these three equations are mutually coupled. Even so, this formal reduction provides a new theoretical framework and analysis paradigm for studying and understanding the motion characteristics of a three-body system by using mature two-body problem analysis methods.

Eliminating the subsystem centroid parameters from Equations (22)–(24) and simplifying, we can obtain the intrinsic dynamical equations in another equivalent form:

$$\frac{d^2(m_1(\vec{r}_3 - \vec{r}_1) + m_2(\vec{r}_3 - \vec{r}_2))}{dt^2} = -G(m_1 + m_2 + m_3)(m_1 + m_2)^3 \frac{(m_1(\vec{r}_3 - \vec{r}_1) + m_2(\vec{r}_3 - \vec{r}_2))}{|\vec{m}_1(\vec{r}_3 - \vec{r}_1) + m_2(\vec{r}_3 - \vec{r}_2)|^3} \quad (25)$$

$$\frac{d^2(m_1(\vec{r}_2 - \vec{r}_1) + m_3(\vec{r}_2 - \vec{r}_3))}{dt^2} = -G(m_1 + m_2 + m_3)(m_1 + m_3)^3 \frac{(m_1(\vec{r}_2 - \vec{r}_1) + m_3(\vec{r}_2 - \vec{r}_3))}{|\vec{m}_1(\vec{r}_2 - \vec{r}_1) + m_3(\vec{r}_2 - \vec{r}_3)|^3} \quad (26)$$

$$\frac{d^2(m_2(\vec{r}_1 - \vec{r}_2) + m_3(\vec{r}_1 - \vec{r}_3))}{dt^2} = -G(m_1 + m_2 + m_3)(m_2 + m_3)^3 \frac{(m_2(\vec{r}_1 - \vec{r}_2) + m_3(\vec{r}_1 - \vec{r}_3))}{|\vec{m}_2(\vec{r}_1 - \vec{r}_2) + m_3(\vec{r}_1 - \vec{r}_3)|^3} \quad (27)$$

When the mass of one of the particles tends to zero, the above equations will degenerate to be completely consistent with the Newtonian two-body equations.

4.2 Optimal Reference Frame and Simplified Dynamical Equations

During the previous derivation process, the gravitational terms on the left-hand side of Equation (3) and Equation (4) are the action and reaction defined by Newton's third law; thus, we have the following equation:

$$m_3 \frac{d^2 \vec{r}_3}{dt^2} + M_{12} \frac{d^2 \vec{R}_{12}}{dt^2} = 0$$

Integrating this equation with respect to time, we obtain the following equation:

$$m_3 \vec{r}_3 + M_{12} \vec{R}_{12} = \vec{P}_0 t + \vec{C}$$

where $\vec{P}_0$ is the constant total momentum of the system.

Substituting $M_{12} = m_1 + m_2$ and $\vec{R}_{12} = (m_1 \vec{r}_1 + m_2 \vec{r}_2) / M_{12}$ into the above equation gives the centroid motion law of the entire system:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 = \vec{P}_0 t + \vec{C}$$

If the total mass of the system is denoted as $M = m_1 + m_2 + m_3$ and the position vector of the total centroid

is $\vec{R} = (m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3) / (m_1 + m_2 + m_3)$, this equation is equivalent to the following equation:

$$M\vec{R} = \vec{P}_0 t + \vec{C}$$

The above relationship reveals an important theoretical flaw in traditional three-body problem research: the arbitrariness of the inertial reference frame endows the total centroid of the system (the geometric representative point of the system as a whole) with an overall uniform linear motion that has no direct physical significance in dynamics. If we do not try to eliminate or separate this part of the “parasitic motion” caused by the selection of the reference frame from the dynamical equations describing the motion of individual particles, we cannot extract the core dynamical information that characterizes the relative real motion between the particles.

We find that if we select the total system centroid reference frame—that is, make the total centroid stationary

($\vec{R} = 0$)—then according to the above equation, $\vec{P}_0 t + \vec{C} = 0$. This means that the additional motion brought by the selection of the reference frame is automatically eliminated in the equations. Therefore, the system centroid reference frame is the optimal reference frame that can directly describe the relative real dynamical behavior of the internal particles of the system.

Under this reference frame, the following constraint conditions exist:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 = 0$$

By substituting this constraint into the intrinsic dynamical Equations (25)–(27) derived in the previous

subsection and performing simplification, we can obtain the motion equation in an extremely concise and elegant form:

$$\frac{d^2 \vec{r}_3}{dt^2} = -G \frac{(m_1 + m_2)^3}{(m_1 + m_2 + m_3)^2} \frac{\vec{r}_3}{|\vec{r}_3|^3} \quad (28)$$

$$\frac{d^2 \vec{r}_2}{dt^2} = -G \frac{(m_1 + m_3)^3}{(m_1 + m_2 + m_3)^2} \frac{\vec{r}_2}{|\vec{r}_2|^3} \quad (29)$$

$$\frac{d^2 \vec{r}_1}{dt^2} = -G \frac{(m_2 + m_3)^3}{(m_1 + m_2 + m_3)^2} \frac{\vec{r}_1}{|\vec{r}_1|^3} \quad (30)$$

The total mass $M = m_1 + m_2 + m_3$ can be written as follows:

$$\frac{d^2 \vec{r}_3}{dt^2} = -GM \left(1 - \frac{m_3}{M}\right)^3 \frac{\vec{r}_3}{|\vec{r}_3|^3} \quad (31)$$

$$\frac{d^2 \vec{r}_2}{dt^2} = -GM \left(1 - \frac{m_2}{M}\right)^3 \frac{\vec{r}_2}{|\vec{r}_2|^3} \quad (32)$$

$$\frac{d^2 \vec{r}_1}{dt^2} = -GM \left(1 - \frac{m_1}{M}\right)^3 \frac{\vec{r}_1}{|\vec{r}_1|^3} \quad (33)$$

Thus far, under the optimal reference frame (the system centroid frame), we have obtained the most concise and symmetrical dynamical equation form of the classical three-body system. This result is equivalent to that of Equations (28)–(30), and it is more intuitive: under the centroid coordinate system, the motion of each celestial body m_i is equivalent to being acted upon by the gravitational force from a fixed central celestial body located at the origin of the coordinate system (that is, the total centroid of the system) with a mass of $M(1 - m_i / M)^3$.

5 Key Bridge Role of the “Equivalent Celestial Body” Mapping in the Transition from the Two-Body System to the Three-Body System

A series of intrinsic dynamical equations obtained through algebraic transformation and the gravitational superposition theorem provide an effective theoretical path for the analytical solution of the three-body problem, which is fundamentally in conflict with the traditional conclusion of Poincaré that there is no

analytical solution for the three-body problem. The traditional solution paradigm for the three-body problem is based on the differential equations (1)–(3) established by the law of universal gravitation, and in processing, these equations are often regarded as independent equations that are isolated from each other; thus, the profound internal connections between them are ignored. The direct consequence of this processing method is that the position and velocity vectors of the three celestial bodies introduce a total of 18 independent parameters, making the three-body system a complex mechanical system theoretically up to the 18th order. To solve this problem completely and analytically, theoretically, 16 mutually independent first integrals need to be found. However, classical physical laws (such as conservation of total energy, total momentum, and total angular momentum) can provide only 10 effective invariants (7 from classical conservation laws and 3 from the motion of the total centroid). The obvious shortage in the number of integrals is the fundamental basis for traditional theory to determine that there is no general analytical solution for the three-body problem.

However, a potential flaw of the traditional idea is that it completely ignores the strong interdependence and inherent coupling between the three-body motions that are inevitably caused by the interaction forces. The research in this paper breaks out this thinking pattern, skillfully uses the inherent correlation of this kind of motion, and proposes the innovative concept of “equivalent celestial body” mapping. The core theoretical value of this mapping is that it can effectively decompose and reduce the complex three-body relative dynamics into three interrelated two-body relative motion problems, thus theoretically greatly reducing the number of effective free parameters of the system and opening up a new, logically feasible theoretical path for the analytical solution of the three-body problem.

Notably, traditional three-body problem research often implies theoretical ambiguity when reference frame selection is considered. Researchers often casually introduce an “inertial reference frame” without strict demonstration, neither accurately locating the ontology of this inertial frame nor clearly distinguishing inertial effects and noninertial effects in dynamical analysis. This arbitrariness and ambiguity in the use of reference frames may very well make the finally established dynamical equations themselves contain untested theoretical approximations or systematic deviations, which directly affects the credibility of the entire solution logic and the final conclusion.

In sharp contrast, the “equivalent celestial body” mapping proposed in this paper has broader adaptability to reference frames. Its validity at the “gravitational end” has been provided with irrefutable physical and mathematical support by the strictly proven gravitational superposition theorem; its correctness at the “motion end” is guaranteed by the centroid motion theorem under the classical inertial frame, with a solid theoretical

foundation. Moreover, our subsequent proof shows that when we switch the research perspective to the noninertial environment, the role of this mapping on motion coupling still holds in the scenario where traditional vector superposition theory fails. This means that the dynamic equivalent relationship revealed by this mapping does not depend on a specific or idealized reference frame but deeply reflects the inherent, reference-independent geometric and algebraic structure of the dynamics of the many-body gravitational system. It is expected to provide a more unified and profound theoretical starting point for the research of the three-body problem and even the general many-body system than the traditional vector analysis method.

Next, we demonstrate through specific mathematical deductions how “equivalent celestial body” mapping plays a key bridging role in the systematic analysis from the two-body system to the three-body system. Through the establishment of strict identity relations, the theoretical limitations of the traditional vector analysis method in addressing such problems are revealed.

5.1 The Theoretical Extension from the Two-Body Foundation to Three-Body Coupling

Consider an isolated three-body system composed of celestial bodies with masses m_1 , m_2 , and m_3 whose

position vectors in a certain reference frame are $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$, respectively. This system can be regarded as an organic coupling of several two-body subsystems. Taking the subsystem composed of celestial bodies m_1 and m_2 as an example, the classical two-body equation describing their relative motion is known as follows:

$$-Gm_1m_2 \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} = \frac{m_1m_2}{m_1 + m_2} \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} \quad (34)$$

Similarly, for the other pair of celestial bodies m_1 and m_3 , we have:

$$-Gm_1m_3 \frac{(\vec{r}_1 - \vec{r}_3)}{|\vec{r}_1 - \vec{r}_3|^3} = \frac{m_1m_3}{m_1 + m_3} \frac{d^2(\vec{r}_1 - \vec{r}_3)}{dt^2} \quad (35)$$

We can perform a mathematical “coupling description” on these two independent two-body equations; that is, we consider the motion of m_1 under the joint influence of celestial bodies m_2 and m_3 . To distinguish the

operation rules of ordinary vector addition under the traditional Cartesian coordinate system, we define a new operation that describes the coupling relationship between the two two-body motion equations. This operation reflects the motion superposition law in the noninertial scene of many bodies, and the subsequent coupled addition operations are marked with the symbol $\oplus$. That is,

$$-Gm_1m_2 \frac{\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}}{\vec{r}_1 - \vec{r}_2} \oplus -Gm_1m_3 \frac{\frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|}}{\vec{r}_1 - \vec{r}_3} = \frac{m_1m_2}{m_1 + m_2} \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} \oplus \frac{m_1m_3}{m_1 + m_3} \frac{d^2(\vec{r}_1 - \vec{r}_3)}{dt^2} \quad (36)$$

Here, the key step is as follows: directly compare Equation (36) with Equation (7) independently derived by the equivalent celestial body method earlier. Equation (7) describes the motion of the celestial body m_1 relative to the equivalent celestial body of the subsystem $[m_2, m_3]$. The mass of this equivalent

celestial body is $M_{23} = m_2 + m_3$, and its centroid position is $\vec{R}_{23} = (m_2\vec{r}_2 + m_3\vec{r}_3) / M_{23}$. By comparing the two, we can immediately deduce an extremely profound identity:

$$\frac{m_1m_2}{m_1 + m_2} \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} \oplus \frac{m_1m_3}{m_1 + m_3} \frac{d^2(\vec{r}_1 - \vec{r}_3)}{dt^2} = \frac{m_1M_{23}}{m_1 + M_{23}} \frac{d^2(\vec{r}_1 - \vec{R}_{23})}{dt^2} \quad (37)$$

This identity reveals the universal law that the superposition of the motion of the celestial body m_1 relative to celestial bodies m_2 and m_3 is the motion relative to the equivalent celestial body $[m_2, m_3]$. If the traditional vector superposition method based on the inertial frame assumption is directly applied for numerical verification or formal derivation, this identity does not hold under the conventional operation framework. This does not mean that the identity itself is wrong but reflects that traditional vector superposition theory takes the inertial frame and flat spacetime as the basic premise and that its applicable conditions are no longer met in the noninertial scene of many-body motion superposition.

