
The perimeter of an ellipse

Edgar Valdebenito

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ABSTRACT: In this note, we recall exact formulas for calculating the perimeter of an ellipse and give an example of an ellipse whose perimeter is π .

I. Introduction

The ellipse with semi-major axis a and semi-minor axis b ($b \leq a$) does not have a simple algebraic expression for its perimeter. The exact formula involves the complete elliptic integral of the second kind, $E(e)$, where e is the eccentricity, i.e.,

$$P = 4 a E(e) \quad (1)$$

where

$$E(e) = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta = \int_0^1 \sqrt{\frac{1 - e^2 x^2}{1 - x^2}} dx, \quad e = \sqrt{1 - \frac{b^2}{a^2}} \quad (2)$$

It is known that $E(k)$ can be expressed in terms of the Gauss hypergeometric function:

$$E(k) = \frac{\pi}{2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, k^2\right) \quad (3)$$

See [10, p.861, Eq.8.114.1], [19, p.490, Eq.19.5.2].

II. Some exact formulas for the perimeter of an ellipse

Let a and b be the semi-major and semi-minor axes of an ellipse with perimeter P and whose eccentricity is $e = \sqrt{1 - \frac{b^2}{a^2}}$.

Entry 1. we have

$$P = 2 \pi a {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, 1 - \frac{b^2}{a^2}\right) \quad (4)$$

$$P = \pi (a + b) {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}, 1, \left(\frac{a - b}{a + b}\right)^2\right) = \pi (a + b) {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}, 1, 1 - \frac{4ab}{(a + b)^2}\right) \quad (5)$$

$$P = \pi \sqrt{2(a^2 + b^2)} {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}, 1, \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2\right) \quad (6)$$

$$P = 4a \left(1 + \sum_{n=0}^{\infty} 2^{-4n} \binom{2n}{n}^2 \left(\frac{2n+1}{2n+2}\right) \left(\frac{b}{a}\right)^{2n+2} \left(\ln\left(\frac{4a}{b}\right) - \left(\sum_{k=1}^n \frac{2}{(2k-1)(2k)}\right) - \frac{1}{(2n+1)(2n+2)}\right)\right) \quad (7)$$

$$P = \frac{16\pi a^2 b^2}{(a+b)^3} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}, 1, 1 - \frac{4ab}{(a+b)^2}\right) \quad (8)$$

$$P = 4a\pi \sum_{n=0}^{\infty} \frac{2^{-8n}}{1-4n} \binom{4n}{2n}^2 \left(1 - \frac{b^2}{a^2}\right)^{2n} - 2\pi \sqrt{2a^2 - b^2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2 - b^2}{2a^2 - b^2}\right) \quad (9)$$

$$P = 2\pi \sqrt{2a^2 - b^2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2 - b^2}{2a^2 - b^2}\right) - a \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{2^{-8n}}{4n+1} \binom{4n+2}{2n+1}^2 \left(1 - \frac{b^2}{a^2}\right)^{2n+1} \quad (10)$$

$$P = 4a\pi {}_4F_3\left(\begin{matrix} -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \\ \frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \middle| \left(1 - \frac{b^2}{a^2}\right)^2\right) - 2\pi \sqrt{2a^2 - b^2} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2 - b^2}{2a^2 - b^2}\right) \quad (11)$$

$$P = 2\pi \sqrt{ab} \left({}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, -\frac{(a-b)^2}{4ab}\right) - {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, -\frac{(a-b)^2}{4ab}\right) \right) \quad (12)$$

$$P = \sqrt{2(a^2 + b^2)} \sum_{n=0}^{\infty} \frac{2^{-2n}}{1-2n} \binom{2n}{n} \left(\left(\frac{a^2}{a^2 + b^2} \right)^n + \left(\frac{b^2}{a^2 + b^2} \right)^n \right) \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} {}_2F_1\left(1, \frac{3}{2}, k + \frac{3}{2}, \frac{1}{2}\right) \quad (13)$$

$$P = 4 \left(\frac{2}{3} \right)^{3/2} \sqrt{a^2 + b^2} \sum_{n=0}^{\infty} \binom{2n}{n} 12^{-n} (2n+1) \sum_{k=0}^n \frac{(-2)^k}{2k+1} \binom{n}{k} \left({}_2F_1\left(-\frac{1}{2}, 1, k + \frac{3}{2}, \frac{a^2}{a^2 + b^2}\right) + {}_2F_1\left(-\frac{1}{2}, 1, k + \frac{3}{2}, \frac{b^2}{a^2 + b^2}\right) \right) \quad (14)$$

