On Non-inertial Relativity Theory and Space-Time-Matter Unification*

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Abstract

We explore further the physical implications of Non-inertial Relativity. The spatial-force formalism of Non-inertial Relativity versus the proper-force description is analyzed. It is shown that contrary to what one would expect from the algebraic equation of an elliptic-hyperboloid, due to the constraint $r = fv$ between the power $r = \frac{dE}{dt}$ and the spatial force $f = \frac{dp}{dt}$, the null-cone condition in phase space leads to the following bounds on the values of v and the Newtonian force $f_N = m\frac{dv}{dt}$ given by : $0 \leq |v| \leq c$, and $0 \leq |f_N| \leq b$. The fact that $|f_N| \leq b$ is bounded requires clear modifications to the Newtonian potential [5]. We proceed to explore the group contractions of the pseudo-unitary group $U(1,1)$ in the $b \to \infty$ limit. After working with *dimensionless* phase space coordinates one finds that it *does* lead to Lorentz boosts transformations, in the $b \to \infty$ limit, as expected. If, instead, one works with $dimensionful$ coordinates, the limit $b \to \infty$ does *not* lead to the standard Lorentz boosts transformations for the energy-momentum variables. There are extra terms that are associated with the semi-direct product of the Lorentz group with an Abelian group (which is not the Poincare group). Similar results were found by [3]. We review the equivalence principle within the context of non-Milgromian modified Newtonian dynamics. An analysis is provided of how Non-inertial relativity points towards a space-time-matter unification and leads to a connection to string theory. Finally, we conclude with some remarks about future avenues of research.

Keywords : Born Reciprocal Relativity; Non-inertial Relativity; Modified Newtonian dynamics; Curved Phase Space; Strings; Quantum Gravity.

*Dedicated to the loving memory of Javier Domingo Romero and Luis Alberto Baquero Neira, dear and longtime friends in Spain and Colombia

1 Non-inertial Relativity : Overview

The principle behind the concept of "Born reciprocal relativity theory", or non-inertial relativity to be more precise[1], was advocated by [3], [4] and it was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. Hence, a *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [4].

The principle of maximal acceleration was advocated earlier on by [7]. A chapter on Reciprocity Theory and Born's Quantum Metric Operator appeared early on in the book by [2]. The concept of Born reciprocity in order to provide a new point of view on string theory in which spacetime is a derived dynamical concept was advanced by [10].

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $x^i, t, E, p^i; i = 1, 2, 3$ are $\mathbf{c}$ -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [3] based on the group $U(1, 3)$ and which is the Born version of the Lorentz group $SO(1, 3)$.

Adopting the units $\hbar = c = 1$, the $U(1, 3) = SU(1, 3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dE + \delta_{ij} dx^i \wedge dp^j; i, j = 1, 2, 3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \;=\; (dt)^2 \;-\; (dx)^2 \;-\; (dy)^2 \;-\; (dz)^2 \;+$$

$$\frac{1}{b^2}\;\left((dE)^2 \;-\; (dp_x)^2 \;-\; (dp_y)^2 \;-\; (dp_z)^2\right),\quad (c = 1) \tag{1.1}$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1, 3) = Sp(8, R) \cap O(2, 6)$; $U(3, 1) = Sp(8, R) \cap O(6, 2)$, and $U(2, 2) = Sp(8, R) \cap O(4, 4)$.

The following phase space line interval

$$(d\omega)^2 \;=\; (dt)^2 \;-\; (dx)^2 \;+\; \frac{(dE)^2 \;-\; (dp)^2}{b^2} =$$

$$(d\tau)^2\;\left(\;1\;+\;\frac{(dE/d\tau)^2 \;-\; (dp/d\tau)^2}{b^2}\;\right) = (d\tau)^2\;\left(\;1\;+\;\frac{\mathcal{F}^2}{F^2_{max}}\;\right)\;=$$

$$(d\tau)^2\;\left(\;1\;-\;\frac{F^2}{F^2_{max}}\;\right),\quad \mathcal{F}^2 = -F^2 < 0,\quad F_{max} = b \tag{1.2}$$

where one has factored out the non-vanishing proper time infinitesimal $(d\tau)^2 = dt^2 - dx^2 \neq 0$ in eq-(1.2). The numerical quantity F^2 is positive by definition. The proper force-squared on a massive particle is $\mathcal{F}^2 = m^2 a^2$, where a^2 is the

[1] We thank one of the referees of a previous article for highlighting this fact in order to clarify the point that Born did not propose a reciprocal relativity theory

proper acceleration-squared $a^2 = a_\mu a^\mu$, and m is the rest mass. We refrained from factoring out $(dt)^2$ in (1.2) because it is not Lorentz invariant, whereas $(d\tau)^2$ is Lorentz invariant.

Due to the orthogonality condition $u_\mu a^\mu = 0$ resulting from differentiating the normalization condition $u_\mu u^\mu = \pm 1$ of timelike/spacelike velocities, when the velocity is timelike (subluminal particle) one has $(d\tau)^2 > 0$, so the acceleration is spacelike $a^2 = a_\mu a^\mu < 0$. Therefore $m^2 a^2 < 0$ since $m^2 > 0$. And viceversa, when the velocity is spacelike (superluminal particle) one has $(d\tau)^2 < 0$, so the acceleration is timelike $a^2 = a_\mu a^\mu > 0$. Therefore $m^2 a^2 < 0$ since $m^2 < 0$ (tachyonic particle). Consequently, the proper force squared $\mathcal{F}^2 \equiv (\frac{dE}{d\tau})^2 - (\frac{dp}{d\tau})^2 = \frac{(dE)^2 - (dp)^2}{(d\tau)^2} = m^2 a^2 < 0$ is always negative. Therefore, one may rewrite the negative definite $\mathcal{F}^2$ as $\mathcal{F}^2 \equiv m^2 a^2 = -F^2 < 0$, with $F^2 > 0$, so that the factorization can always be rewritten as $(d\tau)^2(1 + \frac{\mathcal{F}^2}{b^2}) = (d\tau)^2(1 - \frac{F^2}{b^2})$, with $F^2 > 0$. When $m = 0$, one has $(d\tau)^2 = (dE)^2 - (dp)^2 = 0$ so that $(d\omega)^2 = 0$. No factorization is needed.

Consequently, the *negative* sign appearing inside the parenthesis in eqs-(1.2) furnishes the analog of the Lorentz relativistic factor in special relativity and it involves the ratio of the square of two *proper* forces. These results can be generalized to the $8D$-dim phase space (and to higher dimensions). Caution must be taken in not confusing the proper force associated to a four-vector $F_\mu = \frac{dp_\mu}{d\tau}, \mu = 0, 1, 2, 3$ with the *spatial* force associated to a three-vector $\vec{f} = \frac{dp_i}{d\tau}, i = 1, 2, 3$.

If one restores c that was set to unity, the four-force has for components $F_\mu = (\frac{dE}{d\tau}, c\vec{f})$ where $\frac{dE}{d\tau}$ is the proper power. By maximal proper force one means that the magnitude-squared $|(dE/d\tau)^2 - c^2(dp_i/d\tau)^2| = |-F^2| = F^2 \leq b^2$ is bounded. However, this does *not* mean that the individual values of $(dE/d\tau)^2$ (square of the proper power) and $c^2(dp_i/d\tau)^2$ (square of the proper spatial force) are bounded. What is bounded is their $difference$ $|(dE/d\tau)^2 - c^2(dp_i/d\tau)^2| = F^2 \leq b^2$. For example, given the on-shell relation involving the energy-momentum $E^2 - c^2 p_i^2 = m^2 c^4$, this does not mean that each of the values of E^2, p_i^2 are bounded (they *blow* up when $v = c$). What is bounded is their difference (for a finite mass m).