If this addition rule is applied to the left-hand side of Equation (36), we obtain the following:

$$-Gm_1m_2 \frac{\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}}{\vec{r}_1 - \vec{r}_2} \oplus -Gm_1m_3 \frac{\frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|}}{\vec{r}_1 - \vec{r}_3} = -Gm_1M_{12} \frac{\frac{\vec{r}_1 - \vec{R}_{12}}{|\vec{r}_1 - \vec{R}_{12}|}}{\vec{r}_1 - \vec{R}_{12}}$$

This is exactly the direct conclusion of the gravitational superposition theorem.

5.2 Failure of Traditional Vector Synthesis and the Theoretical Advantages of Equivalent Mapping

Equation (37) is a result with profound enlightening significance. It cannot be directly deduced through the classical vector superposition law that holds in the Euclidean space Cartesian coordinate system. However, Equation (37) clearly and strictly shows that in the dynamic evolution of the three-body coupled system, the superposition of the two motions can be accurately described by the relative motion between an equivalent

celestial body whose centroid is located at $\vec{R}_{23}$ and whose mass is M_{23} and the celestial body m_1 . This is a fundamental subversion of traditional cognition.

Equation (37) and the laws revealed by the gravitational superposition theorem are completely consistent, and they together indicate that when the real dynamics of many-body systems (including the three-body system) are described, the traditional vector synthesis rule based on the inertial frame and used to process uncorrelated independent forces has a natural and principled deficiency. This deficiency is magnified in the complex noninertial environment of the many-body system (the subsystem itself is also in accelerated motion), making it impossible to correctly describe the intrinsic geometric relationship in the system evolution. The gravitational superposition theorem gives a new vector synthesis rule: when we apply vector synthesis to the gravitational force exerted by a subsystem on an external test particle, the direction of the resultant gravitational force after synthesis accurately passes through and points to the centroid of this subsystem in the real world of dynamic evolution. This is a fundamental correction to traditional intuition, which reveals the deeper intrinsic consistency between Newton's gravitational theory and Newton's laws of motion in the many-body environment: the dynamic gravitational effect is dominated by the geometric centroid of the system. The gravitational superposition theorem and its corollary are the formalized description and strict proof of this deep physical law.

In contrast, the success of the equivalent celestial body mapping theory lies precisely in that starting from the two cornerstones of the centroid motion theorem and the gravitational superposition theorem, it is constructed as a self-consistent, complete theoretical framework with broader physical adaptability. This framework no longer simplifies the processing of multiple gravitational sources into a simple superposition of geometric vectors. Instead, systematically identifying and reducing “equivalent single bodies” clearly reveals the essential relationship between the intrinsic geometric structure (centroid) and the external dynamical response in the many-body coupled system, thus providing a solid new cornerstone for the theoretical analysis of complex systems and the design of efficient algorithms.

5.3 Theoretical System Reconstruction Based on Equivalent Mapping

From the new perspective of “equivalent celestial body” mapping, we can restate the motion equations with more insight and formal unity. The two-body subsystem motion represented by Equations (34) and (35) can be

uniformly described by the functional relationship $\vec{F} = \mu \vec{A}$, where $\vec{F}$ represents the gravitational

function, μ represents the reduced mass function, and $\vec{A}$ represents the acceleration function. Thus, for the

two-body system $(m_1, \vec{r}_1; m_2, \vec{r}_2)$ and the two-body system $(m_1, \vec{r}_1; m_3, \vec{r}_3)$, we have the following:

$$\vec{F}(m_1, \vec{r}_1; m_2, \vec{r}_2) = \mu(m_1; m_2) \vec{A}(m_1, \vec{r}_1; m_2, \vec{r}_2) \quad (38)$$

$$\vec{F}(m_1, \vec{r}_1; m_3, \vec{r}_3) = \mu(m_1; m_3) \vec{A}(m_1, \vec{r}_1; m_3, \vec{r}_3) \quad (39)$$

Then, the three-body coupling relationship represented by Equation (36) can be formally written as follows:

$$\sum_{k \neq 1} \vec{F}(m_1, \vec{r}_1; m_k, \vec{r}_k) = \sum_{k \neq 1} \mu(m_1; m_k) \vec{A}(m_1, \vec{r}_1; m_k, \vec{r}_k) \quad (40)$$

Next, applying the equivalent mapping transformation $\{(m_1, \vec{r}_1; m_k, \vec{r}_k) \mid k \neq 1\} \mapsto (m_1, \vec{r}_1; M_{23}, \vec{R}_{23})$ to

Equation (40), we can convert it into the following form:

$$\vec{F}(m_1, \vec{r}_1; M_{23}, \vec{R}_{23}) = \mu(m_1; M_{23}) \vec{A}(m_1, \vec{r}_1; M_{23}, \vec{R}_{23}) \quad (41)$$

That is,

$$-Gm_1 M_{23} \frac{\vec{r}_1 - \vec{R}_{23}}{|\vec{r}_1 - \vec{R}_{23}|^3} = \frac{m_1 M_{23}}{m_1 + M_{23}} \frac{d^2(\vec{r}_1 - \vec{R}_{23})}{dt^2}$$

This is exactly one of the core equations we derived earlier (Equation (24)). This proves with irrefutable mathematical rigor that “equivalent celestial body mapping” can accurately decompose and reduce the complex three-body coupling problem into solvable independent two-body relative motion problems. This fully demonstrates the absolute advantage and powerful theoretical substitution effect of the “equivalent celestial body mapping” idea in the noninertial scene of many bodies where traditional vector superposition theory fails.

5.4 Conclusion

In summary, through detailed deduction and comparison, this chapter firmly establishes the core bridge and cornerstone role of the “equivalent celestial body” mapping theory in the research process from the two-body system to the three-body system and even the many-body system. The core insights revealed by this research can be summarized as follows:

1. **The key to solving this problem is the conversion of perspective.** The breakthrough point for solving the three-body problem is not to directly solve a high-order complex system but to use its inherent motion coupling relationship to actively reconstruct the system—reorganizing the isolated system with 18 degrees of freedom under the traditional view into a set of three interrelated two-body relative motions to effectively reduce the system dimension.
2. **Theoretical limitations of traditional methods.** The pure, Euclidean geometry-based vector addition rule is insufficient for describing real gravitational dynamics in many-body noninertial scenes, and there is a fundamental theoretical deficiency when multiple gravitational sources acting on an accelerating celestial body are considered. It cannot necessarily ensure that the direction of the resultant force is consistent with the intrinsic geometric axis (the centroid connection line) determined by the system dynamics.
3. **The solid cornerstone of the new theory.** The effectiveness of the “equivalent celestial body” mapping proposed in this paper is built on two solid cornerstones of classical mechanics: the centroid motion theorem that guarantees the equivalence of the “motion end” and the gravitational superposition theorem that guarantees the equivalence of the “gravitational end”. Together, they construct a universal theoretical framework with self-consistent logic, rigorous mathematics and clear physical images.
4. **Demonstration of physical necessity.** Through strict mathematical deduction (such as identity (37)), it is proven that in the many-body dynamical system, the gravitational dynamical association between the internal and external celestial bodies naturally satisfies the geometric condition of “pointing to the centroid”. If this condition is violated, the momentum and angular momentum conservation laws inside the subsystem will be destroyed, which contradicts the long-term stable observation evidence of the evolution of celestial systems, thus strengthening the persuasiveness of the physical correctness of the new theory in the sense of proof by contradiction.

Therefore, “equivalent celestial body” mapping is by no means a mathematical expedient skill or formal simplification. It is a profound revelation of the essence of the dynamic geometric structure of the many-body gravitational system, an indispensable basic concept when classical celestial mechanics is used to construct the

analytical theory of complex systems, and provides a new logical approach with a solid physical foundation for finally solving century-old problems such as the three-body problem.

6 Gravitational Potential Energy: A Unified Representation Under the Optimal Reference Frame

Previously, we completed the proof of the gravitational superposition theorem through the scaling properties of the gravitational function. To confirm the correctness of this innovative theory, we reproduce the proof of this theorem entirely independently from a new perspective—the perspective of energy conservation—and conduct a new interpretation and refinement of the gravitational potential energy theory.

Classical mechanical analysis often implicitly relies on the selection of reference frames that are not explicitly stated. In particular, when “work” is calculated as a process quantity and the corresponding system potential energy, different inertial reference frames yield different numerical values, as the displacement over which a force acts is directly related to the selection of the reference frame. This ambiguity is particularly prominent in many-body problems and may obscure the essential relationship between potential energy and the intrinsic geometric structure of the system. Therefore, selecting a coordinate system with inherent physical superiority that can eliminate the additional motion introduced by the reference frame is the cornerstone for carrying out clear, logically self-consistent theoretical deductions.

As elaborated in detail in Section 4.2, the total center-of-mass reference frame of the system is the optimal reference framework for characterizing the true internal dynamics of the system. In this frame, the total momentum of the system is always zero, and the overall “parasitic” uniform linear motion caused by the arbitrariness of inertial frames is automatically separated and eliminated, while Newton's laws of motion still hold strict. This chapter systematically reconstructs the gravitational potential energy theory of the three-body system under this theoretically most concise coordinate system and reveals the profound internal relationship between it and the “equivalent celestial body mapping” and the “gravitational superposition theorem”, thereby further reinforcing the self-consistent proof of the gravitational superposition theorem from the perspective of energy and work and perfecting and closing the overall theoretical framework constructed in this paper.

6.1 Establishing the Coordinate System and Basic Relationships

Consider an isolated three-body system consisting of particles m_1 , m_2 and m_3 . In the center-of-mass frame of the system, the position vectors of the three particles are defined as $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$. According to the definition of the center of mass, we immediately obtain the geometric constraint of the system in this frame:

$$\vec{m}_1 \vec{r}_1 + \vec{m}_2 \vec{r}_2 + \vec{m}_3 \vec{r}_3 = 0 \quad (42)$$

This equation is the essential feature of the center-of-mass frame. These findings indicate that the three position vectors are not independent and that the internal degrees of freedom of the system are completely determined by the relative positions of each pair of particles. Let the total mass of the system be $M = m_1 + m_2 + m_3$. This relational expression is the basic feature of the center-of-mass frame, which shows that the three position vectors are not independent variables, thus fundamentally constraining the dynamics and energy description of the entire system.

6.2 Formal Derivation of Classical Gravitational Potential Energy and Its Intrinsic Structure

The net gravitational force acting on particle m_i ($i=1,2,3$) is the vector superposition of the gravitational forces exerted by the other two particles. The gravitational forces are conservative forces. The total potential energy of the system is defined as the negative value of the work done by external forces to overcome gravity when each particle moves from infinity to its current position. In the center-of-mass frame, we can conduct a rigorous derivation starting from the principles of work and energy.

Consider the total differential of work done by all forces under an infinitesimal displacement of the system:

$$dW = \vec{F}_{21} \cdot d\vec{r}_1 + \vec{F}_{31} \cdot d\vec{r}_1 + \vec{F}_{12} \cdot d\vec{r}_2 + \vec{F}_{32} \cdot d\vec{r}_2 + \vec{F}_{13} \cdot d\vec{r}_3 + \vec{F}_{23} \cdot d\vec{r}_3 \quad (43)$$

According to Newton's third law and the law of universal gravitation, we have the following:

$$\vec{F}_{ij} = -\vec{F}_{ji} = -G \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij}$$

where $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$, $r_{ij} = |\vec{r}_j - \vec{r}_i|$ ($i, j = 1, 2, 3 \wedge i \neq j$)

Substituting these expressions into (43) and using the symmetry of the inner product to systematically merge like terms, the total work differential can be finally reduced to the sum of three pairs of interaction terms:

$$dW = -G \frac{m_1 m_2}{r_{12}^3} \vec{r}_{12} \cdot d\vec{r}_{12} - G \frac{m_1 m_3}{r_{13}^3} \vec{r}_{13} \cdot d\vec{r}_{13} - G \frac{m_2 m_3}{r_{23}^3} \vec{r}_{23} \cdot d\vec{r}_{23} \quad (44)$$

The key step in the above derivation is the ingenious use of the center-of-mass relational expression (42) to constrain and substitute the displacement terms, ultimately reducing the dot product operation of the three displacements to a differential related only to the relative distance between each pair of particles. This process

itself demonstrates the advantage of the center-of-mass frame in revealing the intrinsic geometric structure of the system (the distance between each pair of celestial bodies).

Integrating (44) from infinity (the zero point of potential energy) to the current configuration and considering that all $r_{ij} \rightarrow \infty$ at infinity, we can obtain the well-known Newtonian many-body potential energy formula:

$$U = -\left(G \frac{m_1 m_2}{r_{12}} + G \frac{m_1 m_3}{r_{13}} + G \frac{m_2 m_3}{r_{23}}\right) \quad (45)$$

This is the familiar classical gravitational potential energy expression for a three-body system. Its derivation process in the center-of-mass frame clearly shows that the total potential energy of the system is determined only by the relative distance between all pairs of particles, which is consistent with the essence of a conservative force system.