$$P = 4a \sqrt{2 \left(1 + \sqrt{1 - \frac{b^2}{a^2}} \right)} \quad (15)$$

$$\sum_{n=0}^{\infty} \frac{2^{-3n}}{2n+1} \binom{2n}{n} {}_2F_1\left(-\frac{1}{2}, 1, n + \frac{3}{2}, \sqrt{1 - \frac{b^2}{a^2}}\right) {}_2F_1\left(-\frac{1}{2}, -n, \frac{1}{2} - n, \frac{2\sqrt{a^2 - b^2}}{a + \sqrt{a^2 - b^2}}\right)$$

$$P = \pi \sqrt{2(a^2 + b^2)} \sum_{n=0}^{\infty} \binom{2n}{n} \binom{4n}{2n} \frac{2^{-6n}}{1-4n} \left(\frac{a^2 - b^2}{a^2 + b^2} \right)^{2n} \quad (16)$$

$$P = \pi \sqrt{2(a^2 + b^2)} \sum_{n=0}^{\infty} \binom{2n}{n} \binom{4n}{2n} \frac{(-1)^n 2^{-6n}}{1-4n} \left(\frac{a^2 - b^2}{a^2 + b^2} \right)^{2n} - \quad (17)$$

$$2\pi \sqrt{2(a^2 + b^2)} \sum_{n=0}^{\infty} \binom{4n+2}{2n+1} \binom{8n+4}{4n+2} \frac{2^{-12n-6}}{8n+3} \left(\frac{a^2 - b^2}{a^2 + b^2} \right)^{4n+2}$$

$$P = \pi \sqrt{a^2 - b^2 + \sqrt{a^4 + 6a^2b^2 + b^4}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2a^2}{a^2 + b^2 + \sqrt{a^4 + 6a^2b^2 + b^4}} \right)^n$$

$$\sum_{k=0}^n \frac{(-1/2)_{n-k} (n-k - \frac{1}{2})_k}{(n-k)! k!} \binom{2k}{k} \left(\frac{b^2 (a^2 + b^2 + \sqrt{a^4 + 6a^2b^2 + b^4})}{2a^2 (a^2 - b^2 + \sqrt{a^4 + 6a^2b^2 + b^4})} \right)^k \quad (18)$$

$${}_2F_1\left(\frac{1}{2} + k, k - n, 1 + k, 2 \left(\frac{a^2 + b^2 + \sqrt{a^4 + 6a^2b^2 + b^4}}{a^2 - b^2 + \sqrt{a^4 + 6a^2b^2 + b^4}} \right)\right)$$

$$P = 8\sqrt{2} a \sum_{n=0}^{\infty} 2^{-3n} \sum_{k=0}^n \frac{2^{5k}}{1-4k^2} \left(1 - \frac{b^2}{a^2}\right)^k \frac{\binom{2k}{k} \binom{2n-2k}{n-k}^2}{\binom{2n+2k+2}{n+k+1} \binom{n+k+1}{n-k}} = \quad (19)$$

$$\begin{aligned}
& 4 \sqrt{2} a \sum_{n=0}^{\infty} \frac{2^{-3n}}{2n+1} \binom{2n}{n} {}_4F_3 \left(\begin{matrix} -\frac{1}{2}, \frac{1}{2}, 1, -n \\ \frac{1}{2} - n, \frac{1}{2} - n, \frac{3}{2} + n \end{matrix} \middle| -8 \left(1 - \frac{b^2}{a^2} \right) \right) \\
P = & 2 \pi a {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 1, 1 - \frac{b^2}{a^2} \right) - \frac{4a}{3} \left(1 - \frac{b^2}{a^2} \right) {}_F_1 \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, 1, 1 - \frac{b^2}{a^2} \right)
\end{aligned} \tag{20}$$

$$\begin{aligned}
P = & 4a \sqrt[4]{1 - \frac{b^2}{a^2}} {}_F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \sqrt{1 - \frac{b^2}{a^2}}, \left(1 - \frac{b^2}{a^2} \right)^{3/2} \right) - \\
& 4b \sqrt{1 - \sqrt{1 - \frac{b^2}{a^2}}} {}_F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, 1 - \sqrt{1 - \frac{b^2}{a^2}}, -\frac{1 - \frac{b^2}{a^2}}{1 + \sqrt{1 - \frac{b^2}{a^2}}} \right)
\end{aligned} \tag{21}$$

$$P = 4a \sqrt{2 \left(1 + \sqrt{1 - \frac{b^2}{a^2}} \right)} \sum_{n=0}^{\infty} \frac{\left(1 - \frac{b^2}{a^2} \right)^{n/2}}{1 - 4n^2} {}_F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, n + \frac{3}{2}, \frac{1}{2}, \frac{\sqrt{1 - \frac{b^2}{a^2}}}{1 + \sqrt{1 - \frac{b^2}{a^2}}} \right) \tag{22}$$

$$P = \sqrt{2(a^2 + b^2)} {}_F_1 \left(1, \frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{a^2}{a^2 + b^2} \right) + \sqrt{2(a^2 + b^2)} {}_F_1 \left(1, \frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{b^2}{a^2 + b^2} \right) \tag{23}$$

