

Brief Trick when Summing Lambert Series over Their Bases

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Turning its Double Sum into Dirichlet Series

Abstract

This short note describes a mundane rearrangement of double sum, yet often encountered when working with Lambert series or q-analogues, leading into some Dirichlet series which often ease the computation, or at least, provides some other perspective or insight. Similar point like “series recentering” in a similar fashion as going from Taylor series to Maclaurin one—therefore eliminating the need of Binomial Expansion—was made.

1. Intro

This section details the sum of Riemann zeta values evaluated at integer > 1 minus 1, and its contribution on the simplification of certain double sum which usually arises from Lambert series.

1.1. One Zeta

The sum of zeta function evaluated at every integer > 1 minus 1, converges as of follows

Proposition 1:

$$\sum_{n=1}^{\infty} (\zeta(n+1) - 1) = 1$$

Proof: simply shift the index n from 1 into $n = 2$, since all terms are non-negative, such sum can be interchanged as

$$\sum_{n=1}^{\infty} (\zeta(n+1) - 1) = \sum_{n=2}^{\infty} (\zeta(n) - 1) = \sum_{n=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k^n} = \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{k^n}$$

When $k \geq 2$, observe that their inner sum converges to a geometric series

$$\sum_{n=2}^{\infty} \frac{1}{k^n} = \frac{k^{-2}}{1 - k^{-1}} = \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

leading to the cancellation chain of

$$\sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1$$

therefore the only surviving terms are $1/k$ at $k = 2$ and $k = \infty$, giving $1/(2-1) + 1/\infty = 1$. Therefore, after shifting k one obtains

$$\sum_{k=1}^{\infty} (\zeta(k+1) - 1) = 1$$

which is simply proposition 1 with the index n being renamed into k .

1.2. When Sophomore's Dream Fades to Nothing

Working with the following double sum is cumbersome and leads into heavier computational workload:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+1)^n} - \frac{1}{m+1} \right)$$

Also, the term $(m+1)^n$ introduces Binomial Expansion terms, while it is true that Sophomore's dream comes true when working with field of positive characteristics, in char 0 the only case of vanishing Binomial Expansion terms, is when $(m+0)^n$, however

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+0)^n} - \frac{1}{m+0} \right)$$

diverges, yet shifting the index m to 2 achieves nothing since it reverted back into the former

$$\sum_{m=2}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+0)^n} - \frac{1}{m+0} \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+1)^n} - \frac{1}{m+1} \right)$$

leading one undergoes merry-go-round ride.

2. Fin

In order to perform simplification of the double summation series mentioned in section 1.2. consider the following

Lemma: Let $f(n)$ be regular for positive integer n , and f has polynomial growth. Whenever $f(1) = 1$ the following trick can be applied

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+1)^n} - \frac{1}{m+1} \right) = 1 + \sum_{k=1}^{\infty} (f(k) - 1)(\zeta(k) - 1)$$

such expression allows the double sum on the left hand side to be converted into single sum, but how does one arrive into such? Recall the proposition 1, one wonders if merely

$$\sum_{k=1}^{\infty} (\zeta(k+1) - 1) = 1$$

then what if $f(k+1)$ is being attached as weighing coefficient for such series

$$\sum_{k=1}^{\infty} f(k+1)(\zeta(k+1) - 1)$$

?

2.1. Using Fubini's Theorem

Since the term $f(1) = 1$ is needed, it echoes a classic condition for a series with invertible Dirichlet multiplication, however as one progresses with plethora of arithmetic functions (say differences between them, their quotient when built as rational functions, etc. etc.) there are cases when some function $g(1)$ being nonzero yet not equal to 1. Let any series with nonzero term $g(1) = k$ can be simply scaled down by dividing with k , i.e. doing $f(n) = \frac{g(n)}{k}$ therefore, when $f(n)$ having $f(1) = 1$, revisiting

$$\sum_{k=1}^{\infty} f(k+1)(\zeta(k+1) - 1)$$

the following proposition can be stated

Proposition 2: by shifting the index k to 2 instead, it turned out that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+1)^n} - \frac{1}{m+1} \right) = \sum_{k=2}^{\infty} f(k)(\zeta(k) - 1)$$