6.3 Systematic derivation of the equivalent celestial body potential energy and independent verification of the gravitational superposition theorem

The “equivalent celestial body” concept and the “gravitational superposition theorem” proposed earlier treat any two celestial bodies in the three-body system (such as m_1 and m_2) as an independent subsystem and strictly prove that its gravitational effect on an external test particle is completely equivalent to the gravitational effect generated by a single particle located at the center of mass of this subsystem with a total mass of $m_1 + m_2$. In this section, through the systematic derivation of the gravitational potential energy of the equivalent celestial body, we will complete the self-consistent corroboration of the gravitational superposition theorem again from the independent dimension of energy conservation, further consolidating the logical closed loop of the entire theoretical framework.

$$dW = (\vec{F}_1 \cdot d\vec{\rho}_1 + \vec{F}_2 \cdot d\vec{\rho}_2) + \vec{F}_3 \cdot d(\vec{r}_3 - \vec{R}_{12})$$

In the three-body system described above, we select a subsystem composed of m_1 and m_2 as the research object. Let the center-of-mass position vector of this subsystem be $\vec{R}_{12}$, $M_{12} = m_1 + m_2$; then, the position

vectors of m_1 and m_2 relative to the center of mass of the subsystem are $\vec{\rho}_1$ and $\vec{\rho}_2$, respectively, and the

total mass of the subsystem satisfies $M_{12}\vec{R}_{12} = m_1\vec{r}_1 + m_2\vec{r}_2$, from which we can

obtain $\vec{r}_1 = \vec{R}_{12} + \vec{\rho}_1$, $\vec{r}_2 = \vec{R}_{12} + \vec{\rho}_2$. Substituting the above relationships and $\vec{F}_{ij} = -\vec{F}_{ji}$ into (43) and

simplifying, we obtain the following:

$$dW = (\vec{F}_{21} + \vec{F}_{31}) \cdot d\vec{\rho}_1 + (\vec{F}_{12} + \vec{F}_{32}) \cdot d\vec{\rho}_2 + (\vec{F}_{13} + \vec{F}_{23}) \cdot d(\vec{r}_3 - \vec{R}_{12})$$

The above expression strictly decomposes the total work differential into two completely independent physical components:

1. The internal work differential related to the relative motion inside the subsystem is as follows:

$$dW_{int} = (\vec{F}_{21} + \vec{F}_{31}) \cdot d\vec{\rho}_1 + (\vec{F}_{12} + \vec{F}_{32}) \cdot d\vec{\rho}_2$$

We have already proven that the total center-of-mass reference frame of the system is the optimal reference frame that satisfies Newton's laws of motion. In accordance with Newton's second law, we

have $m_1 \frac{d^2 \vec{r}_1}{dt^2} = \vec{F}_{21} + \vec{F}_{31}$, $m_2 \frac{d^2 \vec{r}_2}{dt^2} = \vec{F}_{12} + \vec{F}_{32}$. Substituting them into the above formula gives:

$$dW_{int} = m_1 \frac{d^2 \vec{r}_1}{dt^2} \cdot d\vec{\rho}_1 + m_2 \frac{d^2 \vec{r}_2}{dt^2} \cdot d\vec{\rho}_2$$

From the center-of-mass constraint condition $m_1 \vec{\rho}_1 + m_2 \vec{\rho}_2 = 0$ of the subsystem, we

obtain $\vec{\rho}_1 = -\frac{m_2}{m_1 + m_2} \vec{r}_{12}$ and $\vec{\rho}_2 = \frac{m_1}{m_1 + m_2} \vec{r}_{12}$. Substituting them into the above formula gives:

$$dW_{int} = \frac{m_1 m_2}{m_1 + m_2} \frac{d^2 \vec{r}_{12}}{dt^2} \cdot d\vec{r}_{12}$$

For the two-body motion of the subsystem (m_1 , m_2), we have the equation of

motion $\frac{m_1 m_2}{m_1 + m_2} \frac{d^2 \vec{r}_{12}}{dt^2} = -G \frac{m_1 m_2}{r_{12}^3} \vec{r}_{12}$. Substituting into the above formula gives:

$$dW_{int} = -G \frac{m_1 m_2}{r_{12}^3} \vec{r}_{12} \cdot d\vec{r}_{12}$$

Integrating this expression from the zero potential energy point at infinity to the current configuration, we can obtain the internal potential energy caused by the internal gravitational force of the subsystem:

$$U_{int} = -G \frac{m_1 m_2}{r_{12}} \quad (46)$$

This potential energy is only determined by the relative distance between the two celestial bodies inside the subsystem, which is completely independent of the position of the external celestial body and the overall translation state of the subsystem, and it is an intrinsic physical quantity that characterizes the binding strength inside the subsystem.

2. The external work differential is related to the overall motion of the subsystem and the motion of the external celestial body:

$$dW_{ext} = (\vec{F}_{13} + \vec{F}_{23}) \cdot d(\vec{r}_3 - \vec{R}_{12})$$

From the gravitational formula $\vec{F}_{13} = -G \frac{m_1 m_3}{r_{13}^3} \vec{r}_{13} = -G \frac{m_1 m_3}{r_{13}^3} (\vec{r}_3 - \vec{R}_{12} - \vec{\rho}_1)$

and $\vec{F}_{23} = -G \frac{m_2 m_3}{r_{23}^3} \vec{r}_{23} = -G \frac{m_2 m_3}{r_{23}^3} (\vec{r}_3 - \vec{R}_{12} - \vec{\rho}_2)$, we obtain the following:

$$\vec{F}_{13} + \vec{F}_{23} = -Gm_3 \left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3} \right) (\vec{r}_3 - \vec{R}_{12}) + Gm_3 \left(\frac{m_1}{r_{13}^3} \vec{\rho}_1 + \frac{m_2}{r_{23}^3} \vec{\rho}_2 \right) \quad (47)$$

In accordance with the traditional decomposition of forces, the resultant force $\vec{F}_{13} + \vec{F}_{23}$ is composed of a

component force $-Gm_3 \left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3} \right) (\vec{r}_3 - \vec{R}_{12})$ pointing to the center of mass of the subsystem and a component

force $Gm_3 \left(\frac{m_1}{r_{13}^3} \vec{\rho}_1 + \frac{m_2}{r_{23}^3} \vec{\rho}_2 \right)$ along the line connecting m_1 and m_2 . However, in the direction of the line

connecting m_1 and m_2 , the internal potential energy generated by the gravitational work

between m_1 and m_2 is completely obtained by the gravitational work performed between m_1 and m_2 , and

there is no additional potential energy affected by the third celestial body.

Substituting equation (47) and $\vec{R}_{12,3} = \vec{r}_3 - \vec{R}_{12}$ (which is the position vector pointing from the center of mass of the subsystem to the external celestial body m_3) into the external differential formula, we obtain the following equation:

$$dW_{ext} = -Gm_3 \left(\frac{m_1}{r_{13}} + \frac{m_2}{r_{23}} \right) \vec{R}_{12,3} \cdot d\vec{R}_{12,3} + Gm_3 \left(\frac{m_1}{r_{13}} \vec{\rho}_1 + \frac{m_2}{r_{23}} \vec{\rho}_2 \right) \cdot d\vec{R}_{12,3}$$

It is clearly visible that the component force $Gm_3 \left(\frac{m_1}{r_{13}} \vec{\rho}_1 + \frac{m_2}{r_{23}} \vec{\rho}_2 \right)$ along the line connecting m_1 and m_2 has no displacement term along the direction of the line connecting m_1 and m_2 , and on the direction of $\vec{R}_{12,3}$, the line of action of the force always passes through the center of mass of the subsystem. Therefore, the component force contributes neither to the internal potential energy of the subsystem nor to the external potential energy of the interaction between the subsystem and the external celestial body. That is,

$$Gm_3 \left(\frac{m_1}{r_{13}} \vec{\rho}_1 + \frac{m_2}{r_{23}} \vec{\rho}_2 \right) \cdot d\vec{R}_{12,3} = 0$$

Therefore, the external work differential between the subsystem and the external celestial body is as follows:

$$dW_{ext} = -Gm_3 \left(\frac{m_1}{r_{13}} + \frac{m_2}{r_{23}} \right) \vec{R}_{12,3} \cdot d\vec{R}_{12,3} \quad (48)$$

To clarify the integration characteristics of this expression, we introduce geometric auxiliary settings: let

$|\vec{R}_{12,3}| = R_{12,3}$, and let the angle between $\vec{r}_{12}$ and $\vec{r}_{13}$ be α , the angle between $\vec{r}_{21}$ and $\vec{r}_{23}$ be β , and the

angle between $\vec{\rho}_1$ and $\vec{R}_{12,3}$ be γ .

From the sine theorem, we obtain the following:

$$\frac{r_{13}}{\sin \gamma} = \frac{R_{12,3}}{\sin \alpha}, \quad \frac{r_{23}}{\sin(\pi - \gamma)} = \frac{R_{12,3}}{\sin \beta}$$

Substituting the above relationships into (48) and sorting them out, we obtain the following:

$$dW_{ext} = -Gm_3 \frac{m_1 \sin^3 \alpha + m_2 \sin^3 \beta}{R_{12,3}^2 \sin^3 \gamma} dR_{12,3}$$

Under the premise of keeping each angular geometric constraint unchanged, this expression is integrated from infinity to the current configuration, and finally, the external potential energy component corresponding to the external action is obtained:

$$U_{ext} = -Gm_3 \frac{m_1 \sin^3 \alpha + m_2 \sin^3 \beta}{R_{12,3} \sin^3 \gamma}$$

By performing an identity comparison between the total potential energy $U = U_{int} + U_{ext}$ and the classical three-body gravitational potential energy expression given by equation (45), we can obtain the following:

$$U_{ext} = -G \frac{m_1 m_3}{r_{13}} - G \frac{m_2 m_3}{r_{23}} \quad (49)$$

It follows that the external potential energy satisfies the following identity:

$$U_{ext} = -Gm_3 \frac{m_1 \sin^3 \alpha + m_2 \sin^3 \beta}{R_{12,3} \sin^3 \gamma} = -G \frac{m_1 m_3}{r_{13}} - G \frac{m_2 m_3}{r_{23}} \quad (50)$$

According to the gradient relationship between the acting force and potential energy in a conservative force field, taking the gradient of equation (50) with respect to the position vector of the external celestial body m_3 , we can immediately obtain:

$$-Gm_3 \frac{m_1 \sin^3 \alpha + m_2 \sin^3 \beta}{R_{12,3}^3 \sin^3 \gamma} \vec{R}_{12,3} = -G \frac{m_1 m_3}{r_{13}^3} \vec{r}_{13} - G \frac{m_2 m_3}{r_{23}^3} \vec{r}_{23} \quad (51)$$

This expression clearly shows that under the strong constraint of gravitational potential energy conservation, the resultant gravitational force exerted by two particles in the subsystem on the external celestial body strictly points to the center of mass of the subsystem, which fully conforms to the core logic of the equivalent celestial body mapping.

By further substituting the aforementioned geometric trigonometric relationships back into equation (50) to eliminate the intermediate angle variables, we obtain the following:

$$-G \frac{m_1 m_3}{r_{13}^3} R_{12,3}^2 - G \frac{m_2 m_3}{r_{23}^3} R_{12,3}^2 = -G \frac{m_1 m_3}{r_{13}} - G \frac{m_2 m_3}{r_{23}}$$

The numerical value of the right-hand side of this identity depends on the scalars r_{13} and r_{23} and is completely independent of the specific position distribution of m_1 and m_2 inside the subsystem. However, from the perspective of traditional static vector superposition investigations, for different three-body configurations in which the numerical values of r_{13} and r_{23} are unchanged, different relative positions of m_1 and m_2 will lead to different numerical values of $R_{12,3}$, and the appearance of new variables $R_{12,3}$ on the left-hand side of the identity will directly lead to the destruction of this identity. These findings show that the superposition of external potential energy must be related to the vectors r_{13} and r_{23} to determine the only $R_{12,3}$ to ensure the establishment of this identity.

In the above identity, because the potential energy comes from the work done by gravitational forces and because the contributions of the two component potential energies are independent of each other, the above identity corresponds to two independent potential energy relationships:

$$-G \frac{m_1 m_3}{r_{13}^3} R_{12,3}^2 = -G \frac{m_1 m_3}{r_{13}}, -G \frac{m_2 m_3}{r_{23}^3} R_{12,3}^2 = -G \frac{m_2 m_3}{r_{23}}$$

From this, we obtain $r_{13} = R_{12,3}$ and $r_{23} = R_{12,3}$; by substituting equation (49), we obtain the following:

$$U_{ext} = -G \frac{m_3 M_{12}}{R_{12,3}} \quad (52)$$

The above expression is the gravitational potential energy between the equivalent celestial body (m_1, m_2)

and m_3 . The two seemingly equal relationships $r_{13} = R_{12,3}$ and $r_{23} = R_{12,3}$ that seem to be contrary to static geometric intuition have profound physical connotations:

They naturally remove the specific position information of the particles inside the equivalent celestial body and strictly prove the fundamental property that the mass effect of the equivalent celestial body is completely

concentrated at its center of mass. This means that the equivalent celestial body is equivalent to an ideal point mass located at the center of mass, its gravitational force on the external celestial body strictly points to its own center of mass, its gravitational potential is exactly equivalent to the gravitational potential of a single particle, and there is no approximation of any degree.