$$\begin{aligned}
P = & 2 \sqrt{2} a {}_F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \left(1 - \frac{b^2}{a^2} \right) \right) + \\
& \frac{4a^2}{\sqrt{a^2 + b^2}} {}_F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{a^2}{a^2 + b^2}, \frac{a^2 - b^2}{a^2 + b^2} \right) - \frac{2 \sqrt{2} (a^2 - b^2)}{\sqrt{a^2 + b^2}}
\end{aligned} \tag{24}$$

$$P = 8a \sqrt{\frac{a}{a+b}} {}_F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{a}{a+b}, \frac{a-b}{a} \right) - 4(a-b) \tag{25}$$

$$P = 4a \sum_{n=0}^{\infty} \frac{1}{1 - 4n^2} \left(\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{b^2}{a^2}} \right)^n {}_F_1 \left(n+1, 2, -\frac{1}{2}, n + \frac{3}{2}, \frac{1}{2}, \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{b^2}{a^2}} \right) \tag{26}$$

$$P = \frac{2\pi \sqrt{2} a^2}{\sqrt{a^2 + b^2}} \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} 2^{-2n} \left(\frac{a^2 - b^2}{a^2 + b^2} \right)^n {}_2F_1 \left(-n, -\frac{1}{2}, 1, 2 \right) \tag{27}$$

$$P = \pi \sqrt{2(a^2 + b^2)} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n}}{1 - 2n} \binom{2n}{n} \left(\frac{a^2 - b^2}{a^2 + b^2} \right)^n {}_2F_1 \left(-n, \frac{1}{2}, 1, 2 \right) \tag{28}$$

$$P = \frac{2\pi b^2}{a} \sum_{n=0}^{\infty} \binom{2n}{n}^2 2^{-4n} (2n+1) \left(1 - \frac{b^2}{a^2} \right)^n \tag{29}$$

$$P = \frac{4 \sqrt{2} b^2}{a} \sum_{n=0}^{\infty} \frac{2^{-3n}}{2n+1} \binom{2n}{n} {}_F_1 \left(n + \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, n + \frac{3}{2}, \frac{i \sqrt{a^2 - b^2}}{b + i \sqrt{a^2 - b^2}}, -\frac{i \sqrt{a^2 - b^2}}{b - i \sqrt{a^2 - b^2}} \right) \tag{30}$$

$$P = \frac{\pi (a+b)^2}{2a} \sum_{n=0}^{\infty} \left(\frac{a-b}{2a} \right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{n-2k} \left(\frac{2^{-2k}}{1-2k} \binom{2k}{k} \right)^2 \tag{31}$$

$$P = \frac{\pi (a+b)^2}{2a} \sum_{n=0}^{\infty} \left(\frac{a-b}{2a} \right)^{2n} \left(g(n) + \left(\frac{a-b}{2a} \right) h(n) \right) \tag{32}$$

where

$$g(n) = {}_4F_3 \left(\begin{matrix} -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} - n, -n \\ \frac{1}{2}, 1, 1 \end{matrix} \middle| 1 \right) \quad (33)$$

$$h(n) = {}_4F_3 \left(\begin{matrix} -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} - n, -n \\ \frac{1}{2}, 1, 1 \end{matrix} \middle| 1 \right) \quad (34)$$

$$P = \frac{\pi (a^2 + b^2)^{3/2}}{\sqrt{2} a^2} \sum_{n=0}^{\infty} \left(\frac{a^2 - b^2}{2 a^2} \right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{n-2k} \left(\frac{(-1/4)_k (1/4)_k}{(k!)^2} \right) \quad (35)$$

$$P = \frac{\pi (a^2 + b^2)^{3/2}}{\sqrt{2} a^2} \sum_{n=0}^{\infty} \left(\frac{a^2 - b^2}{2 a^2} \right)^{2n} \left(g(n) + \left(\frac{a^2 - b^2}{2 a^2} \right) h(n) \right) \quad (36)$$

where

$$g(n) = {}_4F_3 \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4}, \frac{1}{2} - n, -n \\ \frac{1}{2}, 1, 1 \end{matrix} \middle| 1 \right) \quad (37)$$

$$h(n) = {}_4F_3 \left(\begin{matrix} -\frac{1}{2}, \frac{1}{4}, -\frac{1}{2} - n, -n \\ \frac{1}{2}, 1, 1 \end{matrix} \middle| 1 \right) \quad (38)$$

$$P = 4a + \frac{b^2}{2a} \ln \left(\frac{4a}{b} \right) \left({}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 2, \frac{b^2}{a^2} \right) + \frac{b^2}{a^2} {}_2F_1 \left(\frac{3}{2}, \frac{3}{2}, 3, \frac{b^2}{a^2} \right) \right) - \quad (39)$$