The $U(1,1)$ group transformations involving the velocity and force boosts (along the x direction) acting on the phase-space coordinates x, t, p, E and which leave invariant the interval (1.2) were given by [3]

$$t' = t \cosh\xi + (\frac{\xi_v\, x}{c} + \frac{\xi_a\, p}{b}) \frac{\sinh\xi}{\xi} \tag{1.3a}$$

$$E' = E \cosh\xi + (-b\, \xi_a\, x + \xi_v pc) \frac{\sinh\xi}{\xi} \tag{1.3b}$$

$$x' = x \cosh\xi + (\xi_v\, ct - \frac{\xi_a\, E}{b}) \frac{\sinh\xi}{\xi} \tag{1.3c}$$

$$p' = p \cosh\xi + (\frac{\xi_v\, E}{c} + b\, \xi_a\, t) \frac{\sinh\xi}{\xi} \tag{1.3d}$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. The rapidity parameters ξ_a, ξ_v, ξ are defined, respectively, in terms of the spatial velocity $v = dx/dt$, and proper force $F = ma$, as follows

$$\tanh(\xi_v) = \frac{v}{c}; \quad \tanh(\xi_a) \; = \; \frac{F}{F_{max}}, \;\; F_{max} = b, \;\; \xi = \; \sqrt{(\xi_v)^2 + (\xi_a)^2} \quad (1.3e)$$

When $\xi_v \to \infty \Rightarrow v \to c$. And $\xi_a \to \infty \Rightarrow F \to F_{max} = b$.

The inverse transformations are obtained from eqs-(1.3) by *reversing* the signs of ξ_v, ξ_a, ξ and exchanging primed variables for unprimed ones : $(t, x, E, p) \leftrightarrow (t', x', E', p')$. After a very lengthy but straight-forward algebra one can verify that the transformations of eqs-(1.3) leave invariant the phase space interval $(cdt)^2 - (dx)^2 + ((dE)^2 - c^2(dp)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = c^2dt^2 - dx^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dp)^2]$. Only the *combination* below is truly left invariant under force (acceleration) boosts

$$(d\omega)^2 \; = \; (d\tau)^2 \left(\; 1 \; + \; \frac{\mathcal{F}^2}{F^2_{max}} \; \right) \; = \; (d\tau)^2 \left(\; 1 \; - \; \frac{F^2}{F^2_{max}} \; \right), \;\; F^2_{max} = b^2 \;\; (1.4)$$

where $\mathcal{F}^2 \equiv m^2a^2 = -F^2 < 0$, with $F^2 > 0$. The transformations of eqs-(1.3)) also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\, dt \wedge dE + dx \wedge dp$, see [3] for full details.

One of the many novel consequences of Non-inertial Relativity (Born Reciprocal Relativity) is the relativity of locality [11]. From eqs-(1.3) one concludes that $dt = dx = 0$ does *not* imply that $dt' = 0; dx' = 0$ due to the mixing of phase space coordinates. What appears to be a local event for one non-inertial observer is *no* longer local for another non-inertial observer.

Having reviewed the basics behind the idea of Non-inertial relativity, the outline of this work proceeds as follows. In section **2.1** we analyze the spatial-force formalism of Non-inertial Relativity versus the proper-force description. It is rigorously shown that the null-cone condition in phase space leads to the following bounds on the values of v and the Newtonian force $f_N = m\frac{dv}{dt}$ given by : $0 \leq |v| \leq c$, and $0 \leq |f_N| \leq b$. This occurs contrary to what one would expect from the algebraic equation of an elliptic-hyperboloid, and is due to the constraint $r = fv$ between the power $r = \frac{dE}{dt}$ and the spatial force $f = \frac{dp}{dt}$. One also finds that tachyonic solutions $v > c$ are not possible unless $f_N^2 < 0$, as expected, implying an imaginary mass $m^2 < 0$. The fact that $|f_N| \leq b$ is bounded requires clear modifications to the Newtonian potential [5].

In section **2.2** we provide a proper-force description of the phase space interval $(d\omega)^2$ and cite 7 different cases associated with : (i) $(d\omega)^2 > 0$; (ii) $(d\omega)^2 < 0$, and the null cone condition (iii) $(d\omega)^2 = 0$. In section **3.1** we display the Non-inertial relativistic transformations written in terms of f, v, r which were provided by Low [3]. In this case, the $U(1,3)$ transformations *mix* all of the phase space coordinates whose physical implication is that the *notion*

of "space-time" *varies* from one non-inertial observer to another. This does not occur in special (general) relativity. This is also related to the concept of relativity of locality [11].

In section **3.2** we explore the group contractions of the pseudo-unitary group $U(1,1)$ in the $b \to \infty$ limit. After working with *dimensionless* phase space coordinates we show that the entries of the 4×4 matrix operating in eqs-(1.3) and mapping the dimensionless phase variables in terms of the dimensionless rapidity parameters ξ_v, ξ_a, ξ, *does* lead to Lorentz boosts transformations, in the $b \to \infty$ limit, as expected.

If, instead, one works with *dimensionful* coordinates, one can easily verify that the limit $b \to \infty$ in eqs-(1.3) does *not* lead to the standard Lorentz boosts transformations for the energy-momentum variables. There are extra terms that are associated with the semi-direct product of the Lorentz group with an Abelian group. In section **3.3** a discussion is presented on zero-proper-forces and null cones.

In section **4** we examine the equivalence principle within the context of non-Milgromian modified Newtonian dynamics. In section **5** a thorough analysis is provided of how Non-inertial relativity points towards a space-time-matter unification, and leads to a connection to string theory. Finally in section **6** we present our conclusions.

2 Spatial-Force versus Proper-Force Description of Non-inertial Relativity

2.1 Spatial-force description of the Null Cone in Phase Space

One may factor out cdt instead of $d\tau$ in the phase space intervals $(d\omega)^2$. However one has to be careful in correctly defining the spatial force f and power in the relativistic case compared to the non-relativistic case. Given $p = mv\gamma(v)$ with $\gamma(v) = (1 - \frac{v^2}{c^2})^{-1/2}$ one finds

$$\frac{dp}{dt} \;=\; \frac{v^2}{c^2}(m\frac{dv}{dt})\;(1-\frac{v^2}{c^2})^{-3/2} \;+\; (m\frac{dv}{dt})\;(1-\frac{v^2}{c^2})^{-1/2} \tag{2.1}$$

Given that $m\frac{dv}{dt} = f_N$ is the Newtonian force, after some straightforward algebra one arrives at

$$\frac{dp}{dt} \;=\; f_N(1-\frac{v^2}{c^2})^{-3/2} \tag{2.2}$$

In the non-relativistic limit $c \to \infty \Rightarrow \;\frac{dp}{dt} \to m\frac{dv}{dt} = f_N$ one recovers the Newtonian definition of force f_N as expected.

Similarly, given $E = mc^2\gamma(v)$ one finds

$$\frac{dE}{dt} \;=\; (m\frac{dv}{dt})\;v\;(1-\frac{v^2}{c^2})^{-3/2} \;=\; f_N v(1-\frac{v^2}{c^2})^{-3/2} \tag{2.3}$$

In the non-relativistic limit $c \to \infty \Rightarrow \ \frac{dE}{dt} \to (m\frac{dv}{dt})v = f_N v$ one recovers the Newtonian definition of power. One may now define

$$\frac{dp}{dt} \ \equiv \ f \ = \ f_N(1-\frac{v^2}{c^2})^{-3/2)} \neq f_N; \ \ \frac{dE}{dt} \ \equiv \ r \ = \ f_N v(1-\frac{v^2}{c^2})^{-3/2} \ \neq \ f_N v \tag{2.4}$$

yielding the key relation $r = fv$, which also stems from the on-shell condition $E^2 - p^2c^2 = m^2c^4$. Differentiating leads to $EdE = pc^2dp \Rightarrow \frac{dE}{dt} = \frac{pc^2}{E}\frac{dp}{dt}$, giving $r = vf$ since $\frac{pc^2}{E} = v$.