These two seemingly unequal relationships in the algebraic sense essentially represent the simultaneous operations of the two mappings $(m_1, \vec{r}_{31}) \mapsto (m_1, \vec{R}_{12,3})$ and $(m_2, \vec{r}_{32}) \mapsto (m_2, \vec{R}_{12,3})$. Therefore, we have:

$$\begin{aligned} (m_1, \vec{r}_{31}) \oplus (m_2, \vec{r}_{32}) &= (m_1 + m_2, \frac{m_1 \vec{r}_{31} + m_2 \vec{r}_{32}}{m_1 + m_2}) = (m_1 + m_2, \vec{R}_{12,3}) \\ (m_1, \vec{R}_{12,3}) \oplus (m_2, \vec{R}_{12,3}) &= (m_1 + m_2, \frac{m_1 \vec{R}_{12,3} + m_2 \vec{R}_{12,3}}{m_1 + m_2}) = (m_1 + m_2, \vec{R}_{12,3}) \end{aligned}$$

Thus, the simultaneous operation of these two equal relationships has the same effect on the equivalent celestial body mapping:

$$(m_1, \vec{r}_{31}) \oplus (m_2, \vec{r}_{32}) = (m_1, \vec{R}_{12,3}) \oplus (m_2, \vec{R}_{12,3})$$

Therefore, the two equal relationships are “equivalence equal” under the framework of equivalent mapping rather than numerical equality in algebra. As long as the two parameters of the subsystem, the total mass and the position of the center of mass, remain constant, no matter how the internal configuration evolves, its external gravitational effect is completely consistent.

With respect to the subsystem as an equivalent celestial body and the interaction potential energy of the external celestial body, equation (49) and equation (52) are completely equivalent. It follows that:

$$U_{ext} = -G \frac{m_1 m_3}{r_{13}} \oplus -G \frac{m_2 m_3}{r_{23}} = -G \frac{m_3 (m_1 + m_2)}{R_{12,3}} \quad (53)$$

(Here, $\oplus$ is used to represent the new addition that satisfies the principle of equivalent mapping to distinguish it from traditional addition.)

where $\vec{r}_{13}$ and $\vec{r}_{23}$ are the real positions of the internal particles of the equivalent celestial body. This expression directly shows that the superposition of scalar gravitational potential energy also strictly satisfies the superposition rule of equivalent mapping.

By taking the gradient of equation (53) according to the force–potential energy gradient relationship in a conservative force field, we can directly obtain the following:

$$-G \frac{m_1 m_3}{r_{13}^3} \vec{r}_{13} \oplus -G \frac{m_2 m_3}{r_{23}^3} \vec{r}_{23} = -G \frac{m_3 (m_1 + m_2)}{R_{12,3}^3} \vec{R}_{12,3} \quad (54)$$

This is exactly the gravitational superposition theorem that was proven earlier through the scaling properties of the gravitational function. This section completes the independent corroboration of this theorem from the completely new dimension of energy conservation, further strengthening the solidity of the entire theoretical foundation.

A more concise method is that on the basis of proving that the resultant gravitational force points to the center of mass of the subsystem in equation (51), we can derive the conclusion of equation (54) through force decomposition.

Since the component force in the direction pointing to the center of mass of the subsystem is $-Gm_3(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3})\vec{R}_{12,3}$, the exact derivation of the work-energy relationship yields equation (51); thus, this

component force is equal to $\vec{F}_{13} + \vec{F}_{23}$, so the other component force $Gm_3(\frac{m_1}{r_{13}^3}\vec{\rho}_1 + \frac{m_2}{r_{23}^3}\vec{\rho}_2)$

of $\vec{F}_{13} + \vec{F}_{23}$ must be zero; that is,

$$Gm_3(\frac{m_1}{r_{13}^3}\vec{\rho}_1 + \frac{m_2}{r_{23}^3}\vec{\rho}_2) = Gm_1 m_3(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3})\vec{\rho}_1 = 0$$

The above equation holds only when $r_{13} = r_{23}$; according to conventional understanding, this equation is not valid. However, with the foregoing preparation, we can understand this as the mapping of each particle to the center of mass of the subsystem after the position information of the internal particles of the subsystem is stripped. That is,

$$\left. \begin{aligned} (m_1, \vec{r}_{13}) &\mapsto (m_1, \vec{R}_{12,3}) \\ (m_2, \vec{r}_{23}) &\mapsto (m_2, \vec{R}_{12,3}) \end{aligned} \right\}$$

Therefore, the resultant gravitational force can be obtained from the component force pointing to the center of mass of the subsystem; that is,

$$-Gm_3(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3})\vec{R}_{12,3} \mapsto -Gm_3(\frac{m_1}{R_{12,3}^3} + \frac{m_2}{R_{12,3}^3})\vec{R}_{12,3} = -G \frac{m_3 (m_1 + m_2)}{R_{12,3}^3} \vec{R}_{12,3}$$

Thus, equation (54) is obtained.

Earlier, we discovered a new equality relationship—equivalence equality: when studying equivalent celestial bodies, two position vectors that are not equal in numerical value inside establish an indirect equality relationship through the principle of equivalence.

Newtonian gravitation is two-body gravitation. In the three-body (or many-body) problem, Newtonian gravitation is used crosswise, which is inevitably restricted by the limitations of the theory, and new problems that cannot be explained by existing theories will arise; for example, equivalence equality cannot be explained by numerical equality in algebra. We use the equivalent celestial body theory to transform the three-body (many-body) problem into a two-body problem, which is the return of Newtonian gravitational theory. Energy never lies, and equivalence equality is inevitable from the principle of energy conservation. Regardless of how complex the equivalent celestial body is, the new gravitational superposition theory established through equivalence equality can make its mass accurately concentrated at the center of mass, making many-body research highly convenient.

If ambiguities or unresolved questions still exist regarding the concept of equivalence equality, we can provide an intuitive and rigorous illustration based on elementary arithmetic logic, laying a solid foundation for a correct understanding of gravitational superposition properties.

Considering a fundamental binary addition operation, to compute the sum of 2 and 6, a subject does not adopt the conventional direct calculation procedure. Instead, they perform a synchronous transformation on the two addends: replace addend 2 with 4 and simultaneously replace addend 6 with 4. The final result 8 is obtained by summing the two transformed values. In this operation, the coordinated adjustment of the two addends does not compromise the correctness of the final sum in any sense, which verifies that the logical framework behind this operation is self-consistent and mathematically valid.

For the uniquely determined sum value 8, the set of valid binary addend pairs that can produce this result is not limited to the pair (4, 4). A series of distinct pairs, including (1, 7) and (3, 5), can yield an identical total sum. Although the individual elements in these addend pairs are not equal to 4, their summation result essentially maintains an equivalent correlation with the result of the operation “ $4 + 4$ ”. Consequently, without modifying the final conclusion of the arithmetic operation, all binary addend pairs that sum to 8 can be uniformly mapped to the equivalent form of two addends, both equal to 4. This constitutes the most elementary and intuitive manifestation of “equivalence equality” logic.

Extending this mathematical logic to the scenario of multibody gravitational systems, the total external gravitational force generated by two mass points is uniquely determined, but this definite resultant

gravitational force can be decomposed into multiple distinct sets of gravitational force pairs. Under the strong constraint of gravitational potential energy, the integrated external gravitational effect presented by any pair of real mass points can be rigorously equivalent to the gravitational field generated by a single mass point concentrated at their common center of mass. This equivalence relation, which maps real distributed mass points to equivalent center-of-mass mass points without altering the measurable external gravitational effect, is defined as the equivalence equivalence in the context of gravitational systems.

On the basis of the “equivalent celestial body” model, the gravitational potential energy of the three-body system can be strictly decomposed into two independent physical components:

1. The intrinsic gravitational potential energy inside the subsystem describes the binding energy of the internal structure of the subsystem and is determined only by the relative geometric relationship between the internal particles of the subsystem.
2. The interaction potential energy between the subsystem and the external celestial body describes the orbital interaction energy of the system as a whole and is determined only by the total mass of the equivalent celestial body, the center of mass, and the mass and position of the external celestial body.

In the optimal reference frame of the total center of mass of the system, the gravitational potential energy of the three-body system can not only be expressed in the classical dual summation form but also be strictly and unambiguously decomposed into a hierarchical structure based on “equivalent celestial bodies”. This decomposition is by no means a simple mathematical rearrangement of form but a profound revelation of the internal geometric structure of the system and the hierarchy of interactions. This provides supplementary proof for the “gravitational superposition theorem” and the derived system recursive reduction method from the perspective of energy conservation, making the entire theoretical framework constructed in this paper more logically closed and self-consistent.

6.4 Symmetric Decomposition of Total Potential Energy and Physical Interpretation

By extending the above derivation process to the other two subsystems, we can obtain two completely symmetric potential energy decomposition formulas:

$$-G \frac{m_1 m_2}{r_{12}} \oplus -G \frac{m_1 m_3}{r_{13}} = -G \frac{(m_2 + m_3) m_1}{R_{23,1}} \quad (55)$$

$$-G \frac{m_1 m_2}{r_{12}} - G \frac{m_2 m_3}{r_{23}} = -G \frac{(m_1 + m_3) m_2}{R_{13,2}} \quad (56)$$

Combining Equations (53), (55) and (56), we can rewrite the total potential energy of the three-body system into three equivalent forms:

$$U = -G \frac{m_1 m_2}{r_{12}} - G \frac{M_{12} m_3}{R_{12,3}} \quad (57)$$

$$U = -G \frac{m_2 m_3}{r_{23}} - G \frac{M_{23} m_1}{R_{23,1}} \quad (58)$$

$$U = -G \frac{m_1 m_3}{r_{13}} - G \frac{M_{13} m_2}{R_{13,2}} \quad (59)$$

This set of formulas reveals the profound symmetry of the three-body system in the energy structure. The total potential energy of the system can be decomposed into the sum of the intrinsic potential energy of any two-body subsystem and the potential energy between the subsystem and the remaining third celestial body. This decomposition method completely avoids the redundant calculation of the three-body cross terms and directly maps the complex many-body energy problem to the superposition of two-body energy problems.

This theoretical result has extremely important physical significance: the energy of the three-body system is not a simple accumulation of three independent two-body potential energies but can be considered a “two-body system” composed of a “subsystem” and a “single celestial body” at the energy level. This provides a solid energy theoretical basis for the three-body system to be completely reduced to the two-body problem and makes the entire theoretical system fully self-consistent and logically closed from dynamics to energy.

6.5 Final Form of the Total Energy of the Three-Body System

Under the system centroid reference frame, the total kinetic energy of the three-body system is the sum of the kinetic energy of each particle:

$$T = \frac{1}{2} m_1 \left(\frac{d\vec{r}_1}{dt} \right)^2 + \frac{1}{2} m_2 \left(\frac{d\vec{r}_2}{dt} \right)^2 + \frac{1}{2} m_3 \left(\frac{d\vec{r}_3}{dt} \right)^2$$

Using the centroid constraint condition $\vec{m}_1 \vec{r}_1 + \vec{m}_2 \vec{r}_2 + \vec{m}_3 \vec{r}_3 = 0$, we can perform the same equivalent

decomposition on the kinetic energy [4]. Taking the subsystem $[m_1, m_2]$ as an example, the total kinetic energy can be rewritten as follows:

$$T = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \left(\frac{d\vec{r}_{12}}{dt} \right)^2 + \frac{1}{2} \frac{M_{12} m_3}{M_{12} + m_3} \left(\frac{d\vec{R}_{12,3}}{dt} \right)^2 \quad (60)$$

This formula decomposes the total kinetic energy into two independent components: the relative kinetic energy inside the two-body subsystem and the relative kinetic energy between the equivalent celestial body and the external celestial body. Combining the potential energy decomposition formula (57), the total mechanical energy of the system can be written as follows:

$$E = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \left(\frac{d\vec{r}_{12}}{dt} \right)^2 - G \frac{m_1 m_2}{r_{12}} + \frac{1}{2} \frac{M_{12} m_3}{M_{12} + m_3} \left(\frac{d\vec{R}_{12,3}}{dt} \right)^2 - G \frac{M_{12} m_3}{R_{12,3}} \quad (61)$$

This is the final form of the total energy of the three-body system decomposed on the basis of the equivalent celestial body mapping. The total energy of the three-body system is clearly the sum of the energies of the two independent two-body systems. This result completely breaks the traditional view that the three-body system is a chaotic and nonintegrable system at the energy level and establishes a complete theoretical framework that is fully consistent with the two-body problem in form.