$$4a \left({}_2F_1 \left(-\frac{1}{2}, -\frac{1}{2}, 1, \frac{b^2}{a^2} \right) - 1 \right) - 2a \sum_{n=1}^{\infty} \frac{2^{-4n} (2n+1)}{n+1} \binom{2n}{n}^2 \left(\frac{b}{a} \right)^{2n+2} \sum_{k=1}^n \frac{1}{k(2k-1)}$$

$$P = \frac{2\pi a^2}{\sqrt{2a^2 - b^2}} \sum_{n=0}^{\infty} \frac{2^{-6n}}{1-2n} \binom{2n}{n} \binom{4n}{2n} \left(1 - \frac{b^2}{a^2} \right)^{2n} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 2n+1, \frac{a^2 - b^2}{2a^2 - b^2} \right) \quad (40)$$

$$P = 2\pi a {}_2F_1 \left(-\frac{1}{2}, -\frac{1}{2}, 1, 1 - \frac{b^2}{a^2} \right) - \pi a \left(1 - \frac{b^2}{a^2} \right) {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 2, 1 - \frac{b^2}{a^2} \right) \quad (41)$$

$$P = 2\pi a \left(1 - \sqrt{1 - \frac{b^2}{a^2}} \tanh^{-1} \left(\frac{1}{4} \sqrt{1 - \frac{b^2}{a^2}} \right) - \sum_{n=2}^{\infty} \left(\binom{2n}{n}^2 - \binom{2n-2}{n-1}^2 \right) \left(\frac{1}{4} \sqrt{1 - \frac{b^2}{a^2}} \tanh^{-1} \left(\frac{1}{4} \sqrt{1 - \frac{b^2}{a^2}} \right) - \sum_{k=1}^{n-1} \frac{2^{-4k}}{2k-1} \left(1 - \frac{b^2}{a^2} \right)^k \right) \right) \quad (42)$$

$$P = \pi \sqrt{2(a^2 + b^2)} \sum_{n=0}^{\infty} \frac{2^{-2n}}{1-2n} \binom{2n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \left(\frac{a^2}{2(a^2 + b^2)} \right)^k {}_2F_1 \left(\frac{1}{2}, -k, \frac{1}{2} - k, \frac{b^2}{a^2} \right) \quad (43)$$

$$P = \pi \sqrt{2(a^2 + b^2)} \sum_{n=0}^{\infty} \frac{2^{-2n}}{1-2n} \binom{2n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \left(\frac{a^2 - b^2}{2(a^2 + b^2)} \right)^k {}_2F_1 \left(-k, -k, \frac{1}{2} - k, -\frac{b^2}{a^2 - b^2} \right) \quad (44)$$

$$P = \pi \sqrt[4]{2((a^2 + b^2)^2 + 4a^2 b^2)} \sum_{n=0}^{\infty} \frac{(-1)^n (-1/4)_n}{n!} \left(\frac{(a^2 - b^2)^2}{(a^2 + b^2)^2 + 4a^2 b^2} \right)^n {}_2F_1 \left(-n, \frac{1}{4}, 1, 2 \right) \quad (45)$$

$$P = \frac{\pi \sqrt[4]{8} (a^2 + b^2)}{\sqrt{(a^2 + b^2)^2 + 4a^2 b^2}} \sum_{n=0}^{\infty} \frac{(-1)^n (1/4)_n}{n!} \left(\frac{(a^2 - b^2)^2}{(a^2 + b^2)^2 + 4a^2 b^2} \right)^n {}_2F_1 \left(-n, -\frac{1}{4}, 1, 2 \right) \quad (46)$$

Entry 2. for $1 < \frac{a}{b} < \sqrt{2}$ we have

$$P = \frac{2\pi a^2}{b} \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n}^2 2^{-4n} (2n+1) \left(\frac{a^2}{b^2} - 1 \right)^n \quad (47)$$

Entry 3. for $0 < \lambda < 1$ we have

$$P = 4a \lambda F_1 \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \lambda^2, \lambda^2 \left(1 - \frac{b^2}{a^2} \right) \right) + 4(1-\lambda) \sqrt{\frac{a^2 - \lambda^2(a^2 - b^2)}{1 - \lambda^2}} + 4a \int_a^{\infty} \frac{\sqrt{\frac{a^2 - \lambda^2(a^2 - b^2)}{1 - \lambda^2}}}{\sqrt{x^2 - \left(1 - \frac{b^2}{a^2}\right)}} \left(1 - \sqrt{\frac{x^2 - 1}{x^2 - \left(1 - \frac{b^2}{a^2}\right)}} \right) dx \quad (48)$$