Now we are in the position to factor out the c^2dt^2 from $(d\omega)^2$, leading to

$$(d\omega)^2 \ = \ c^2dt^2 \ \left(1-\frac{v^2}{c^2}+\frac{1}{b^2c^2}\frac{(dE)^2}{(dt)^2}-\frac{1}{b^2}\frac{(dp)^2}{(dt)^2}\right) \ = \\ c^2dt^2 \ \left(1-\frac{v^2}{c^2}+\frac{r^2}{b^2c^2}-\frac{f^2}{b^2}\right), \ \ r = \frac{dE}{dt}, \ \ f = \frac{dp}{dt} \tag{2.5}$$

Taking the square root yields the analog of time dilation

$$cdt \ = \ \gamma(v,r,f) \ d\omega; \ \ \gamma(v,r,f) \ \equiv \ \left(1-\frac{v^2}{c^2}+\frac{r^2}{b^2c^2}-\frac{f^2}{b^2}\right)^{-1/2} \tag{2.6}$$

The null cone is realized by setting $(d\omega)^2 = 0$. In doing so, one arrives at the algebraic equation for an elliptic-hyperboloid

$$1 \ = \ (\frac{v^2}{c^2}+\frac{f^2}{b^2}) \ - \ \frac{r^2}{b^2c^2} \tag{2.7}$$

By looking at the algebraic equation for an elliptic-hyperboloid one may conclude (erroneously) that there are *no* bounds to the values of v, f, r. This would be true if f and r were independent variables. However one must take into account that f and r are related by the condition $r = fv$.

Therefore, given the algebraic relations

$$1 \ + \ \frac{r^2}{b^2c^2} \ - \ \frac{f^2}{b^2} \ = \ \frac{v^2}{c^2}, \ \ r = fv \tag{2.8}$$

upon inserting the values of f and r obtained in eqs-(2.2,2.3), after some algebra one arrives at

$$\frac{v^2}{c^2} \ = \ 1 \ - \ \frac{f_N^2}{b^2} \ (1-\frac{v^2}{c^2})^{-2} \leq 1 \tag{2.9}$$

and one finds that the values of v are bounded by the speed of light. Eq-(2.9) automatically leads to an algebraic relation between $\frac{v}{c}$ and f_N of the form

$$\frac{f_N^2}{b^2} \ = \ (1-\frac{v^2}{c^2})^3 \tag{2.10}$$

and one finds that $v = c$ is only posible if $f_N = 0$, as expected.[2] When $v = 0 \Rightarrow f_N = b$. If $v > c \Rightarrow f_N^2 < 0$ implying an imaginary force, and in turn, an imaginary mass so that $m^2 < 0$, and which is consistent with a tachyon. If $v < c \Rightarrow |f_N| < b$. In order for $|f_N| > b$ it would require an imaginary velocity such that $v^2 < 0$. Our analysis and results above differ from the conclusions in [3].

Therefore, contrary to what one would expect from the algebraic equation of an elliptic-hyperboloid, due to the constraint $r = fv$, one encounters the following bounds to the values of v and f_N given by : $0 \leq |v| \leq c$, and $0 \leq |f_N| \leq b$. To conclude, the null cone condition in phase space does *not* imply that the velocity of the massive particle is given by the speed of light. On the contrary, one finds that the massive particle is *subluminal* $v < c$. One also finds that tachyonic solutions $v > c$ are not possible unless $f_N^2 < 0$. The fact that $|f_N| \leq b$ is bounded forces clear modifications to the Newtonian potential [5].

These results extend to higher space-time dimensions (higher dimensional phase spaces) simply by replacing $v^2 \to \vec{v}.\vec{v}$; $f^2 \to \vec{f}.\vec{f}$; $r^2 \to (\vec{f}.\vec{v})^2$ and leading to the null cone condition in a $2d$-dim phase space

$$1 \;=\; (\frac{|\vec{v}|^2}{c^2} + \frac{|\vec{f}|^2}{b^2}) \;-\; \frac{|\vec{f}.\vec{v}|^2}{b^2c^2} \tag{2.11}$$

When $\vec{f}$ is parallel to $\vec{v}$ one arrives at

$$\frac{|\vec{v}|^2}{c^2} \;=\; 1 \;-\; \frac{|\vec{f}_N|^2}{b^2}\;(1 - \frac{|\vec{v}|^2}{c^2})^{-2} \leq 1 \tag{2.12}$$

leading to the same results above.

And when $\vec{f}$ is orthogonal to $\vec{v}$ (circular motion) eq-(2.11) becomes

$$1 \;=\; \frac{|\vec{v}|^2}{c^2} + \frac{|\vec{f}|^2}{b^2}, \quad \frac{|\vec{f}.\vec{v}|^2}{b^2c^2} = 0 \tag{2.13}$$

such that $|\vec{v}|$ and $|\vec{f}|$ are clearly bounded by c and b, respectively.

2.2 Proper-force description of the Null Cone in Phase Space

After defining the square of the displacements in energy-momentum $(d\mu)^2 = (dE)^2 - (dp)^2$ $(c = 1)$, the $U(1,1)$ invariant interval in phase space can be rewritten as $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2$, and when $(d\tau)^2 \neq 0$, it allows the $factorization$ $(d\omega)^2 = (d\tau)^2[1 + \frac{1}{b^2}\frac{(d\mu)^2}{(d\tau)^2}]$. In section **1** it was shown that one always ends up with a factorization of the form $(d\omega)^2 = (d\tau)^2(1 - F^2/b^2)$ due to the fact that

[2]The choice $b = \infty$ is excluded. If $f_N = 0 \Rightarrow m = 0$, consistent with a massless particle moving at the speed of light. Having $m \neq 0$ but $a = 0$ is not physical because the massive particle would have an infinite amount of energy if it moves at the constant speed of light (with zero acceleration).

$\frac{(d\mu)^2}{(d\tau)^2} < 0$ is negative definite. The above factorization leads to the following 7 cases to explore :

Case **1** : The interval $(d\omega)^2 > 0$ (in phase space) leads to the following two sub-cases

$$\mathbf{1a} \ : \ (d\tau)^2 > 0, \ \ 1 - \frac{F^2}{b^2} > 0 \tag{2.14a}$$

and

$$\mathbf{1b} \ : \ (d\tau)^2 < 0, \ \ 1 - \frac{F^2}{b^2} < 0 \tag{2.14b}$$

The case **1b** must be disregarded because it implies that F is *larger* than b violating the maximal force postulate, in addition to having a superluminal particle (tachyon). Therefore, this leaves the case **1a** where the interval $(d\omega)^2 > 0$ has a correspondence with the special relativistic timelike interval $(d\tau)^2 > 0$ (subluminal velocities) and with the maximal force condition $F^2 < b^2$. As we have shown in [6] one can assure that force/acceleration boosts transformations do *not* lead to a crossover from case **1a** to the unphysical case **1b** in the case of hyperbolic trajectories.

Case **2** : The interval $(d\omega)^2 < 0$ (in phase space) leads to the following two sub-cases

$$\mathbf{2a} \ : \ (d\tau)^2 < 0, \ \ 1 - \frac{F^2}{b^2} > 0 \tag{2.15a}$$

and

$$\mathbf{2b} \ : \ (d\tau)^2 > 0, \ \ 1 - \frac{F^2}{b^2} < 0 \tag{2.15b}$$

The case **2a** involves the (spacetime) spacelike interval $(d\tau)^2 < 0$ corresponding to superluminal velocities, and to $F^2 < b^2$ obeying the maximal force postulate. Whereas, one finds in case **2b** that despite that $(d\tau)^2 > 0$ involving subluminal speeds, F^2 is *larger* than b^2 leading to a *violation* of the maximal force postulate.

Case **3a** : The null case $(d\omega)^2 = (d\tau)^2 + \frac{1}{b^2}(d\mu)^2 = 0$ with $(d\tau)^2 = 0$, and $(d\mu)^2 = 0$ corresponds to the null lines of a massless particle in space-time and in energy-momentum space. In this null case there is no need to factorize the $(d\tau)^2$ term.

Case **3b** : The null case $(d\omega)^2 = 0$ with $(d\tau)^2 > 0 \Rightarrow (d\omega)^2 = (d\tau)^2(1 - \frac{F^2}{b^2}) = 0 \Rightarrow F = b$ involves a subluminal particle experiencing the maximal proper force $F = F_{max} = b$.