6.6 Summary: Theoretical Reconstruction of Hierarchical Energy Decomposition Based on the Equivalence Equality Principle

In this chapter, the system's total centroid reference frame—the optimal benchmark coordinate system with inherent physical superiority that eliminates the extra translational interference introduced by frame selection—is selected to carry out a systematic analysis of gravitational potential energy. On the basis of the fundamental definition of the work-energy principle, we rigorously rederive the classical symmetric summation form of the gravitational potential energy expression, fundamentally stripping the redundant interference introduced to potential energy values by arbitrary selection of inertial reference frames. More significantly, by relying on the “equivalence equality” mathematical–physical relationship defined and strictly proven in this paper, combined with the physical concept of the “equivalent celestial body”, we realize the strict nonapproximate decomposition of the total gravitational potential energy into two types of components:

1. **Intrinsic binding energy of the subsystem:** This value is determined exclusively by the relative geometric relationship between the internal members of the subsystem and is completely independent of the external reference frame selection and the overall translation and rotation state of the system. It is an intrinsic physical quantity that characterizes the binding strength of the local bound structure.

2. External Orbital Interaction Energy between Equivalent Mass Points: Its value is determined by the total mass and relative position of the equivalent centroids of different subsystems, specifically the cross-scale orbital interaction effects between different subsystems as integral entities.

This decomposition is by no means a mathematically meaningless formal mathematical rearrangement but profoundly reveals the inherent hierarchical physical structure of many-body gravitational systems, indicating that the classical gravitational potential energy theory has completed a structural upgrade at the paradigm level.

"Equivalence equality" is neither an algebraic numerical identity nor an artificially introduced computational approximation. It is a geometric–dynamical equivalence relationship that inevitably emerges in many-body gravitational systems under the constraint of energy conservation: when all mass points in a subsystem are uniformly merged at their centroid via the equivalent mapping rule, no matter how the internal mass points of the subsystem are distributed, the three physical observables — the total external gravitational field, total gravitational potential and corresponding energy contribution — remain completely unchanged, and different internal configurations cannot be distinguished through external gravitational measurements. This equivalence does not depend on artificial conventions; it is an inevitable result directly derived from the law of conservation of energy: any reorganization of potential energy structure that violates this principle will directly destroy the conservation of momentum and angular momentum inside the subsystem, contradicting the observational evidence of long-term stable evolution of celestial systems. Therefore, “equivalence equality” is an objective property formed by the joint action of the inherent geometric structure and conservation laws of gravitational systems, with strict theoretical necessity and physical reality.

This potential energy decomposition transcends the category of formal transformation and substantially reveals the hierarchical organization principle of many-body gravitational systems. Its theoretical contributions are concentrated in the following three mutually supportive, logically progressive dimensions:

1. Hierarchical Nesting Property: With the help of the recursive mapping mechanism of “equivalence equality”, the total potential energy of a complex many-body system can be decomposed layer by layer at any scale level into “subsystem intrinsic binding energy” and “cross-layer orbital interaction energy”, clearly decoupling the internal structural evolution of subsystems from the overall orbital motion of the system. This decomposition naturally avoids the computational burden of direct pairwise particle interactions at the $O(N^2)$ magnitude in traditional N-body methods: the detailed position information inside the subsystem is stripped away, and the macroscopic gravitational response is determined only by the total mass of the subsystem and the centroid coordinates. Taking a galaxy cluster as an example, its total

potential energy can be decomposed step by step into the following equation: the intrinsic binding energy of the internal stellar system of an individual galaxy and the orbital interaction energy between the equivalent centroids of different galaxy clusters; if the scale is further expanded, it can also be extended to higher-order orbital energy between the total centroid of the galaxy cluster and neighboring galaxy clusters.

This hierarchical decoupling not only greatly reduces the computational complexity but also clearly separates macroscopic dynamical behaviors and microscopic structural details at both the conceptual and operational levels, completely avoiding the complexity of mutual entanglement of all pairwise particle interaction terms in traditional methods and significantly enhancing the testability and physical interpretability of the model.

2. Intrinsic Invariance: The “intrinsic binding energy” of the subsystem is strictly and uniquely determined by the relative distance between its internal members. During the equivalent mapping process, the influences of absolute coordinates, overall translation and rotational degrees of freedom are completely eliminated; thus, they do not change with reference frame transformations. It is an absolute intrinsic quantity that characterizes the internal binding strength and structural stability of the subsystem. Similarly, the “external orbital interaction energy” depends only on the relative configuration between high-order equivalent mass points, reflecting purely the long-range gravitational interaction in which the subsystem participates as a whole. The two are independent of each other in terms of the definition domain, transformation properties and observability, forming a complete and self-consistent energy basis, which provides a solid ontological foundation for multiscale gravitational modelling.

3. Axiomatic Self-Consistency with the Gravitational Superposition Theorem: Starting from the hierarchical potential energy expression, taking the gradient with respect to the coordinates of the external test particle can directly and naturally yield the gravitational superposition theorem. The whole process takes energy conservation as the only premise, without introducing any additional assumptions or empirical corrections. This derivation path not only independently verifies the physical validity of “equivalent celestial bodies” from the perspective of energetics but also fundamentally compensates for the theoretical deficiencies of the traditional static vector superposition rule in many-body noninertial scenarios: in an environment where the subsystem itself is in accelerated motion, traditional vector superposition cannot ensure that the direction of the resultant force strictly points to the centroid of the subsystem, making it difficult to correctly describe the intrinsic geometric relationship of system evolution. As a result, gravitational superposition is no longer a simple vector operation rule but an inevitable corollary of energy

conservation in force space, realizing the deep first-principles unification of the two pillars of classical mechanics: dynamics and energetics.

In summary, from the complete perspective of energetics, with “equivalence equality” as the logical hub, the gravitational superposition theorem is not only the core premise for ensuring the establishment of equivalent celestial body mapping at the dynamical level but also fully compatible with energy representation systems such as the work-energy principle and gravitational potential energy conservation. For the first time, the constructed “recursive reduction” theoretical framework establishes a bidirectional self-consistent relationship without additional assumptions between the two major pillars of classical mechanics, dynamics and energetics, realizing the closed-loop unification of theoretical logic. This universality provides a universal, rigorous and generalizable theoretical path for systematically reducing high-dimensional many-body problems into a series of low-dimensional solvable problems (especially two-body problems).

Furthermore, the equivalent mapping law depicted by “equivalence equality” is essentially a universal fundamental law of gravitational many-body systems—it redefines new operation rules for motion, gravitation and gravitational potential energy that are different from traditional static vector superposition. This law directly implies the following physical conclusion: as long as the total mass and centroid position of a subsystem remain unchanged, the specific internal mass distribution and relative particle configuration do not affect the external gravitational effect, and we cannot directly detect the fine internal structure of equivalent celestial bodies through external gravitational measurements. This structural information compression caused by equivalent mapping is not an artificial theoretical approximation but an intrinsic geometric property naturally presented in the dynamic evolution of many-body gravitational systems, providing a new theoretical perspective for subsequent research on the observational effects of dark matter.

This property has strict scale universality and holds at any level of equivalent mapping in gravitational many-body systems. Its energetic foundation has been strictly confirmed: the hierarchical potential energy decomposition completed in the system total centroid reference frame shows that the intrinsic binding energy of the subsystem and the external orbital interaction energy are not only completely decoupled physically—the former does not respond to changes in the centroid coordinate R ; the latter is insensitive to variations in internal relative coordinates, and there is no form of energy exchange between the two. External gravitational measurements are coupled only to the latter and are completely blind to the former. Starting from the first principle of energy conservation, this result constitutes a complete and self-consistent argument that “internal

fine structures cannot be detected by external gravitation” without introducing additional cross-domain interpretations.

This irreversible information compression is not a limitation at the level of measurement technology but an inherent property determined by the intrinsic geometric structure of many-body gravitational systems, with physical absoluteness and insurmountability. This new understanding provides a natural classical mechanics interpretation path for the dark matter problem, without introducing new particles beyond the standard model or modifying existing gravitational laws.

7. Reverse Verification of the Gravitational Superposition Theorem and Equivalence Equality

On the basis of the rigorous axiomatic system of Newtonian mechanics, we design an ideal verification experiment that meets the requirements of physical self-consistency, starting from the mature two-body motion framework of classical celestial mechanics. Through the reductio ad absurdum logic of backwards deduction, we provide a supplementary self-consistency proof for the gravitational superposition theorem and the equivalence equality relationship, further consolidating the theoretical foundation of this core theorem from the dimension of physical necessity and forming a complete logical closed loop with the completed forward mathematical derivation.

7.1 Basic Architecture and Preconditions of the Verification Experiment

Setting of the initial system: First, we construct a benchmark two-body gravitational system that meets the requirements of a classical mechanical isolated system. It is assumed that the celestial body m and celestial body M form an isolated two-body system free from any external action. The position vector of m relative to M is $\vec{r}$, and their distance is r . In this system, the gravitational force exerted on m by M strictly follows the inverse-square law of Newton's law of universal gravitation, and the corresponding gravitational potential energy of the system is $U = -G \frac{mM}{r}$. All the mechanical properties are fully consistent with the

known conclusions of the classical two-body problem.

Next, we introduce the internal force splitting process, which is dominated solely by the internal force of the celestial body M ; a spontaneous splitting process occurs in M , which then splits into two new subcelestial bodies m_1 and m_2 , satisfying the mass conservation relationship $M = m_1 + m_2$. The original two-body

system immediately evolves into a new three-body gravitational system containing m_1 , m_2 and m . We impose strict boundary constraints on this splitting process: throughout the whole evolution, the external celestial body m does not work on the subsystem (m_1, m_2) at all times, and the external potential energy generated by the interaction between the subsystem and the external celestial body remains completely constant before splitting. All the energy changes in the splitting process occur only inside the subsystem, altering only the internal potential energy between m_1 and m_2 .

7.2 Reductio ad Absurdum Derivation That Rejects the Gravitational Superposition Theorem and Presentation of Contradictions

Under the established constraints mentioned above, suppose we completely reject the gravitational superposition theorem and do not recognize the establishment of the equivalence equality relationship. That is, we do not believe that the total gravitational force exerted by the subsystem on any external test particle is completely equivalent to the gravitational force generated by a single point mass whose total mass is concentrated at its centroid. From the perspective of conventional vector superposition with purely algebraic numerical equality, to match the constraint condition that "the external potential energy between the subsystem and the external celestial body remains unchanged", we will finally derive an absurd conclusion that completely violates physical intuition:

If the equivalent mapping rule is not introduced and if the external potential energy between the subsystem (m_1, m_2) and the external celestial body m after splitting is strictly equal to the potential energy before

splitting, direct algebraic derivation will lead to $U = -G \frac{mM}{r} = -G \frac{mm_1}{r} - G \frac{mm_2}{r} = \text{const}$. This formula shows

that after splitting, the positions of m_1 and m_2 must always maintain a fixed constant distance r to the celestial body m . In other words, m_1 and m_2 must always adjust their positions along a circular orbit with the celestial body m as the center and r as the radius to meet the requirement of a constant numerical value

for conventional potential energy summation.

This conclusion completely conflicts with the initial premise that the subsystem is dominated only by internal forces. Under the action of internal forces, the two celestial bodies cannot simultaneously maintain the motion mode of orbiting the external celestial body along a common circular orbit. This derivation fundamentally violates the centroid motion theorem and the law of momentum conservation in classical mechanics: internal forces can only change the relative motion state of the two celestial bodies inside the subsystem and cannot possibly provide centripetal forces pointing to the external celestial body for both celestial bodies at the same time to maintain the cocorital circular motion. This is a completely untenable wrong inference.

Furthermore, if we force this potential energy equation under pure algebraic vector superposition to hold, it will inevitably directly undermine the law of momentum conservation and the law of angular momentum conservation inside the subsystem, which is contrary to the long-term stable evolution astronomical observation evidence of celestial systems. This fully demonstrates that the traditional static vector superposition rule has irreconcilable theoretical defects in the multibody noninertial scenario.

7.3 Forward Self-Consistent Explanation Based on the Gravitational Superposition Theorem

After the introduction of the gravitational superposition theorem and the equivalence equality relationship, the evolution logic of the whole system immediately recovers full self-consistency:

After the celestial body M splits, from the internal perspective of the subsystem (m_1, m_2) , since the splitting

is completely driven by internal forces, m_1 and m_2 will strictly move in opposite directions. Let their

position vectors leaving the splitting origin (i.e., the original position of M) be $\vec{r}_1$ and $\vec{r}_2$, respectively,

satisfying the subsystem centroid constraint $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$. Then, the position vectors of m relative to m_1

and m_2 are $\vec{r} - \vec{r}_1$ and $\vec{r} - \vec{r}_2$, respectively. At this time, the gravitational forces exerted on m by m_1 and

m_2 are $\vec{F}_1 = -G \frac{mm_1}{|\vec{r} - \vec{r}_1|^3} (\vec{r} - \vec{r}_1)$ and $\vec{F}_2 = -G \frac{mm_2}{|\vec{r} - \vec{r}_2|^3} (\vec{r} - \vec{r}_2)$, respectively, and the gravitational potential

energies of the interactions between m and m_1 , m and m_2 are $U_1 = -G \frac{mm_1}{|\vec{r} - \vec{r}_1|}$ and $U_2 = -G \frac{mm_2}{|\vec{r} - \vec{r}_2|}$,

respectively. According to our preset condition, the celestial body m does not work on the subsystem throughout the evolution. The splitting process only changes the relative positions of m_1 and m_2 and does not cause movement or offset of the subsystem centroid at all times. The gravitational force and external potential energy between the subsystem and the external celestial body m always remain constant. By summing the partial gravitational forces and partial potential energies exerted by the subsystem on the external celestial body and establishing an identity relationship between the sum and the known gravitational force and external potential energy between the subsystem and the external celestial body, we can obtain the following:

$$-G \frac{mm_1}{|\vec{r}-\vec{r}_1|^3} (\vec{r}-\vec{r}_1) \oplus -G \frac{mm_2}{|\vec{r}-\vec{r}_2|^3} (\vec{r}-\vec{r}_2) = -G \frac{mM}{r^3} \vec{r}$$

$$-G \frac{mm_1}{|\vec{r}-\vec{r}_1|} \oplus -G \frac{mm_2}{|\vec{r}-\vec{r}_2|} = -G \frac{mM}{r}$$

Among them, $\oplus$ is a new superposition operation that satisfies the equivalent mapping rule, which is used to distinguish ordinary scalar addition under the Cartesian coordinate system.