Entry 4. for $0 < \lambda < 1$ we have

$$P = 4a \sum_{n=0}^{\infty} \frac{2^{-2n} \lambda^{2n+1}}{2n+1} \binom{2n}{n} F_1 \left(2n+1, -\frac{1}{2}, -\frac{1}{2}, 2n+2, \lambda \sqrt{1 - \frac{b^2}{a^2}}, -\lambda \sqrt{1 - \frac{b^2}{a^2}} \right) + 4a \sqrt{1 - \lambda^2} \sqrt{1 - \lambda^2 \left(1 - \frac{b^2}{a^2} \right)} \sum_{n=0}^{\infty} \frac{2^{-2n} (1 - \lambda^2)^n}{2n+1} \binom{2n}{n} {}_2F_1 \left(-\frac{1}{2}, 1, n + \frac{3}{2}, \frac{(1 - \lambda^2)(a^2 - b^2)}{a^2 - \lambda^2(a^2 - b^2)} \right) \quad (49)$$

Entry 5. for $a > b \sqrt{2}$ we have

$$P = 2a \sqrt{\frac{a^2 + 2b^2}{a^2 - b^2}} \sum_{n=0}^{\infty} \frac{2^{-2n}}{2n+1} \binom{2n}{n} \left(\frac{a^2}{2(a^2 - b^2)} \right)^n {}_2F_1 \left(-\frac{1}{2}, 1, n + \frac{3}{2}, \frac{a^2}{a^2 + 2b^2} \right) + 2\sqrt{2(a^2 - 2b^2)} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n}}{1 - 2n} \binom{2n}{n} \left(\frac{b^2}{a^2 - b^2} \right)^n {}_2F_1 \left(n, \frac{1}{2}, \frac{3}{2}, \frac{a^2 - 2b^2}{2(a^2 - b^2)} \right) \quad (50)$$

Entry 6. for $0 < \lambda < 1/2$ we have

$$P = 2\pi a \sqrt{1 - \lambda} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n}}{1 - 2n} \binom{2n}{n} \left(\frac{\lambda}{1 - \lambda} \right)^n {}_2F_1 \left(\frac{1}{2}, -n, 1, \frac{1}{\lambda} \left(1 - \frac{b^2}{a^2} \right) \right) \quad (51)$$

Entry 7. for $0 < \lambda < \frac{a}{\sqrt{a^2 - b^2}} - 1$ we have

$$P = 4\sqrt{2} b \sqrt{\lambda} \sum_{n=0}^{\infty} \frac{2^{-3n} \lambda^n}{2n+1} \binom{2n}{n} F_1 \left(n + \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, n + \frac{3}{2}, -\frac{\lambda \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}, \frac{\lambda \sqrt{a^2 - b^2}}{a + \sqrt{a^2 - b^2}} \right) + 4a \sum_{n=0}^{\infty} \frac{2^{-2n} (1 - \lambda)^{2n+1}}{2n+1} \binom{2n}{n} {}_2F_1 \left(-\frac{1}{2}, -n, \frac{1}{2} - n, 1 - \frac{b^2}{a^2} \right) \quad (52)$$

Entry 8. for $\sqrt{2} b < a$ we have

$$P = 4a \sqrt{1 - \frac{b^2}{a^2}} - a \sqrt{1 - \frac{b^2}{a^2}} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-4n}}{n+1} \binom{2n}{n}^2 \left(\frac{b^2}{a^2 - b^2} \right)^{n+1} \left(\ln \left(\frac{b^2}{16(a^2 - b^2)} \right) - \frac{1}{n+1} + \sum_{k=1}^n \frac{2}{k(2k-1)} \right) \quad (53)$$

Entry 9. for $0 < s < 1$ we have

$$P = 4a \sum_{n=0}^{\infty} \frac{2^{-2n} s^{2n+1}}{2n+1} \binom{2n}{n} {}_2F_1 \left(-\frac{1}{2}, -n, \frac{1}{2} - n, 1 - \frac{b^2}{a^2} \right) + 4 \int_0^{\frac{\pi}{2} - \sin^{-1}(s)} \sqrt{a^2 (\sin x)^2 + b^2 (\cos x)^2} dx \quad (54)$$

$$P = 4a \sum_{n=0}^{\infty} \frac{2^{-2n} s^{2n+1}}{2n+1} \left(\frac{b}{a}\right)^{2n} \binom{2n}{n} {}_2F_1\left(-n, 1-n, \frac{1}{2}-n, 1-\frac{a^2}{b^2}\right) + 4 \int_0^{\frac{\pi}{2} - \sin^{-1}(s)} \sqrt{a^2 (\sin x)^2 + b^2 (\cos x)^2} dx \quad (55)$$

examples:

$$P = 2a \sum_{n=0}^{\infty} \frac{2^{-4n}}{2n+1} \binom{2n}{n} {}_2F_1\left(-\frac{1}{2}, -n, \frac{1}{2}-n, 1-\frac{b^2}{a^2}\right) + 4 \int_0^{\frac{\pi}{3}} \sqrt{a^2 (\sin x)^2 + b^2 (\cos x)^2} dx \quad (56)$$