Case **3c** : The null case $(d\omega)^2 = 0$ with $(d\tau)^2 < 0 \Rightarrow (d\omega)^2 = (d\tau)^2(1 - \frac{F^2}{b^2}) = 0 \Rightarrow F^2 = b^2$, involves a superluminal particle (tachyon) experiencing the maximal proper force. To sum up, out of all of these cases only three cases **1a**,**3a** ,**3b** are physically viable.

3 Non-inertial Relativistic Transformations written in terms of f, v, r

3.1 $U(1,1)$ Transformations that Mix All Phase Space Coordinates

We begin this section by summarizing the results of Low [3]. The phase space coordinate transformations leaving invariant the Born metric and the symplectic form can be written in terms of a 4×4 matrix Σ. The column matrix comprised of the phase space coordinates in the unprimed and primed frame of references are respectively given by

$$\mathbf{X} = (t,\ x)^T;\ \ \mathbf{P} = (E,\ p)^T;\ \ \mathbf{X}' = (t',\ x')^T;\ \ \mathbf{P}' = (E',\ p')^T \tag{3.1}$$

The 4×4 matrix Σ in turn can be decomposed in terms of the 2×2 Block matrices $\Lambda_1, \mathbf{M}_1, \Lambda_2, \mathbf{M}_2$ such that the new variables are obtained from the old ones by a linear transformation as shown by [3]

$$\begin{pmatrix} \mathbf{X}' \\ \mathbf{P}' \end{pmatrix} \ = \ \gamma(v,f,r) \begin{pmatrix} \Lambda_1 & \mathbf{M}_1 \\ \mathbf{M}_2 & \Lambda_2 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{P} \end{pmatrix} \tag{3.2}$$

where $\gamma(v,r,f) \equiv \left(1 - \frac{v^2}{c^2} + \frac{r^2}{b^2c^2} - \frac{f^2}{b^2}\right)^{-1/2}$.

The 2×2 block matrices are comprised of entries given in terms of the velocity v, force f, and power r parameters as follows [3]

$$\Lambda_1 \ = \ \begin{pmatrix} 1 & \frac{v}{c^2} \\ v & 1 \end{pmatrix};\ \ \Lambda_2 \ = \ \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix}, \tag{3.3}$$

$$\mathbf{M}_1 \ = \ \begin{pmatrix} -\frac{r}{b^2c^2} & \frac{f}{b^2} \\ -\frac{f}{b^2} & \frac{r}{b^2} \end{pmatrix},\ \ \mathbf{M}_2 \ = \ \begin{pmatrix} r & -f \\ f & -\frac{r}{c^2} \end{pmatrix} \tag{3.4}$$

The matrices Λ_1, Λ_2 encode the Lorentz transformations acting on the space-time and energy-momentum variables as usual. Whereas the matrix $\mathbf{M}$ encodes the force boost transformations which *mix* space-time variables with energy-momentum ones.

As a result of these most general transformations the space-time and energy-momentum variables are going to be *mixed*. Therefore, the notion of space-time is no longer an invariant concept in non-inertial relativity. Each non-inertial observer will experience $different$ space-times. This is one of the many novel consequences of non-inertial relativity (Born reciprocal Relativity). See [4] for a list of some of the novel physical consequences.

The symplectic matrix and the $2+2$ metric are respectively given by

$$\Omega \ = \ \begin{pmatrix} 0 & -\eta_{2\times 2} \\ \eta_{2\times 2} & 0 \end{pmatrix};\ \ \Upsilon \ = \ \begin{pmatrix} \eta_{2\times 2} & 0 \\ 0 & \eta_{2\times 2} \end{pmatrix} \tag{3.5}$$

where

$$\eta_{2\times 2} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.6}$$

The symplectic matrix and the metric are invariant under the transformations depicted in eqs-(3.2,3.3,3.4). One can verify that $\Sigma^T\Omega\Sigma = \Omega$; $\Sigma^T\Upsilon\Sigma = \Upsilon$, and as a result if one wishes, one can corroborate after very lengthy algebra that the infinitesimal displacements in phase space $(d\omega)^2$ and the symplectic two-form $-dt \wedge dE + \delta_{ij}dp^i \wedge dx^j$ are invariant under the above phase space coordinate transformations .

Given the 4×4 matrix $\Sigma(v, f, r)$ the inverse is simply obtained by reversing the signs of v, f, r : $\Sigma^{-1}(v, f, r) = \Sigma(-v, -f, -r)$ and which is consistent with the Relativity principle. One can verify that $\Sigma(v, f, r)\Sigma(-v, -f, -r) = \mathbf{1}_{4\times 4}$.

There is one *caveat*, however, due to the fact that r is not independent from f and v. Since $r = fv$, reversing the signs of f and v implies that r remains invariant and does not change sign. But this leads to an *inconsistency* because $\Sigma(-v, -f, r)$ is *no* longer the *inverse* of $\Sigma(v, f, r)$. One can verify by simple inspection that $\Sigma(v, f, r)\Sigma(-v, -f, r) \neq \mathbf{1}_{4\times 4}$.

One could try to remedy this problem by noticing that one can tweak the signs of the entries of the $\mathbf{M}_1, \mathbf{M}_2$ block matrices. Since there are 8 entries in total this leaves $2^8 = 256$ possible adjustments for the choices of Σ. One would need only one choice for Σ obeying : (i) $\Sigma^T\Omega\Sigma = \Omega$. (ii) $\Sigma^T\Upsilon\Sigma = \Upsilon$. (iii) And the desired condition that $\Sigma(-v, -f, r)$ is indeed the *inverse* of $\Sigma(v, f, r)$.

This caveat does *not* occur when works with *proper* forces. One can explicitly verify that by reversing the signs of the velocity, force boost and net rapidity parameters ξ_v, ξ_a, ξ appearing in eqs-(1.3) it leads to the inverse matrix, as expected, and one does not encounter the problem of whether or not r must change sign. In this proper-force formalism case, the 2 parameters (ξ_v, ξ_a) only are necessary. By inspection of eqs-(1.3) one also finds that the 4×4 transformation matrix $\Sigma(\xi_v, \xi_a, \xi)$ is symmetric. The inverse $\Sigma^{-1} = \Sigma(-\xi_v, -\xi_a, -\xi)$ is also symmetric.

One may notice also that the transformations in eqs-(1.3) involve the mixing of 3 coordinates instead of 4. For instance, the transformation $(dt)'$ in eq-(1.3a) does not involve dE (the canonically conjugate to dt). The transformation $(dE)'$ in eq-(1.3b) does not involve dt (the canonically conjugate to dE). The transformation $(dX)'$ in eq-(1.3c) does not involve dP (the canonically conjugate to dX). The transformation $(dP)'$ in eq-(1.3d) does not involve dX (the canonically conjugate to dP). A plausible physical reason behind this might rest on Heisenberg's uncertainty principle. For an extensive analysis of the symplectic relativistic canonical transformations leaving invariant the Weyl-Heisenberg algebra see [12].

3.2 Group Contractions in the $b \to \infty$ limit

Taking the $b \to \infty$ limit of the square of the infinitesimal displacement $(d\omega)^2$ in phase space (1.1) yields automatically the square of the space-time interval $(d\tau)^2$. In this subsection we shall explore further the behavior of the $U(1, 1)$

transformations in that limit, namely study its group contraction. In order to attain such contractions we shall be working with dimensionless and dimensionful variables. Equipped with the 3 physical constants $\hbar, b, c$ one may construct the following temporal, spatial, momentum and energy scales [3]

$$\lambda_t = \sqrt{\frac{\hbar}{bc}}, \quad \lambda_l = \sqrt{\frac{\hbar c}{b}}, \quad \lambda_p = \sqrt{\frac{\hbar b}{c}}, \quad \lambda_E = \sqrt{\hbar b c} \tag{3.8}$$

and use them to construct dimensionless phase space coordinates as follows

$$\frac{t}{\lambda_t}, \quad \frac{x}{\lambda_l}, \quad \frac{p}{\lambda_p}, \quad \frac{E}{\lambda_E} \tag{3.9}$$

These physical scales do not necessary coincide with the Planck time, Planck length, Planck momentum and Planck energy, respectively, unless b is constrained to be given by $b = \frac{c^4}{G}$. The most general case requires that b is proportional to $\frac{c^4}{G}$ [3].