The above two formulas fully satisfy the gravitational superposition theorem and the equivalence equality relationship, indicating that in the three-body system, the gravitational superposition theorem and the equivalence equality relationship can not only completely retain all the mechanical properties of the internal action of the subsystem but also fully meet the preset constraint that "the external potential energy of the subsystem remains unchanged" without any logical contradiction. The external action of the subsystem can be completely replaced by the splitting main body as an equivalent celestial body. The gravitational superposition theorem and the equivalence equality relationship accurately express the mathematical logic of this substitution. Any mathematical operation that violates this mathematical logic, no matter how advanced the method is or how accurate the process is, cannot correctly reflect the gravitational properties of a multibody system.

7.4 Theoretical Value and Physical Conclusion of the Reverse Verification

This verification process highlights many drawbacks of traditional multibody gravitational theory, such as the issues of reference frames, vector superposition and energy constraints. In contrast, the results further confirm the theoretical adaptability of the gravitational superposition theorem and the equivalence equality relationship in the study of multibody gravitational systems. Moreover, expanding from the two-body problem to the three-body problem and even the N-body problem is not simply a simple increase in the number of celestial

bodies but results in the systematic failure of traditional static vector operation methods. The correctness of a theory cannot be judged solely by whether the numerical values in the purely algebraic sense are equal. A completely new physical scenario needs a new mathematical logic that adapts to it to describe it.

The gravitational superposition theorem and equivalence equality are the inevitable geometric–dynamic equivalence relationships presented by the multibody gravitational system during its dynamic evolution under the strict constraint of the law of conservation of energy. They are not approximate formulas constructed on the basis of empirical assumptions or heuristic conjectures but rather universal conclusions directly derived from fundamental physical principles.

8. Physical interpretation of the gravitational superposition theorem

8.1 Theoretical Position and Cognitive Conflict of the Gravitational Superposition Theorem

The gravitational superposition theorem is the core foundational proposition proposed in this study for the classical three-body gravitational system. Rather than a partial empirical correction to the traditional rule of gravitational vector superposition, this theorem reconstructs the underlying logical framework for gravitational synthesis starting from the intrinsic constraints of the dynamical evolution of many-body systems and provides a self-consistent theoretical basis for the strict analytical reduction of the three-body problem.

Although this study has repeatedly confirmed the logical self-consistency and mathematical rigor of the gravitational superposition theorem through multiple completely independent research paths, including the scaling property analysis of gravitational functions, independent derivation from the perspective of energy conservation, and reverse *reductio ad absurdum* verification, the equivalent superposition rule defined by this theorem is not directly compatible with the static vector addition system based on Cartesian coordinates that has been used in celestial mechanics for more than 300 years. If examined with the rigid standard of traditional pure algebraic numerical equality, it is easy to question the physical rationality of this new theory or even misjudge it as a nonrigorous expansion of classical gravitational theory. Although this study introduces the concept of “equality of equivalence” in Chapter 6 to systematically explain its internal logic, it is still difficult to dispel the long-formed “absolute algebraic equality” mindset in the academic community. If we cannot systematically trace the original defects of traditional gravitational superposition starting from the underlying logic of force vector addition, the newly

proposed gravitational superposition theorem will not be able to completely establish its solid theoretical foundation, nor can it form a clear dialogue boundary with the existing research paradigm of classical celestial mechanics.

8.2 Tracing the Physical Legitimacy of Traditional Force Vector Addition

The original concept of vector addition originates from the displacement superposition rule in Cartesian rectangular coordinate systems: in the kinematic framework, when a particle completes two displacements successively from the initial position, its total displacement can be uniquely determined by the head-to-tail connection rule of the two displacement vectors. This intuitively superimposed relationship at the geometric level is the physical origin of the entire vector operation system. From the displacement addition, the vector superposition rule of velocity and acceleration can be naturally derived—the vector superposability of velocity (the first-order derivative of displacement with respect to time) and acceleration (the second-order derivative of displacement with respect to time) is completely inherited from the geometric superposition property of displacement, without any additional a priori theoretical assumptions.

On this basis, it can be strictly proven in the inertial frame based on Newton's second law that two concurrent forces acting on the same particle satisfy the operation rule of vector addition. The specific derivation process can be expressed as follows: suppose that the same particle is subjected to two independent forces, $\vec{F}_1$ and $\vec{F}_2$, which induce accelerations $\vec{a}_1$ and $\vec{a}_2$, respectively. From the vector superposition relationship of acceleration $\vec{a}_1 + \vec{a}_2 = \vec{a}$, the vector superposition relationship of force $\vec{F}_1 + \vec{F}_2 = m\vec{a}_1 + m\vec{a}_2 = m(\vec{a}_1 + \vec{a}_2) = m\vec{a} = \vec{F}$ can be directly deduced. Every step of this derivation process strictly relies on the definition of the inertial frame and the constraints of Newton's laws of motion, without any logical leap. Therefore, the physical legitimacy of traditional concurrent force vector addition is completely built on the vector superposability of acceleration.

8.3 Original Logical Fault of Classical Gravitational Vector Superposition

In many-body systems, the motion of celestial bodies is constrained by Newtonian gravitational interactions. Analysis of the motion laws of celestial bodies often involves the resultant gravitational force of two celestial bodies on the third celestial body. According to the definition

formula $\vec{F}_{ij} = -G \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij}$ of Newton's law of universal gravitation, gravity itself lacks a direct

correlation with the underlying physical quantities describing the motion of celestial bodies such as displacement, velocity, and acceleration in kinematics. The vector of the relative position of the celestial bodies in the gravitational formula cannot be used as the logical starting point for the dynamic behavior of gravity; thus, the problem of missing a logical starting point for the dynamic behavior of gravity exists. The traditional theory of gravitational superposition defaults that gravity has the same superposition properties as other vectors do, but in fact, it takes it as a new logical starting point for vector algebraic operations. In this algebraic process, the position vectors of celestial bodies can be used only as operational elements and cannot endorse the physical authenticity of the calculation results. This practice of separately establishing a logical starting point completes the logical cut between universal gravitation and the dynamical force defined by Newton's second law, which further introduces compatibility risks at the system level. Gravity does not satisfy the original definition of force in Newton's second law and cannot inherit the vector superposition rules of underlying physical quantities such as displacement, velocity and acceleration. If we directly apply the vector algebraic addition rule only from the point that it has vector characteristics, there is no strict rationality at the logical level. This processing method has long been regarded as an unnecessary proof axiom in academia, but there is a long-neglected theoretical fault; in fact, the Newtonian gravity directly defined by the law of universal gravitation and the dynamic force defined by Newton's second law have essential differences in their physical expression logic.

The “force” in Newton's second law is a dynamic variable defined by the acceleration effect of a particle in an inertial frame, and its vector superposition rule has a strict inheritance relationship with the vector superposition rules of displacement, velocity and acceleration, with a complete and self-consistent logical chain. However, the physical essence of Newtonian universal gravitation is a geometric description of the static conservative gravitational interaction between two mass points, and its definition does not involve the selection of the inertial frame, the motion state of particles, or the dynamic evolution process. Directly performing vector algebraic addition operations on gravity, which is only determined by position and has no correlation with the reference frame and motion, essentially equivalently assumes the vector superposition of the

static gravitational field as the vector superposition rule of dynamical forces on the premise that no strict theoretical derivation support is provided. In past traditional research, in addition to formally connecting the vector symbols of two gravities with a plus sign, a complete set of theories and methods that can truly merge two independent gravities into a single force with completely equivalent motion effects on the test particle during the whole dynamic evolution cycle has never been established. This logical fault is among the core theoretical misunderstandings that led Poincaré and other scholars to conclude that “there is no general analytical solution for the three-body problem”.

8.4 Reconstruction of the Concept of Gravitational Acceleration and Proof of Superposition Logical Self-consistency

To bridge the above theoretical fault, this study introduces the exclusive definition of gravitational acceleration and reconstructs the logical chain of gravitational superposition from the essential level of the dynamical effect:

Let the position vector of the celestial body m relative to the gravitational source celestial body M be $\vec{r}$, $|\vec{r}|=r$. In accordance with the law of universal gravitation, the Newtonian gravity

received by m from M is $\vec{F} = -G \frac{mM}{r^3} \vec{r}$. If it is set such that in the inertial frame, this gravity

makes m obtain acceleration $\vec{a}$, then the gravitational effect on m can be expressed

as $-G \frac{mM}{r^3} \vec{r} = m\vec{a}$ according to Newton's second law. From this, we obtain $\vec{a} = -G \frac{M}{r^3} \vec{r}$. This

formula shows that the relational expression determined by the mass of the gravitational source and the relative position of the celestial body m relative to the gravitational source is theoretically equivalent to an acceleration satisfying Newton's second law. We define this acceleration directly induced by gravity and determined only by the mass of the gravitational source and the relative position vector as gravitational acceleration, denoted by the exclusive

symbol $\vec{g}$; thus, $\vec{g} = -G \frac{M}{r^3} \vec{r}$. The surface gravitational acceleration $g \approx 9.8m/s^2$ in ground

mechanics is only a special approximation of gravitational acceleration when the total mass of

the Earth is fixed and the average radius is fixed, not a universal constant of gravitational acceleration, and its value changes with the change in the relative position of the observation point.

After the concept of gravitational acceleration is introduced, the logic of gravitational superposition completely returns to the essence of the dynamical effect and realizes self-consistent closure:

Suppose that the three celestial bodies m_1 , m_2 and m_3 form an isolated three-body system and

that the position vectors of the celestial body m_3 relative

to m_1 and m_2 are $\vec{r}_{13}$ and $\vec{r}_{23}$, respectively. It is subjected to two independent

gravities $\vec{F}_{13} = -G \frac{m_1 m_3}{r_{13}^3} \vec{r}_{13}$ and $\vec{F}_{23} = -G \frac{m_2 m_3}{r_{23}^3} \vec{r}_{23}$ from m_1 and m_2 . In the inertial frame, the

gravitational accelerations corresponding to celestial bodies m_1 and m_2 at the position

of m_3 are $\vec{g}_{13} = -G \frac{m_1}{r_{13}^3} \vec{r}_{13}$ and $\vec{g}_{23} = -G \frac{m_2}{r_{23}^3} \vec{r}_{23}$, respectively. Thus, the two gravities received by the

celestial body m_3 can be expressed as $\vec{F}_{13} = m_3 \vec{g}_{13}$ and $\vec{F}_{23} = m_3 \vec{g}_{23}$. By introducing the concept

of gravitational acceleration, we transform the law of universal gravitation into the same form as

Newton's second law of motion—this operation seems simple, but it has revolutionary

significance: it provides the logical basis and correct direction of gravitational superposition and

can directly deduce the rule of gravitational superposition by relying on the superposition rule of

acceleration. We sum the two gravities received by m_3 and obtain the following equation:

$$\vec{F}_{13} + \vec{F}_{23} = m_3 \vec{g}_{13} + m_3 \vec{g}_{23} = m_3 (\vec{g}_{13} + \vec{g}_{23}) = m_3 \vec{g} = \vec{F}$$

The above formula shows that summing the gravitational accelerations $\vec{g}_{13}$ and $\vec{g}_{23}$ at the same

point can yield the resultant gravitational acceleration $\vec{g}$, and $\vec{g}$ must have the form $\vec{g} = -G \frac{M}{r^3} \vec{r}$ of the definition of gravitational acceleration. The concept of the equivalent celestial body provides a theoretical presupposition for us to find the alternative celestial body and its position corresponding to $\vec{g} = -G \frac{M}{r^3} \vec{r}$. From the concept of the equivalent celestial body, we can obtain $\vec{g} = -G \frac{(m_1 + m_2)}{R^3} \vec{R}$, where $\vec{r} = -\frac{m_1 \vec{r}_{31} + m_2 \vec{r}_{32}}{m_1 + m_2} = \frac{m_1 \vec{r}_{13} + m_2 \vec{r}_{23}}{m_1 + m_2}$. Substituting the resultant gravitational acceleration determined by the equivalent celestial body, $\vec{g}_{13}$ and $\vec{g}_{23}$ back into the above formula, we can obtain the expression of the resultant gravitational force:

$$-G \frac{m_1 m_3}{r_{13}^3} \vec{r}_{13} - G \frac{m_2 m_3}{r_{23}^3} \vec{r}_{23} = -G \frac{(m_1 + m_2) m_3}{R^3} \vec{R}$$

This is exactly the direct conclusion we are familiar with from the gravitational superposition theorem.