$$P = 2\sqrt{2}a \sum_{n=0}^{\infty} \frac{2^{-3n}}{2n+1} \binom{2n}{n} {}_2F_1\left(-\frac{1}{2}, -n, \frac{1}{2}-n, 1-\frac{b^2}{a^2}\right) + 4 \int_0^{\frac{\pi}{4}} \sqrt{a^2 (\sin x)^2 + b^2 (\cos x)^2} dx \quad (57)$$

$$P = 2\sqrt{3}a \sum_{n=0}^{\infty} \frac{2^{-4n} \cdot 3^n}{2n+1} \binom{2n}{n} {}_2F_1\left(-\frac{1}{2}, -n, \frac{1}{2}-n, 1-\frac{b^2}{a^2}\right) + 4 \int_0^{\frac{\pi}{6}} \sqrt{a^2 (\sin x)^2 + b^2 (\cos x)^2} dx \quad (58)$$

Entry 10.

$$P = 4 \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{\left(\frac{a}{n}\right)^2 + b^2 \left(\sqrt{1 - \left(\frac{k}{n}\right)^2} - \sqrt{1 - \left(\frac{k-1}{n}\right)^2}\right)^2} \quad (59)$$

Entry 11.

$$P = \pi(a+b) \sum_{n=0}^{\infty} \frac{2^{-2n}}{1-2n} \binom{2n}{n} \left(\frac{a-b}{a+b}\right)^n \sum_{k=0}^n \binom{n}{k} \sum_{m=0}^k (-1)^m \left(\frac{a-b}{a+b}\right)^m \binom{k}{m} f(n, k, m) \quad (60)$$

where

$$f(n, k, m) = \begin{cases} 1 & m = 2k - n \\ 0 & m \neq 2k - n \end{cases} \quad (61)$$

Entry 12.

$$P = \pi \sum_{n=0}^{\infty} 2^{-2n} \left(\frac{a-b}{a+b}\right)^n \sum_{k=0}^n \binom{2n-2k}{n-k} \binom{2k}{k} \left(\frac{2(a^2+b^2)}{a+b} f(n, k) - (a-b)(g(n, k) + h(n, k))\right) \quad (62)$$

where

$$f(n, k) = \begin{cases} 1 & n = 2k \\ 0 & n \neq 2k \end{cases}, g(n, k) = \begin{cases} 1 & n = 2k - 1 \\ 0 & n \neq 2k - 1 \end{cases}, h(n, k) = \begin{cases} 1 & n = 2k + 1 \\ 0 & n \neq 2k + 1 \end{cases} \quad (63)$$

Entry 13. for $0 < \lambda < 1 - \frac{b^2}{a^2}$ we have

$$P = 2\pi a {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, \lambda\right) - 2\pi a \sum_{n=1}^{\infty} \frac{2^{-4n}}{2n-1} \binom{2n}{n}^2 \left(1 - \frac{b^2}{a^2} - \lambda\right)^n {}_2F_1\left(n + \frac{1}{2}, n - \frac{1}{2}, n+1, \lambda\right) \quad (64)$$

proof: we use simple methods of analysis, series theory, and hypergeometric transformations.

Remark 1: ${}_2F_1$ is the Gauss hypergeometric function. See [10, p.1005], [19, p.384].

Remark 2: F_1 is the Appell hypergeometric function. See [10, p.1018], [19, p.412].

Remark 3: ${}_pF_q$ is the generalized hypergeometric function. See [10, p.1010], [19, p.404]

Remark 4: Equation (4) is known as the Maclaurin formula (1742).

Remark 5: Equation (5) is known as the Ivory-Gauss-Kummer formula.

Remark 6: Equation (6) is known as the Euler formula (1773).

Remark 7: Equation (7) is known as the Cayley formula (1876).

Remark 8: $(a)_n$ is the Pochhammer symbol.

III. Some integrals and series

Entry 14. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b = \frac{a}{2} \sqrt{2 - \sqrt{3}} \quad (65)$$

then

$$P = P(a, b) = 4a \sqrt[4]{3} \int_0^{(\sqrt{3}+1)/2} \frac{1}{(1 + (\sqrt{3} + 1)x)^2} \sqrt{\frac{1-x+x^2}{x(1+x)}} dx \quad (66)$$

Entry 15. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b = \frac{a}{2} \sqrt{2 + \sqrt{3}} \quad (67)$$

then

$$P = P(a, b) = 4a \sqrt[4]{3} \int_0^{(\sqrt{3}-1)/2} \frac{1}{(1 + (\sqrt{3} - 1)x)^2} \sqrt{\frac{1+x+x^2}{x(1-x)}} dx \quad (68)$$