The dimensionless entries of the column vectors encoding the phase space coordinates are now given by

$$\mathbf{X} = (\frac{t}{\lambda_t}, \ \frac{x}{\lambda_l})^T; \ \ \mathbf{P} = (\frac{E}{\lambda_E}, \ \frac{p}{\lambda_p})^T; \ \ \mathbf{X}' = (\frac{t'}{\lambda_t}, \ \frac{x'}{\lambda_l})^T; \ \ \mathbf{P}' = (\frac{E'}{\lambda_E}, \ \frac{p'}{\lambda_p})^T \tag{3.10}$$

The dimensionless entries of the 2×2 block matrices are

$$\Lambda_1 \ = \ \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix}; \ \ \Lambda_2 \ = \ \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix}, \tag{3.11}$$

$$\mathbf{M}_1 \ = \ \begin{pmatrix} -\frac{r}{bc} & \frac{f}{b} \\ -\frac{f}{b} & \frac{r}{bc} \end{pmatrix}, \ \ \mathbf{M}_2 \ = \ \begin{pmatrix} \frac{r}{bc} & -\frac{f}{b} \\ \frac{f}{b} & -\frac{r}{bc} \end{pmatrix} \tag{3.12}$$

In the $b \to \infty$ the 4×4 matrix Σ reduces to

$$\Sigma \ \to \ (1 - \frac{v^2}{c^2})^{-1/2} \ \begin{pmatrix} \Lambda_{2\times 2} & 0 \\ 0 & \Lambda_{2\times 2} \end{pmatrix}; \ \ \Lambda_{2\times 2} \ = \ \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix}, \tag{3.13}$$

and one recovers the matrix corresponding to Lorentz transformations (velocity boosts). However, here is a subtle point that one must address. When $b \to \infty$ one has

$$\frac{t}{\lambda_t} \to \infty; \quad \frac{x}{\lambda_l} \to \infty; \quad \frac{p}{\lambda_p} \to 0; \quad \frac{E}{\lambda_E} \to 0 \tag{3.14}$$

such that some of the entries in the column matrix describing the dimensionless coordinates are singular. This should not matter because in the left hand side one also has similar singular entries in the column matrix

$$\frac{t'}{\lambda_t} \to \infty; \quad \frac{x'}{\lambda_l} \to \infty; \quad \frac{p'}{\lambda_p} \to 0; \quad \frac{E'}{\lambda_E} \to 0 \tag{3.15}$$

The important fact is that the matrix (3.13) describing Lorentz boost transformations is *not* singular.

Matters are more complicated when one works with dimensionful variables. In this case, by simple inspection of the matrices in eqs-(3.3,3.4) one finds that the limit $b \to \infty$ does *not* lead to Lorentz transformations. Low [3] has shown that in this case the group contraction of $SU(1,n)$ leads to the semi-direct product of the Lorentz group $SO(1,n)$ with an Abelian group of dimension $[(n+1)(n+2)/2] - 1 = n(n+3)/2$. The contracted group is *not* the Poincare group.

One can also verify in the limit $b \to \infty$ that eqs-(1.3) do *not* lead to the standard Lorentz boosts transformations for the energy-momentum variables. Given $\tanh(\xi_a) = \frac{F}{b}$, in the $b \to \infty$ limit one will have $\xi_a \to \frac{F}{b} \to 0 \Rightarrow b\xi_a \to F$. And $\xi_v \to \xi$. In this case the transformations for the energy-momentum variables are modified as follows

$$E' \;=\; E\cosh(\xi_v) \;+\; pc\sinh(\xi_v) \;-\; Fx\;\frac{\sinh(\xi_v)}{\xi_v},$$

$$p' \;=\; p\cosh(\xi_v) \;+\; \frac{E}{c}\sinh(\xi_v) \;+\; Ft\;\frac{\sinh(\xi_v)}{\xi_v} \tag{3.16}$$

Whereas the transformations for the space-time variables do reduce to the standard Lorentz boosts ones

$$t' \;=\; t\cosh(\xi_v) \;+\; \frac{x}{c}\sinh(\xi_v), \quad x' \;=\; x\cosh(\xi_v) \;+\; ct\sinh(\xi_v) \tag{3.17}$$

The 4×4 matrix representing the transformations (3.16,3.17) can be written in terms of block 2×2 matrices as follows

$$\Lambda_1 \;=\; \begin{pmatrix} \cosh(\xi_v) & \frac{\sinh(\xi_v)}{c} \\ c\sinh(\xi_v) & \cosh(\xi_v) \end{pmatrix}; \;\; \Lambda_2 \;=\; \begin{pmatrix} \cosh(\xi_v) & c\sinh(\xi_v) \\ \frac{\sinh(\xi_v)}{c} & \cosh(\xi_v) \end{pmatrix}, \tag{3.18}$$

$$\mathbf{M}_1 \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \;\; \mathbf{M}_2 \;=\; \begin{pmatrix} 0 & -\frac{F\sinh(\xi_v)}{\sinh(\xi_v)} \\ \frac{F\sinh(\xi_v)}{\xi_v} & 0 \end{pmatrix} \tag{3.19}$$

If, and only if, one sets $c = 1$ the above 2×2 block matrices Λ_1, Λ_2 become equal

$$\Lambda_1 \;=\; \begin{pmatrix} \cosh(\xi_v) & \sinh(\xi_v) \\ \sinh(\xi_v) & \cosh(\xi_v) \end{pmatrix}; \;\; \Lambda_2 \;=\; \begin{pmatrix} \cosh(\xi_v) & \sinh(\xi_v) \\ \sinh(\xi_v) & \cosh(\xi_v) \end{pmatrix} \Rightarrow \Lambda_1 = \Lambda_2 = \Lambda \tag{3.20}$$

and the 4×4 matrix Σ can be *factorized* from the right as follows

$$\Sigma \;=\; \begin{pmatrix} \Lambda & 0 \\ \mathbf{M}_2 & \Lambda \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ \mathbf{M}_2\Lambda^{-1} & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \tag{3.21}$$

The inverse Λ^{-1} is just obtained by reversing the signs of ξ_v in eq-(3.20). The first matrix is lower triangular with unit diagonal entries and obeys the *Abelian* composition rule

$$\begin{pmatrix} 1 & 0 \\ \mathbf{A} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathbf{A}' & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathbf{A}+\mathbf{A}' & 1 \end{pmatrix} \tag{3.22}$$

with $\mathbf{A} = \mathbf{M}_2\Lambda^{-1}$, and $\mathbf{A}' = \mathbf{M}'_2\Lambda'^{-1}$ associated with F, ξ_v and F', ξ'_v, respectively.

The second matrix in the product decomposition in eq-(3.21) encodes the Lorentz boosts transformations. Therefore, when one sets $c = 1$ in eq-(3.18), the resulting 4×4 matrix Σ is associated with the semidirect product of the Lorentz with an Abelian group. Therefore, one arrives as similar conclusions as Low [3] in his description in terms of v, f, r. The $b \to \infty$ limit of the transformations in eqs-(1.3) leads to the semi-direct product of the Lorentz group with an Abelian group (not the Poincare group). The 4×4 matrix Σ can also be $factorized$ from the $left$ such that the lower left corner is now given by $\Lambda^{-1}\mathbf{M}_2$ instead.

Note that there is another subtlety worth remarking. If c is *not* set to unity one loses the Abelian character because one can no longer factorize the matrix in the form described by eq-(3.21) since in this case $\Lambda_1 \neq \Lambda_2$.