Proof: when $f(1) = 1$ and $x = m+1$, rewrite the in-before expression as

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{f(n)}{(m+1)^n} - \frac{1}{m+1} \right) = \sum_{x=2}^{\infty} \sum_{n=2}^{\infty} \frac{f(n)}{x^n}$$

because f has polynomial growth along with being regular at positive integer n , when such $f(n)$ happens to met the following condition of absolute convergence,

$$\sum_{n=2}^{\infty} |f(n)| \sum_{x=2}^{\infty} \frac{1}{x^n} = \sum_{n=2}^{\infty} |f(n)| (\zeta(n) - 1) < \infty.$$

their sums can be interchanged (since they are not merely conditionally convergent, they do possess absolute convergence) and thus, Fubini's theorem allows such double sum to be rearranged without requiring any regularisation (observe that the condition $f(1) = 1$ in practise has taken care the pesky harmonic sum, which is the only concern of non-convergence—cancelling them already), yielding

$$\sum_{x=2}^{\infty} \sum_{n=2}^{\infty} \frac{f(n)}{x^n} = \sum_{n=2}^{\infty} f(n) (\zeta(n) - 1)$$

after shifting back x to $m+1$ as well as the index n from 2 back to 1, one arrives at

$$\sum_{x=2}^{\infty} \sum_{n=2}^{\infty} \frac{f(n)}{x^n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+1)^n} - \frac{1}{m+1} \right) = \sum_{k=2}^{\infty} f(k) (\zeta(k) - 1)$$

as stated in proposition 2.

Proposition: notice the following sum involving Riemann's zeta as the Dirichlet series, the left hand side starts with $k = 1$, while the other side starts with $k = 2$.

$$1 + \sum_{k=1}^{\infty} (f(k) - 1)(\zeta(k) - 1) = \sum_{k=2}^{\infty} f(k)(\zeta(k) - 1)$$

Proof: let us expand the multiplication and assign shorthand S for such expanded sum

$$1 + \sum_{k=1}^{\infty} (f(k) - 1)(\zeta(k) - 1) = 1 + \sum_{k=1}^{\infty} (f(k)\zeta(k) - f(k) - \zeta(k) + 1) = 1 + S$$

with $f(1) = 1$, shifting the index from $k = 1$ to $k = 2$, one could separate their $k=1$ components as follows

$$\begin{aligned}
S &= (f(1)\zeta(1) - f(1) - \zeta(1) + 1) + \sum_{k=2}^{\infty} (f(k)\zeta(k) - f(k) - \zeta(k) + 1) \\
&= \zeta(1) - \zeta(1) - 1 + 1 + \sum_{k=2}^{\infty} f(k)\zeta(k) - \sum_{k=2}^{\infty} f(k) - \sum_{k=2}^{\infty} \zeta(k) + \sum_{k=2}^{\infty} 1 \\
&= 0 + \sum_{k=2}^{\infty} f(k)\zeta(k) - \sum_{k=2}^{\infty} f(k) - (\sum_{k=2}^{\infty} \zeta(k) - 1) \\
&= \sum_{k=2}^{\infty} f(k)\zeta(k) - \sum_{k=2}^{\infty} f(k) - (\sum_{k=1}^{\infty} \zeta(k+1) - 1)
\end{aligned}$$

by proposition 1: i.e. the One Zeta thing

$$\begin{aligned}
S &= (\sum_{k=2}^{\infty} f(k)\zeta(k) - \sum_{k=2}^{\infty} f(k)) - 1 \\
S &= -1 + \sum_{k=2}^{\infty} f(k)(\zeta(k) - 1)
\end{aligned}$$

since

$$1 + \sum_{k=1}^{\infty} (f(k) - 1)(\zeta(k) - 1) = 1 + S$$

therefore

$$1 + \sum_{k=1}^{\infty} (f(k) - 1)(\zeta(k) - 1) = \sum_{k=2}^{\infty} f(k)(\zeta(k) - 1)$$

leading to the sum that runs from $k = 1$ to ∞ , rather than from $k = 2$. Finally the lemma

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{f(n)}{(m+1)^n} - \frac{1}{m+1} \right) = 1 + \sum_{k=1}^{\infty} (f(k) - 1)(\zeta(k) - 1)$$

is concluded.

Citations

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