Entry 16. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b = \frac{a}{4} (\sqrt{6} + \sqrt{2}) \quad (69)$$

then

$$P = P(a, b) = 4a \sqrt[4]{3} \int_{\sqrt{3}+1}^{\infty} \frac{x^2 + x + 1}{(\sqrt{3} + x - 1)^2 \sqrt{x^3 - 1}} dx \quad (70)$$

Entry 17. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0 \quad (71)$$

then

$$P = P(a, b) = 4a^2 \int_0^{\infty} \sqrt{\frac{x^2 + b^2}{(x^2 + a^2)^3}} dx \quad (72)$$

Entry 18. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0 \quad (73)$$

then

$$P = P(a, b) = f(a, b) + f(b, a) \quad (74)$$

where

$$f(a, b) = 2a^2 \int_0^\infty \frac{\sqrt{x}}{(x+a^2)^{3/2} (x+b^2)^{1/2}} dx \quad (75)$$

$$f(a, b) = \frac{2\pi a}{a^2 - b^2} \left(a^2 {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2}, 1, 1 - \frac{b^2}{a^2} \right) - b^2 {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 1, 1 - \frac{b^2}{a^2} \right) \right) \quad (76)$$

Entry 19. Let us consider the following ellipse:

$$x^2 + 2y^2 = 1 \quad (77)$$

then

$$P = 4\sqrt{2} \sum_{n=0}^{\infty} \frac{2^{-3n}}{2n+1} \binom{2n}{n} {}_4F_3 \left(\begin{matrix} -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} - n, -n \\ 1, \frac{1}{2} - n, \frac{1}{2} - n \end{matrix} \middle| 1 \right) \quad (78)$$

$$P = 4\sqrt{2} \sum_{n=0}^{\infty} (\sqrt{2} - 1)^{4n} \sum_{k=0}^n \frac{2^{-4k}}{(1-2k)^2} \binom{2k}{k}^2 \left(\frac{1}{4n-4k+1} - \frac{(\sqrt{2}-1)^2}{4n-4k+3} \right) \quad (79)$$

$$P = 2 \cdot \sum_{n=0}^{\infty} 3^{-2n} \sum_{k=0}^n \frac{2^{-6k}}{1-4k} \binom{4k}{2k} \binom{2k}{k} \left(\frac{3}{4n-4k+1} - \frac{1}{4n-4k+3} \right) \quad (80)$$

$$P = \frac{8\pi\sqrt{\pi}}{(\Gamma(1/4))^2} + 4\pi \sum_{n=1}^{\infty} \frac{2^{-5n}n}{2n-1} \binom{2n}{n}^2 \quad (81)$$

Entry 20. Let us consider the following ellipse:

$$x^2 + 3y^2 = 1 \quad (82)$$

then

$$P = 2\sqrt{6} \sum_{n=0}^{\infty} \frac{2^{-4n}}{2n+1} \binom{2n}{n} {}_4F_3 \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4}, -\frac{1}{2} - n, -n \\ 1, \frac{1}{2} - n, \frac{1}{2} - n \end{matrix} \middle| 1 \right) \quad (83)$$

$$P = 2\pi {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{3} \right) - \pi \sum_{n=0}^{\infty} \frac{2^{-3n-2} \cdot 3^{-n-1} (2-2^{-n})}{n+1} \binom{2n}{n} \binom{2n+2}{n+1} \quad (84)$$

Entry 21. Let us consider the following ellipse:

$$x^2 + (2 + \sqrt{3})^2 y^2 = 1 \quad (85)$$

then

$$P = 3\sqrt{6(2-\sqrt{3})} \sum_{n=0}^{\infty} \frac{(3/16)^n}{2n+1} \binom{2n}{n} {}_4F_3 \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4}, -\frac{1}{2} - n, -n \\ 1, \frac{1}{2} - n, \frac{1}{2} - n \end{matrix} \middle| 1 \right) \quad (86)$$

$$P = 4 \left(1 + \sum_{n=0}^{\infty} 2^{-4n} \binom{2n}{n}^2 \left(\frac{2n+1}{2n+2} \right) (2-\sqrt{3})^{2n+2} \left(\ln(8+4\sqrt{3}) - \frac{1}{(2n+1)(2n+2)} - \sum_{k=1}^n \frac{1}{(2k-1)k} \right) \right) \quad (87)$$