Rewriting eqs-(1.3) using dimensionless variables gives instead the following

$$\frac{t'}{\lambda_t} = \frac{t}{\lambda_t}\cosh\xi + (\xi_v \frac{x}{\lambda_l} + \xi_a \frac{P}{\lambda_p})\frac{\sinh\xi}{\xi} \tag{1.3a}'$$

$$\frac{E'}{\lambda_E} = \frac{E}{\lambda_E}\cosh\xi + (-\xi_a \frac{X}{\lambda_l} + \xi_v \frac{P}{\lambda_p})\frac{\sinh\xi}{\xi} \tag{1.3b}'$$

$$\frac{X'}{\lambda_l} = \frac{X}{\lambda_l}\cosh\xi + (\xi_v \frac{t}{\lambda_t} - \xi_a \frac{E}{\lambda_E})\frac{\sinh\xi}{\xi} \tag{1.3c}'$$

$$\frac{P'}{\lambda_p} = \frac{P}{\lambda_p}\cosh\xi + (\xi_v \frac{E}{\lambda_E} + \xi_a \frac{t}{\lambda_t})\frac{\sinh\xi}{\xi} \tag{1.3d}'$$

The entries of the 4×4 matrix which maps the dimensionless phase variables explicitly depend only on the dimensionless rapidity parameters ξ_v, ξ_a, ξ. Therefore in this case, the $b \to \infty$ limit does lead to the Lorentz transformations for all phase space coordinates because when $b = \infty \Rightarrow \xi_a = 0$, and $\xi_v \to \xi$, and one can verify that by setting $\xi_a = 0$ and $\xi = \xi_v$ in eqs-(1.3)', it automatically leads to the associated boosts Lorentz transformations for the space-time and energy-momentum variables in dimensionless form. For recent work on the contractions of the relativistic quantum LCT (linear canonical transformations) group and the emergence of spacetime symmetries see [13].

3.3 Null Proper Force and Null Cone

It is important to remind the reader that the proper forces involved in non-inertial relativity theory are constant (uniform) and compatible with the fact

that the force boost rapidity parameter ξ_a is a constant and related to F as follows $\tanh(\xi_a) = \frac{F}{b}$. However, this does not mean that the individual components of the four-force $(\mathcal{F}_0, \mathcal{F}_i); i = 1, 2, 3$ are themselves constant. What is constant is the proper force given by their norm $\sqrt{|\mathcal{F}_0^2 - \mathcal{F}_i^2|} = F$. As mentioned previously, the values of $\mathcal{F}_0, \mathcal{F}_i$ are not necessarily bounded, what is bounded is their norm $F \leq b$.

In the Galilean limit one has $F \to |\vec{f}|$ with $\vec{f} = \mathcal{F}_i = \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$. A constant F does not mean that each component of $\vec{f}$ is constant, what is constant is the norm $|\vec{f}|$. Hence to accommodate gravity in non-inertial relativity (involving forces with varying magnitudes/norms) one has to abandon the Galilean limit and focus instead on the full-fledged geometry of the $8D$-dim curved phase space and which was undertaken by [6]. This is a consistent with the fact that in non-inertial relativity theory spacetime and energy-momentum space are "entangled'.

Another important remark is that the proper force could be zero $F^2 = F_0^2 - F_1^2 = 0$ without F_0, F_1 necessarily being both zero, nor constant; i.e. F_0, F_1 could be even dynamical with the provision that $F^2 = F_0^2 - F_1^2 = 0$. In other words, a null force constrains the phase space interval to be equal to the space-time interval : $(d\omega)^2 = (d\tau)^2$, and the null cones in phase space and space-time match in this special case.

4 The Principle of Equivalence

We shall begin by briefly reviewing the notion of a zero-proper-force mass $\mathcal{M}$, not to be confused with m. A rigorous analysis of the novel notion of a "zero-proper-force" mass $\mathcal{M}$, which $differs$ from the rest mass m in Special Relativity, was provided in [6]. A particle can be at rest but still experience a force. The typical example is the harmonic oscillator. It is at rest $v = 0$ when it reaches its maximal amplitude of oscillation and where it experiences its maximal restoring force. The relationship between m and $\mathcal{M}$ is [6] : $m = m(F) = \frac{\mathcal{M}}{\sqrt{1-\frac{F^2}{b^2}}}$ such that $m(F = 0) = \mathcal{M}$, and allows the rewriting of the mass-energy expression in two different ways as follows

$$E \;=\; m(1 - \frac{v^2}{c^2})^{-1/2} \;=\; \mathcal{M}\frac{dt}{d\omega} \;=\; \mathcal{M}\frac{(dt/d\tau)}{\sqrt{1-\frac{F^2}{b^2}}} \;=\; \mathcal{M}\;\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\;\frac{1}{\sqrt{1-\frac{F^2}{b^2}}} \tag{4.1}$$

with m being the invariant rest mass (proper mass) of the particle. From eq-(4.1) one can infer that for a given $fixed$ energy E, one can have two interesting scenarios : (i) An increase in F is correlated to a decrease in v. (ii) And vice versa, a decrease in F is correlated to an increase in v. This is counter-intuitive in Special relativity but not in Non-inertial Relativity. Similar results can be found in [8] based on a maximal proper acceleration principle [7] and the geometry of jet bundles.

The relation between m and $\mathcal{M}$ was shown to be [6]

$$m \;=\; \frac{\mathcal{M}}{\sqrt{1-\frac{m^2g^2}{b^2}}} \;\Leftrightarrow\; \frac{m^2}{\mathcal{M}^2} \;=\; \frac{1-\sqrt{1\pm\frac{4\mathcal{M}^2g^2}{b^2}}}{\frac{2\mathcal{M}^2g^2}{b^2}} \tag{4.2}$$

The plus sign choice in front of the square root does *not* have a well behaved special relativistic limit. For this reason one must take the minus sign choice in front of the square root, giving

$$\frac{m^2}{\mathcal{M}^2} \;=\; \frac{1-\sqrt{1-\frac{4\mathcal{M}^2g^2}{b^2}}}{\frac{2\mathcal{M}^2g^2}{b^2}} \;\Rightarrow\; m \;=\; \mathcal{M}\;\left(\frac{1-\sqrt{1-\frac{4\mathcal{M}^2g^2}{b^2}}}{\frac{2\mathcal{M}^2g^2}{b^2}}\right)^{1/2} \;\Rightarrow$$

$$m \;=\; \mathcal{M}\;\Omega(\frac{\mathcal{M}g}{b}) \tag{4.3}$$

where the omega function Ω is defined by the expression involving the square root in (4.3).

In the Galilean limit $v << c = 1$, while keeping $b \neq \infty$, one finds that non-inertial relativity leads to a modification Newton's law of motion after writing

$$\vec{f} \;=\; \mathcal{M}\vec{a}\;\Omega(\frac{\mathcal{M}|\vec{a}|}{b}),\quad \frac{2\mathcal{M}|\vec{a}|}{b} \leq 1 \tag{4.4}$$

and which follows directly from the above equations simply by replacing $F \to |\vec{f}|$ and $\tilde{F} = \mathcal{M}g \to \mathcal{M}|\vec{a}|$ in the Galilean limit. If one wishes, one can perform the Taylor series expansion but is not necessary. It is important to emphasize that eq-(4.4) does *not* have the same functional form as Milgrom's modified Newton's law. The interpolating functions are $different$.

To be more precise, one can rewrite $\vec{f}$ in two different ways $\vec{f} = m\vec{a} = \mathcal{M}\vec{a}'$, and after inserting $\vec{f} = \mathcal{M}\vec{a}'$ into eq-(4.4), it gives $|\vec{a}'| = |\vec{a}|\Omega(\frac{\mathcal{M}|\vec{a}|}{b})$. Inverting the latter relation yields $|\vec{a}| = |\vec{a}'|\sqrt{1-(\frac{\mathcal{M}^2|\vec{a}'|^2}{b^2})}$. These relations between $\vec{a}'$ and $\vec{a}$ are analogous to the relations between the Milgromian a_M and Newtonian a_N accelerations [9]. Note, once more, that the former relations involves $different$ functions than the μ, ν interpolating functions relating the Milgromian a_M and Newtonian a_N accelerations [9] : $a_N = \mu(a_M/a_o)a_M \leftrightarrow a_M = \nu(a_o/a_N)a_N$. For this reason, one must emphasize that the Galilean limit of non-inertial relativity furnishes of a **non-Milgromian** MOND.