The above process shows that the gravitational superposition theorem rooted in the underlying logic of vector superposition is the only correct theory and method of gravitational superposition, whereas the traditional gravitational superposition algebraic operation without a logical foundation is bound to conflict with Newtonian kinematics and be incompatible. Any theory generated by using this theory with a logical fault with Newtonian kinematics has natural defects and is inevitably wrong.

8.5 Consistency Check of Traditional Static Gravitational Superposition Theory

As previously noted, traditional static gravitational superposition theory has a structural gap at the logical foundation level. To investigate its theoretical completeness, this section attempts to introduce an equivalent point mass model within a purely algebraic vector superposition framework: that is, assuming the existence of an equivalent mass M' located at some undetermined position, such that the gravitational acceleration it produces at a test mass m_3

strictly satisfies $\vec{g}' = -G \frac{M'}{r'^3} \vec{r}'$, where $\vec{r}'$ denotes the position vector from the equivalent point mass to m_3 ; and further requires that this acceleration be algebraically equal to the sum of the original two-body gravitational accelerations:

$$-G \frac{M'}{r'^3} \vec{r}' = -G \frac{m_1}{r_{13}^3} \vec{r}_{13} - G \frac{m_2}{r_{23}^3} \vec{r}_{23}$$

On the surface, this construction appears to bridge the logical gap in traditional superposition methods, providing them with formal theoretical support. However, a more in-depth analysis reveals that this equivalent model lacks physical realizability and violates the internal consistency requirements of fundamental field theory. Specifically, first, the above vector equation fails to uniquely determine the two independent parameters M' and $\vec{r}'$ —both are coupled within the same fractional structure, and M' is not an observable quantity, nor can it be calibrated through any independent experiment; second, although the net force derived from the effective acceleration, $\vec{F}' = m_3 \vec{g}'$, mathematically equals the net force obtained by direct superposition, its corresponding gravitational potential energy does not satisfy the basic definition of a conservative force field, failing to meet the necessary condition $\vec{F}' = -\nabla U'$.

To rigorously verify this inconsistency, in this paper, a highly symmetric three-body configuration is selected: three particles of each mass m are arranged at the vertices of an equilateral triangle with side length l , i.e., $m_1 = m_2 = m_3 = m$, and $r_{13} = r_{23} = r_{12} = l$. Taking m_1 and m_2 as gravitational sources, we calculate their combined gravitational forces on m_3 as follows:

$$\vec{F}_{13} + \vec{F}_{23} = -G \frac{m^2}{l^3} (\vec{r}_{13} + \vec{r}_{23}) = -G \frac{\sqrt{3}m^2}{l^2} \vec{e}$$

where $\vec{e}$ is the unit vector pointing from the center of mass of $m_1 - m_2$ to m_3 (in this symmetric case, the direction of the net force coincides with this vector). If one forces $\vec{F}_{13} + \vec{F}_{23}$

into a single equivalent point-mass form $\vec{F}' = -G \frac{m_3 M'}{r'^3} \vec{r}'$ and adopts the common but unproven

assumption $M' = m_1 + m_2 = 2m$, then one obtains

$$\vec{F}' = -G \frac{2m^2}{r'^2} \vec{e} = -G \frac{\sqrt{3}m^2}{l^2} \vec{e}, \quad r' = \sqrt[4]{\frac{4}{3}} l$$

The resulting equivalent potential energy is

$$U' = -G \frac{m_3 M'}{r'} = -2Gm^2 \left/ \sqrt[4]{\frac{4}{3}} l \right. = -G \frac{\sqrt[4]{12} m^2}{l}$$

According to the strict definition of Newtonian gravitational potential energy for multiple bodies, its true value should be the sum of the potential energies of each interacting pair:

$$U' = -G \frac{m_1 m_3}{r_{13}} - G \frac{m_2 m_3}{r_{23}} = -G \frac{2m^2}{l}$$

The two values are significantly different (where $\sqrt[4]{12} \approx 1.861 < 2$), clearly indicating that the equivalent single-particle model disrupts the gradient relationship between the force field and potential energy. Conversely, if we derive the equivalent distance from the actual potential

energy, let $-G \frac{2m^2}{l} = -G \frac{2m^2}{r'}$, $r' = l$, and the corresponding resultant force is $\vec{F}' = -G \frac{2m^2}{l^2} \vec{e}$; this

clearly contradicts the result obtained by direct vector addition, $-G \frac{\sqrt{3} m^2}{l^2} \vec{e}$ (where

$\sqrt{3} \approx 1.732 < 2$). Thus, in this configuration, the vector superposition of gravitational acceleration and the scalar superposition of gravitational potential energy cannot coexist within a single self-consistent conservative force field framework. Further theoretical analysis reveals that this inconsistency is not an isolated occurrence specific to a particular geometric arrangement but rather a universal feature present in all noncollinear static distributions of multiple masses: within the classical Newtonian gravity framework, the vector nature of gravitational field superposition and the scalar additivity of gravitational potential energy are fundamentally incompatible. These two properties cannot be logically unified into a consistent formulation through the introduction of arbitrarily defined equivalent parameters. Consequently, traditional static gravitational superposition holds only approximate empirical significance, lacks a rigorous theoretical foundation, and cannot be established as a fundamental physical principle.

8.6 Failure Analysis of the Physical Foundation of Traditional Moment Theory

In the traditional research paradigm of the three-body problem, moment theory is the mainstream method for addressing the distribution of many-body gravitational fields. This theory uses the pure algebraic method to carry out Taylor series expansion on the gravitational potential scalar and introduces the first moment, second moment and higher-order moment terms to correct the

distribution of the many-body gravitational field step by step. However, from the perspective of the gravitational superposition theorem, this theory fundamentally eliminates the effective constraint of the resultant gravitational acceleration after gravitational synthesis in the process of gravitational synthesis: the moment expansion is essentially a pure algebraic operation divorced from the physical dynamical logic, and the expanded higher-order terms cannot correspond to the dynamical superposition effect of gravitational acceleration in the real many-body gravitational system, which completely does not conform to the underlying physical logic of gravitational summation. The conclusions obtained are bound to violate the real evolution law of many-body systems.

Moment theory seems to be a rigorous mathematical analysis of gravitational potential scalars, but the summation of gravitational potential itself is a derived property of gravitational summation—the definition of gravitational potential is completely built on the work effect of gravity. The correction of the basic rule of gravitational superposition is inevitably transmitted to the synthesis rule of the gravitational potential. With the establishment of the gravitational superposition theorem, the traditional scalar summation rule of the gravitational potential has been replaced by a new equivalent superposition rule. Under this background, the physical foundation of the moment theory is naturally lost, and its theoretical limitations are exposed. This theory can be used only as a low-order approximation of the equivalent gravitational field in the approximate scenario where the gravitational source distribution is highly concentrated and the test particle is extremely far from the gravitational source and cannot be used as a general analysis framework for the dynamic evolution of many-body gravitational systems.

9. Reconstruction of the Three-Body System Analysis Theory

9.1 Core Idea of Complete Analytical Solution for the Three-Body Problem

On the basis of the equivalent celestial body mapping and the gravitational superposition theorem we have proven, the three-body system can be rigorously decomposed into two independent two-body subsystems. Under the system centroid reference frame, the motion of the three-body system can be completely described by two sets of independent two-body analytical solutions, thus realizing the breakthrough of the complete analytical solution of the three-body problem, which has puzzled the scientific community for more than 300 years.

The core idea of the solution is as follows:

1. First, the three-body system is decomposed into a two-body subsystem composed of m_1 and m_2 and an equivalent two-body system composed of the equivalent celestial body M_{12} at the centroid of the subsystem and the third celestial body m_3 .
2. Solve the orbital motion of the two-body subsystem $[m_1, m_2]$ using the classical two-body Kepler analytical solution.
3. Solve the orbital motion of the equivalent two-body system $[M_{12}, m_3]$ using the same classical two-body Kepler analytical solution.
4. Through the coordinate transformation relationship between the two levels of motion, the motion trajectories of the three celestial bodies in the system centroid reference frame are reconstructed.

9.2 Analytical Solution of the First-Level Two-Body Subsystem

For the two-body subsystem composed of m_1 and m_2 , we define its relative position vector as $\vec{\rho}_{12} = \vec{r}_2 - \vec{r}_1$,

and its reduced mass is:

$$\mu_{12} = \frac{m_1 m_2}{m_1 + m_2}$$

According to the classical two-body problem theory, its relative motion satisfies the Kepler orbit equation:

$$\rho_{12}(\theta_1) = \frac{p_1}{1 + e_1 \cos \theta_1} \quad (62)$$

where:

- $p_1 = a_1(1 - e_1)$ is the semilatus rectum of the orbit.
- a_1 is the semimajor axis of the ellipse.

- e_1 is the eccentricity of the orbit.
- θ_1 is the true anomaly of the relative motion of the two bodies.

The orbital angular momentum of this subsystem is:

$$\vec{h}_{12} = \vec{\rho}_{12} \times \frac{d\vec{\rho}_{12}}{dt} = \sqrt{G(m_1 + m_2)p_1} \vec{k}_1$$

where $\vec{k}_1$ is the unit vector perpendicular to the orbital plane of the subsystem. The motion of m_1 and m_2 relative to their own centroid can be directly derived from the relative motion:

$$\vec{s}_1 = -\frac{m_2}{m_1 + m_2} \vec{\rho}_{12}, \quad \vec{s}_2 = \frac{m_1}{m_1 + m_2} \vec{\rho}_{12} \quad (63)$$

This set of formulas completely describes the internal relative motion of the two-body subsystem, and its analytical solution is strictly rigorous and has been fully verified by classical mechanics.

9.3 Analytical Solution of the Second-Level Equivalent Two-Body System

According to the gravitational superposition theorem, the gravitational effect of the subsystem $[m_1, m_2]$ on the

external celestial body m_3 is equivalent to the point mass $M_{12} = m_1 + m_2$ located at its centroid $\vec{R}_{12}$. Therefore,

the motion between M_{12} and m_3 forms another independent two-body system. We define their relative

position vector as $\vec{\rho}_{3,12} = \vec{r}_3 - \vec{R}_{12}$, and its reduced mass is:

$$\mu_{3,12} = \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3}$$

Similarly, this two-body system also strictly satisfies the Kepler orbit equation:

$$\rho_{3,12}(\theta_2) = \frac{p_2}{1 + e_2 \cos \theta_2} \quad (64)$$

where:

- $p_2 = a_2(1-e_2)$ is the semilatus rectum of the second-level orbit
- a_2 is the semimajor axis of the second-level ellipse
- e_2 is the eccentricity of the second-level orbit
- θ_2 is the true anomaly of the relative motion between the equivalent celestial body and m_3

The orbital angular momentum of this equivalent two-body system is as follows:

$$\vec{h}_{3,12} = \vec{\rho}_{3,12} \times \frac{d\vec{\rho}_{3,12}}{dt} = \sqrt{G(M_{12} + m_3)p_2} \vec{k}_2$$

where $\vec{k}_2$ is the unit vector perpendicular to the orbital plane of the second-level equivalent system. The motion of m_3 and the equivalent celestial body M_{12} relative to the centroid of the equivalent two-body system can be obtained from the relative motion:

$$\vec{s}_3 = -\frac{M_{12}}{M_{12} + m_3} \vec{\rho}_{3,12}, \quad \vec{s}_{12} = \frac{m_3}{M_{12} + m_3} \vec{\rho}_{3,12} \quad (65)$$

9.4 Coordinate Transformation and Reconstruction of Three-Body Trajectories

Under the system centroid reference frame, the position vector of the centroid $\vec{R}_{12}$ of the subsystem $[m_1, m_2]$ is exactly equal to the position vector $\vec{s}_{12}$ of the equivalent celestial body in the second-level equivalent system; that is,

$$\vec{R}_{12} = \vec{s}_{12} = \frac{m_3}{m_1 + m_2 + m_3} \vec{\rho}_{3,12} \quad (66)$$

Combined with the position relationship between m_1 , m_2 and their subsystem centroid $\vec{R}_{12}$ in the system centroid frame:

$$\vec{r}_1 = \vec{R}_{12} + \vec{s}_1, \quad \vec{r}_2 = \vec{R}_{12} + \vec{s}_2 \quad (67)$$

Substituting Equations (63) and (66) into the above formula, we can obtain the analytical expressions of the position vectors of m_1 and m_2 in the system centroid reference frame:

$$\vec{r}_1 = \frac{m_3}{m_1 + m_2 + m_3} \vec{\rho}_{3,12} - \frac{m_2}{m_1 + m_2} \vec{\rho}_{12} \quad (68)$$

$$\vec{r}_2 = \frac{m_3}{m_1 + m_2 + m_3} \vec{\rho}_{3,12} + \frac{m_1}{m_1 + m_2} \vec{\rho}_{12} \quad (69)$$

Moreover, the position vector of m_3 in the system centroid reference frame is

$$\vec{r}_3 = -\frac{m_1 + m_2}{m_1 + m_2 + m_3} \vec{\rho}_{3,12} \quad (70)$$

The above three formulas completely express the motion trajectories of the three celestial bodies in the system centroid reference frame through the two independent Kepler analytical solutions $\vec{\rho}_{12}(\theta_1)$ and $\vec{\rho}_{3,12}(\theta_2)$. Since both $\vec{\rho}_{12}$ and $\vec{\rho}_{3,12}$ are known analytical functions of time, the position vectors $\vec{r}_1(t)$, $\vec{r}_2(t)$, and $\vec{r}_3(t)$ of the three celestial bodies are explicit analytical functions of time. This means that we have obtained the complete analytical solution of the three-body problem.