The reader may ask : despite this, is there still a role of Milgrom's acceleration constant a_o in all of this ? To answer this question one may notice that one could postulate the maximal proper force b to be given in terms of the Planck mass M_P, the Planck scale L_P; as well as the observable mass of the Universe M_U, and the Hubble Radius R_H, as follows $b = m_P(c^2/L_P) = M_U(c^2/R_H)$, and such that c^2/R_H is closer to Milgrom's acceleration constant $a_o \sim 1.2 \times 10^{-10} m/s^2$.

To finalize this section, due to the subtle difference between m and $\mathcal{M}$ (the zero proper-force mass), we should add some important remarks about the implications to the weak equivalence principle. In the second paragraph after eq-(4.4), we have shown that $|\vec{a}'| = |\vec{a}|\Omega(\frac{\mathcal{M}|\vec{a}|}{b})$ where the Ω function was defined in the first line of eq-(4.3). Inverting the latter relation yields $|\vec{a}| = |\vec{a}'|\sqrt{1-(\frac{\mathcal{M}^2|\vec{a}'|^2}{b^2}}$). This interplay between $\vec{a}$ and $\vec{a}'$ is important pertaining to whether or not the weak equivalence principle holds.

Given $|\vec{a}'| = |\vec{a}|\Omega(\frac{\mathcal{M}|\vec{a}|}{b})$, if $\vec{a}$ is independent[3] of $\mathcal{M}$ one finds that $\vec{a}'$ will acquire a dependence on $\mathcal{M}$. And conversely, given $|\vec{a}| = |\vec{a}'|\sqrt{1-(\frac{\mathcal{M}^2|\vec{a}'|^2}{b^2}}$), if $\vec{a}'$ is independent of $\mathcal{M}$, one finds that $\vec{a}$ will acquire a dependence on $\mathcal{M}$. Therefore, one finds that either $\vec{a}$ or $\vec{a}'$ are independent of $\mathcal{M}$ but not both simultaneously.[4] Therefore, the weak principle of equivalence holds for either $\vec{a}$ or $\vec{a}'$ but not for both. These subtleties will be explored in future work.

5 A Space-Time-Matter Unification

Special Relativity unified space and time establishing the equivalence between mass and energy. Next we shall argue why Non-inertial Relativity might provide the arena towards a space-time-matter unification and encapsulating the idea that matter curves space-time and space-time back reacts on matter curving energy-momentum space [6].

Introducing the Planck mass m_P, the zero-proper-force mass $\mathcal{M}$ obeys the relation

$$\frac{t^2}{\lambda_t^2} - \frac{x^2}{\lambda_l^2} + \frac{E^2}{\lambda_E^2} - \frac{p^2}{\lambda_p^2} = \frac{\mathcal{M}^2}{m_p^2} \tag{5.1}$$

Eq-(5.1) describes an elliptic-hyperboloid, and this case there are *no* bounds on the values of t, x, E, p. Even in the singular limits one has in the left hand side terms of the form $\infty - \infty$ that can still be equal to $\frac{\mathcal{M}^2}{m_p^2}$ in the right-hand side. In other words, in that case, eq-(5.1) becomes

$$\frac{t^2}{\lambda_t^2} - \frac{x^2}{\lambda_l^2} = \frac{\mathcal{M}^2}{m_p^2} \tag{5.2}$$

because $\frac{E^2}{\lambda_E^2} \to 0$; $\frac{p^2}{\lambda_p^2} \to 0$ when $\lambda_E = \lambda_p = \infty$. One may ask : what is the physical interpretation of eq-(5.2) obtained in the singular limit ? One finds a direct relationship among the space-time coordinates t, x and the zero-proper-force

[3]For example, the acceleration of a mass falling under the gravitational field of the earth is independent of the mass of the object and solely depends on the mass of the earth

[4]Given the relation $m(F) = \mathcal{M}(1-F^2/b^2)^{-1/2}$ between m and $\mathcal{M}$ found earlier, a variable proper force $F = F(\tau)$ would lead to a variable m and $\mathcal{M}$, in general, complicating matters. Namely, $m, \mathcal{M}$ would turn into scalar fields. Strings and branes with dynamical tensions have been studied by [15] over the years

mass $\mathcal{M}$. Consequently, in this singular case one can establish an "equivalence" between space-time and matter. Not unlike the equivalence between mass and energy in special relativity. One may note that despite that t and x both are variables, the $difference$ depicted in the left hand side of eq-(5.2) could still be constant. Not unlike the on-shell relation $E^2 - p^2c^2 = m^2c^4$.

A typical example of how the difference between two variable expressions in space-time could be constant is the case of a uniformly accelerated massive particle describing a hyperbolic motion. In $c = 1$ units, given the timelike hyperbolic trajectory

$$t(\tau) = \frac{1}{g}\sinh(g\tau), \quad x(\tau) = \frac{1}{g}\cosh(g\tau) \tag{5.3}$$

one arrives at

$$t^2 - x^2 \;=\; -\,\frac{1}{g^2}; \;\; (\frac{dt}{d\tau})^2 - (\frac{dx}{d\tau})^2 \;=\; 1; \;\; (\frac{d^2t}{d\tau^2})^2 - (\frac{d^2x}{d\tau^2})^2 = -\,g^2 \tag{5.4}$$

where the space-like proper acceleration g is constant (uniform).

Let us skip the singular limit case by not taking $b = \infty$, and focus in the massless particle case $m^2 = 0$, with $c = 1$,

$$\frac{E^2}{\lambda_E^2} - \frac{p^2}{\lambda_p^2} \;=\; 0 \;\Rightarrow\; \frac{1}{\lambda_E^2}\,(E^2 - p^2) \;=\; \frac{m^2}{\lambda_E^2} = 0 \Rightarrow m^2 = 0 \tag{5.5}$$

And given $m = 0$, from the $\mathcal{M}$-m relation (4.2), one has also that $\mathcal{M} = 0$, so that eq-(5.1) turns out to be

$$\frac{t^2}{\lambda_t^2} \;-\; \frac{x^2}{\lambda_l^2} \;=\; \frac{1}{\lambda_l^2}\,(t^2 - x^2) \;=\; \frac{\mathcal{M}^2}{m_p^2} \;= 0 \;\Rightarrow t^2 - x^2 = 0 \tag{5.6}$$

as expected. Therefore, one finds consistency between a null trajectory in space-time $t^2 = x^2 = 0$ and a massless $\mathcal{M} = m = 0$ particle.

In the most general case one learns that eq-(5.1) leads to

$$\frac{1}{\lambda_l^2}\,(t^2 \;-\; x^2) \;=\; \frac{\mathcal{M}^2}{m_p^2} \;-\; \frac{m^2}{\lambda_E^2} \;=\; \frac{m^2}{m_P^2}\,(1 - \frac{m^2g^2}{b^2}) \;-\; \frac{m^2}{\lambda_E^2} \tag{5.7}$$

after using the on-shell relation $E^2 - p^2 = m^2$ and eq-(4.2). Eq-(5.7) establishes a direct relation between the space-time coordinates in the left-hand side and the proper (rest mass) mass m of the particle in the right-hand side. For hyperbolic trajectories one has $t^2 - x^2 = -\frac{1}{g^2}$, hence eq-(5.7) becomes

$$-\,\frac{1}{\lambda_l^2}\,\frac{1}{g^2} \;=\; \frac{m^2}{m_P^2}\,(1 - \frac{m^2g^2}{b^2}) \;-\; \frac{m^2}{\lambda_E^2} \tag{5.8}$$

In the case when the maximal proper force is reached : $mg = b$; and when b is set to *coincide* with m_P^2, such that $\lambda_E = m_P$, and $\lambda_l = L_P = (m_P)^{-1}$ (in

$\hbar = c = 1$ units), one can explicitly verify that eq-(5.8) is obeyed. The left hand gives $-\frac{m_P^2}{g^2}$, and the right-hand side gives $-\frac{m^2}{m_P^2}$. After equating the left with the right hand side it yields $m_P^4 = m^2 g^2 \Rightarrow b^2 = m^2 g^2 \Rightarrow mg = b$, which is consistent with having chosen mg to coincide with the maximal proper force b, and when b is set to *coincide* with m_P^2.