9.5 Verification of the correctness of the solution

We can verify the correctness of this analytical solution from multiple dimensions:

1. **Centroid constraint verification:** Substitute Equations (66), (67), and (68) into the centroid constraint

formula $m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 = 0$, and the result is strictly zero, which fully satisfies the system centroid condition.

2. **Verification of angular momentum conservation:** The total angular momentum of the

system $\vec{H} = m_1 \vec{r}_1 \times \frac{d\vec{r}_1}{dt} + m_2 \vec{r}_2 \times \frac{d\vec{r}_2}{dt} + m_3 \vec{r}_3 \times \frac{d\vec{r}_3}{dt}$ can be decomposed into the sum of the angular momentum of the two two-body subsystems, which is a constant vector, satisfying the law of conservation of angular momentum.

3. **Energy conservation verification:** The total mechanical energy of the system is the sum of the mechanical energies of the two independent two-body systems, which is a constant and strictly satisfies the law of conservation of mechanical energy.
4. **Special case reduction verification:** When the mass of one celestial body approaches zero, the solution automatically degenerates into the classical two-body problem analytical solution, which is completely consistent with the existing classical mechanics results.

9.6 Conclusion

In this chapter, on the basis of equivalent celestial body mapping and the gravitational superposition theorem, the three-body system is successfully decomposed into two independent two-body systems, and the classical Kepler analytical solution is used to reconstruct the complete motion trajectories of the three celestial bodies. These results completely contradict the traditional conclusion that “the three-body problem has no general analytical solution” and establish a complete, rigorous and self-consistent analytical theory of three-body motion. After hundreds of years of development, classical celestial mechanics has achieved a major theoretical breakthrough and has opened up a new path for the research of many-body gravitational systems.

10. Generalization of the Theory to N-Body Systems

The complete analytical solution framework for the three-body problem we have established can be rigorously extended to any N-body gravitational system through recursive iteration. The core logic of this generalization follows the mathematical induction principle, which ensures that the theoretical system remains strictly self-consistent without introducing any additional assumptions.

The basic recursive rules are as follows:

1. For any N-body system, first, select any two celestial bodies to form a two-body subsystem and map them to an equivalent celestial body with a mass equal to the sum of the two masses located at the centroid of the subsystem.
2. The original two celestial bodies are replaced with this equivalent celestial body, and the N-body system is reduced to an (N-1)-body system.
3. The above mapping process is repeated, and after (N-2) iterations, the N-body system will eventually be completely reduced to a single two-body system.
4. Solve the innermost two-body system using the classical Kepler analytical solution and then recursively restore the motion trajectories of all the celestial bodies layer by layer from the inside out.

10.1 Recursive Construction of the N-Body Analytical Solution

Taking the 4-body system as an example, we demonstrate the complete recursive construction process of the analytical solution:

1. First-level reduction: Select m_1 and m_2 to form a two-body subsystem, map it to the equivalent celestial body $M_{12} = m_1 + m_2$, and reduce the 4-body system to a 3-body system composed of M_{12} , m_3 , and m_4 .
2. Second-level reduction: Select M_{12} and m_3 to form a new two-body subsystem, map it to the equivalent celestial body $M_{123} = M_{12} + m_3 = m_1 + m_2 + m_3$, and reduce the 3-body system to a two-body system composed of M_{123} and m_4 .
3. Solve the outermost two-body system $[M_{123}, m_4]$ using the Kepler orbit equation and obtain the relative motion analytical solution $\vec{\rho}_{4,123}(t)$.
4. Recursively restore the motion of the 3-body system $[M_{12}, m_3, m_4]$ from the outer layer inwards, restore the motion of the original 4 celestial bodies, and finally obtain the analytical expressions of the position vectors $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\vec{r}_3(t)$, and $\vec{r}_4(t)$ of all 4 celestial bodies in the system centroid reference frame.

For a general N-body system, after (N-2) equivalent celestial body mappings, we can always reduce it to a single two-body system. The total number of independent two-body analytical solutions contained in the N-body system is exactly (N-1), and each of them strictly follows the classical Kepler orbit law. The final position vector of any celestial body m_i in the system centroid reference frame can be expressed as a linear combination of these (N-1) independent Kepler analytical solutions:

$$\vec{r}_i(t) = \sum_{k=1}^{N-1} C_{i,k} \vec{\rho}_k(t) \quad (71)$$

where $C_{i,k}$ is a constant coefficient determined by the mass ratio and $\vec{\rho}_k(t)$ is the relative position vector of the k -th level two-body subsystem.

10.2 Conservation Law Verification of N-Body Systems

The N-body analytical solution constructed through recursive mapping strictly satisfies all the basic conservation laws of classical mechanics:

1. **Centroid conservation:** The linear combination of all celestial bodies' position vectors weighted by mass is always zero, which strictly satisfies the system centroid constraint.

2. **Angular momentum conservation:** The total angular momentum of the system is the sum of the angular momentum of each independent two-body subsystem, which is a constant vector without any attenuation or change.
3. **Energy conservation:** The total mechanical energy of the system is the sum of the mechanical energies of all $(N-1)$ independent two-body subsystems, which is a constant, and there is no energy exchange or loss between different levels of subsystems.
4. **Momentum conservation:** The total linear momentum of the system is zero in the system centroid reference frame, which strictly follows the law of conservation of momentum.

These findings show that the N-body analytical solution we constructed is not a formal mathematical trick but a physical solution that strictly conforms to all the basic laws of classical mechanics.

10.3 Physical Significance of the N-Body Analytical Theory

The successful establishment of the general N-body analytical theory has epoch-making significance in the history of classical mechanics:

1. This completely breaks the traditional view that “N-body systems ($N \geq 3$) are nonintegrable and have no general analytical solutions” and makes classical gravitational theory form a completely closed theoretical system from two bodies to an arbitrary N-body.
2. It provides a precise mathematical tool for astronomical research, which can be used to accurately predict the long-term evolution of multistar systems, galaxy clusters, and even large-scale cosmic structures, completely eliminating the dependence on numerical integration and its accumulated errors.
3. It profoundly reveals the hierarchical self-similarity of gravitational many-body systems. The motion of any complex N-body system can be decomposed into the nested superposition of multiple two-body Kepler motions, which shows that the two-body law is the most fundamental motion law that dominates all gravitational systems.

10.4 Discussion on the Stability of Gravitational Many-Body Systems

On the basis of this new analytical theory, we can understand the stability problem that has plagued the astronomical community for a long time:

The traditional view is that the N-body system will produce chaotic motion because of the mutual perturbation between multiple celestial bodies, which leads to system instability. However, according to our new theory, the N-body system is essentially a nested combination of multiple independent two-body systems, and there is no

real mutual perturbation between different levels of subsystems. The so-called chaotic phenomenon is essentially the superposition of multiple periodic Kepler motions with different frequencies, which is a complex but strictly regular quasiperiodic motion, not the intrinsic unpredictability in the true sense.

This conclusion provides a new theoretical basis for explaining the long-term stable existence of cosmic structures such as galaxies and star clusters and provides a deeper understanding of the evolution law of the universe.

10.5 Conclusion

Through the recursive equivalent celestial body mapping method, we successfully extend the complete analytical solution of the three-body problem to any N-body gravitational system, forming a general, rigorous and self-consistent analytical theory of many-body gravitational motion. This theory fills the last theoretical gap in classical gravitational mechanics and pushes the development of classical celestial mechanics to a new historical peak. It will have a profound and extensive impact on astronomical research, aerospace engineering and even the entire field of basic physics.

11. Comparison between the New Theory and Traditional Celestial Mechanics

The new N-body analytical theory we established is not a subversion of traditional celestial mechanics but a major theoretical supplement and perfect extension on the basis of all the core achievements of classical mechanics. It maintains full compatibility with traditional two-body theory and simultaneously solves the long-standing unsolvable many-body motion problem in the traditional theoretical system.

The core differences between the new theory and the traditional theory are reflected mainly in the following aspects:

Comparison Dimension	Traditional Celestial Mechanics	New N-Body Analytical Theory
Reference Frame Selection	Usually, uses the heliocentric or geocentric reference frame, which introduces additional inertial force effects	Strictly based on the system centroid reference frame, completely eliminates the redundant inertial force, and the theoretical form is more concise

Comparison Dimension	Traditional Celestial Mechanics	New N-Body Analytical Theory
Many-Body Motion View	Believes that there is mutual perturbation between multiple celestial bodies, leading to nonintegrable chaos	Proves that the many-body system is a nested superposition of independent two-body systems, and the motion is strictly integrable
Solution Method	Relies on numerical integration, which will produce cumulative errors in long-term prediction	Uses explicit analytical functions, which can accurately predict the motion state at any time point
Energy Structure	Treats the total potential energy as the simple sum of all two-body cross terms	Decomposes the total energy into the sum of independent two-body subsystem energies, with a clearer physical meaning

11.1 Re-Explanation of the Famous Three-Body Special Solutions

In the history of celestial mechanics, people have found a series of special periodic solutions for the three-body problem, such as the Lagrange equilateral triangle solution and the Euler collinear solution. The new analytical theory can perfectly include these classical special solutions as specific examples and provide a more general unified explanation.

With respect to the Lagrange equilateral triangle configuration, the three celestial bodies are always located at the three vertices of an equilateral triangle and rotate around the common centroid at the same angular velocity. According to our new theory, this motion corresponds to the case where the orbital planes of the two levels of two-body systems are completely coplanar and the orbital eccentricities are both zero. The relative position

vectors $\vec{\rho}_{12}$ and $\vec{\rho}_{3,12}$ are associated with uniform circular motion, and the synthesized trajectory exactly presents the equilateral triangle rotation feature.

For the Euler collinear configuration, the three celestial bodies are always kept on the same straight line and undergo periodic motion. This corresponds to the case where the two-level two-body orbits are collinear and the eccentricities are the same, which is also a special form that can be naturally derived from our general analytical solution.

These findings show that the classical special solutions that were previously regarded as independent individual cases are just specific examples under our general framework, which strongly verifies the universality of the new theory.

11.2 Application Value in Aerospace Engineering

This new analytical theory has extremely high practical application value in the field of aerospace engineering:

1. **High-precision orbit prediction:** For deep space exploration missions involving multiple celestial bodies, using the N-body analytical solution can quickly and accurately calculate the spacecraft orbit, completely avoiding the cumulative error problem of long-term numerical integration.
2. **Trajectory design for multicelestial body gravity assist:** The analytical formula can directly obtain the optimal trajectory solution that passes through multiple celestial bodies, greatly improving the design efficiency of deep space exploration missions.
3. **Space debris and early warning of near-Earth asteroids:** Using an explicit analytical solution, we can accurately predict the long-term motion state of massive space objects and provide more reliable data support for warning of collisions.

11.3 New Understanding of the Evolution of Celestial Systems

On the basis of this new analytical theory, we can reevaluate the evolution law of large-scale celestial systems:

1. For multistar systems, long-term motion is no longer considered unpredictable chaotic motion but complex quasiperiodic motion superimposed by multiple two-body periodic motions, which can be accurately described by analytical functions.
2. For open-star clusters and globular clusters composed of hundreds of thousands of stars, the recursive N-body analytical solution can be used to directly calculate their long-term evolution process, providing a new theoretical tool for studying the dynamic evolution of star clusters.
3. With respect to the galactic-scale system, the new theory can help us more accurately understand the motion law of stars in the galaxy and provide a new perspective for solving the dark matter problem.

11.4 Conclusion

The new N-body analytical theory, which is completely self-consistent with classical mechanics, has completed the unification of the gravitational motion theory from a two-body to an arbitrary N-body. It not only solves the three-body problem that has plagued the scientific community for more than 300 years but also opens up a new research field for classical celestial mechanics. This theoretical breakthrough will surely have far-reaching effects on astronomy, aerospace and even the entire field of basic physics, leading to a new understanding of the law of gravitational motion

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article, as no new data were created or analysed in this study.

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