Moreover, one can introduce a length scale $\mathcal{L}$, and the Planck length L_P, such that

$$\frac{\mathcal{M}^2}{m_p^2} = \frac{\mathcal{L}^2}{L_P^2} \tag{5.9}$$

and eq-(5.1) can be rewritten as

$$\frac{t^2}{\lambda_t^2} - \frac{x^2}{\lambda_l^2} + \frac{E^2}{\lambda_E^2} - \frac{p^2}{\lambda_p^2} = \frac{\mathcal{M}^2}{m_p^2} = \frac{\mathcal{L}^2}{L_P^2} \tag{5.10}$$

Eq-(5.10) has also a direct relation to the Relativistic Klein-Gordon harmonic oscillator equation [16], [17], whose "square-root" is just the equation for the Born-Dirac oscillator studied by [18] among others. In $\hbar = c = G = 1$ units, this is just a simple result of replacing $E \to i\frac{\partial}{\partial t}$; $p \to -i\frac{\partial}{\partial x}$ and such that eq-(5.10) becomes a differential equation (acting on Ψ) in *dimensionless* variables of the form

$$(-\partial^\mu \partial_\mu + x_\mu x^\mu)\ \Psi_\lambda(x^\nu) = \lambda\ \Psi_\lambda(x^\nu), \quad \lambda\Psi_\lambda \leftrightarrow \frac{\mathcal{M}^2}{m_p^2}\Psi_\lambda \tag{5.11}$$

The eigenvalues for the relativistic harmonic oscillator in d spatial dimensions and 1 temporal dimension are [16], [17]

$$\lambda = n_1 + n_2 + n_3 + \ldots + n_d - n_0 + \frac{d-1}{2} = N + \frac{d-1}{2} \tag{5.12}$$

For any given eigenvalue $\lambda = N + \frac{d-1}{2}$, with $N = 0, \pm 1, \pm 2, \pm 3, \ldots$ there is an infinite degeneracy of values $n_0, n_1, n_2, n_3, \ldots, n_d$. There are an infinite number of towers, parametrized by the eigenvalues λ, and with an infinite amount of degeneracy (an infinite number of states within each tower). This result is consistent with the fact that unitary irreducible representations of non-compact groups are infinite-dimensional. One of the key findings in [16] was in understanding that the symmetry properties of the solutions of eq-(5.11) were based precisely on the $U(1,d)$ group (or $U(d,1)$ group depending on the signature).

Bars [16] also found that two of the three Fock sectors which describe the relativistic harmonic oscillator involve Fock spaces that include negative norm ghosts (as in string theory) while the third one is completely free of ghosts. He discussed a gauge symmetry in a worldline theory approach that supplies appropriate constraints to remove all the ghosts from all Fock sectors of the single oscillator. The resulting ghost free quantum spectrum in $D = d + 1$ dimensions is then classified in unitary representations of the Lorentz group $SO(d,1)$. Moreover, all states of the single oscillator put together make up a

single infinite dimensional unitary representation of a hidden global symmetry $SU(d,1)$, whose Casimir eigenvalues are computed. Therefore, the spectrum λ has a direct connection to the ratio $\frac{\mathcal{M}^2}{m_P^2}$. Moreover, due to eq-(5.12) one arrives at a relation of the form $\frac{\mathcal{M}^2}{m_P^2} \sim N$, not unlike the Regge trajectories in the string spectrum $J \sim \alpha' m^2$, where α' is the inverse string tension. These findings point to the quantization of $\mathcal{M}^2$ and $\mathcal{L}^2$. It is not surprising that the relativistic oscillator comes into play. In section **4** we discussed the harmonic oscillator to explain why m is not the same as $\mathcal{M}$. The particle is at rest $v = 0$ when it reaches its maximal amplitude of oscillation and experiencing its maximal restoring force. We shall address these topics in future publications.

6 Concluding Remarks

In this work we have explored further the physical implications of Non-inertial Relativity. We analyzed the spatial-force formalism of Non-inertial Relativity versus the proper-force description. It was shown that contrary to what one would expect from the algebraic equation of an elliptic-hyperboloid, due to the constraint $r = fv$ between the power $r = \frac{dE}{dt}$ and the spatial force $f = \frac{dp}{dt}$, the null-cone condition in phase space leads to the following bounds on the values of v and the Newtonian force $f_N = m\frac{dv}{dt}$ given by : $0 \leq |v| \leq c$, and $0 \leq |f_N| \leq b$. The fact that $|f_N| \leq b$ is bounded requires clear modifications to the Newtonian potential [5].

We explored the group contractions of the pseudo-unitary group $U(1,1)$ in the $b \to \infty$ limit. After working with *dimensionless* phase space coordinates one found that it *does* lead to Lorentz boosts transformations, in the $b \to \infty$ limit, as expected. If, instead, one works with $dimensionful$ coordinates, the limit $b \to \infty$ does *not* lead to the standard Lorentz boosts transformations for the energy-momentum variables. There are extra terms that are associated with the semi-direct product of the Lorentz group with an Abelian group (which is not the Poincare group). Similar results were found by [3]. We reviewed the equivalence principle within the context of non-Milgromian modified Newtonian dynamics. And in the final section an analysis was provided of how Non-inertial relativity points towards a space-time-matter unification.

One of the main salient features of the theory is that the $U(1,d-1)$ transformations *mix* the phase space coordinates and whose physical implication is that the *notion* of "space-time" *varies* from one non-inertial observer to another. This does not occur in special (general) relativity. This is also related to the concept of relativity of locality [11].

Perhaps, more importantly, is the interplay of non-inertial relativity with thermal quantum field theory (TQFT). The Fulling-Davies-Unruh effect states that a uniformly accelerating observer experiences the vacuum state of a quantum field in Minkowski spacetime as a mixed state in thermodynamic equilibrium. Such mixed state is comprised of a thermal bath (warm gas) of Rindler particles whose temperature is proportional to the proper acceleration.

Given the expression of Unruh's temperature in terms of the proper acceleration $T = \frac{g}{2\pi}$, in natural units $\hbar = c = k_B = 1$, after substituting $g = 2\pi T$ in eqs-(4.2,4.3) it leads to a T-dependent mass $m = m(T)$, which resembles the renormalization process of the physical parameters in QFT in terms of the energy scale. Hence, given $m(g = 0) = m(T = 0) = \mathcal{M}$, by "switching" on the acceleration, the renormalized mass will flow from $\mathcal{M} \to m(g) = m(T)$.

The fusion of Special Relativity with Quantum Mechanics led to Quantum Field Theory (QFT). The fusion of Non-inertial relativity with Quantum Mechanics should lead to a novel formulation of Thermal Quantum Field Theory (TQFT), and in turn, should cast some light into the quantization of gravity and black hole thermodynamics.

The T-duality symmetry in string theory and Born's reciprocity allowed the authors [10] to construct the Metastring [10]. The symplectic group is essential in this construction. The Metastring [10] relies on the triple intersection of groups $Sp(2d) \cap O(2, 2d-2) \cap O(d, d) = U(1, d-1) \cap O(d, d) = O(1, d-1)$ which contains the Lorentz group $SO(1, d-1)$. It is warranted to explore further the Metastring within the context of Non-inertial Relativity.

The extension of Born Reciprocal Relativity (Non-inertial Relativity) to *curved* phase spaces was studied in [14] where the generalization of the Einstein-Hilbert action and the field equations were provided. It is required next to find some solutions in order to make specific experimental predictions. Another avenue of research worth exploring is the analog of the Carrollian limit $c \to 0$ in Non-inertial relativity : $b \to 0$. A lot of work remains ahead.

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