Entry 22. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0, \quad e = \sqrt{1 - \frac{b^2}{a^2}} \quad (88)$$

then

$$P = 8a \sum_{n=0}^{\infty} \frac{(1-e)^n}{1-4n^2} F_1 \left(n+1, n, -\frac{1}{2}, n+\frac{3}{2}, -1, e \right) \quad (89)$$

$$P = 4a \sum_{n=0}^{\infty} \frac{1}{1-4n^2} \left(\frac{1-e}{2} \right)^n F_1 \left(n+1, 2, -\frac{1}{2}, n+\frac{3}{2}, \frac{1}{2}, \frac{1+e}{2} \right) \quad (90)$$

$$P = 4a \sum_{n=0}^{\infty} \frac{2^{-2n}}{1-2n} \binom{2n}{n} (1-e)^n f(e, n) = 4a \left(1 + \frac{1-e}{\sqrt{e}} \tanh^{-1}(e) - \sum_{n=1}^{\infty} \frac{2^{-2n}}{2n-1} \binom{2n}{n} (1-e)^n f(e, n) \right) \quad (91)$$

where

$$f(e, n) = \int_0^1 \left(\frac{x}{1+x} \right)^n \sqrt{\frac{1-ex}{1-x}} dx \quad (92)$$

$$f(e, n) = (1-e) \int_1^{\infty} \frac{\sqrt{x}}{(x-e)^2} \left(\frac{x-1}{2x-1-e} \right)^n dx \quad (93)$$

$$f(e, n) = (1-e) 2^{-n} \int_0^1 \frac{1}{(1-ex)^2 \sqrt{x}} \left(\frac{1-x}{1-x(\frac{1+e}{2})} \right)^n dx \quad (94)$$

$$f(e, 0) = 1 + \frac{1-e}{\sqrt{e}} \tanh^{-1}(e) \quad (95)$$

$$f(1, n) = \frac{2^{-n}}{n+1} {}_2F_1 \left(1, n, n+2, \frac{1}{2} \right) \quad (96)$$

Entry 23. Let us consider the following ellipse:

$$\frac{x^2}{64} + y^2 = 1, \quad e = \frac{\sqrt{63}}{8} \quad (97)$$

then

$$P = 32 \sum_{n=0}^{\infty} \frac{1}{1-4n^2} \left(\frac{8-\sqrt{63}}{16} \right)^n F_1 \left(n+1, 2, -\frac{1}{2}, n+\frac{3}{2}, \frac{1}{2}, \frac{8+\sqrt{63}}{16} \right) \quad (98)$$

$$P = 4\sqrt{63} - \sqrt{63} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-4n}}{n+1} \binom{2n}{n}^2 63^{-n-1} \left(\ln \left(\frac{1}{1008} \right) - \frac{1}{n+1} + \sum_{k=1}^n \frac{2}{k(2k-1)} \right) \quad (99)$$

IV. Example of an ellipse whose perimeter is Pi

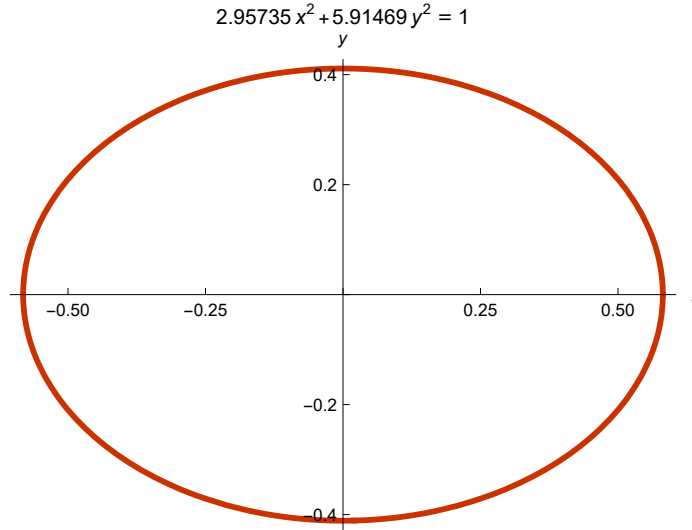
Entry 24. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (100)$$

where ($b < a$)

$$a = \frac{2 \pi^{3/2} (\Gamma(1/4))^2}{8 \pi^2 + (\Gamma(1/4))^4}, \quad b = \frac{\sqrt{2} \pi^{3/2} (\Gamma(1/4))^2}{8 \pi^2 + (\Gamma(1/4))^4} \quad (101)$$

then the perimeter of the ellipse is $P = \pi$.



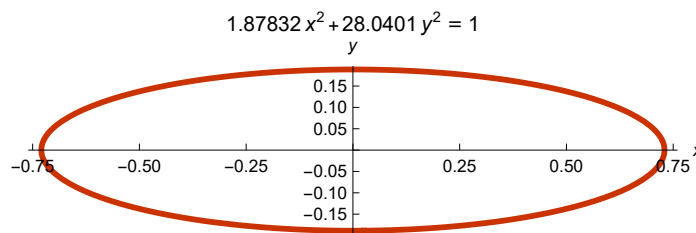
Entry 25. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (102)$$

where ($b < a$)

$$a = \frac{\sqrt{\pi} \sqrt{3} \sqrt[4]{27} \Gamma(\frac{1}{6}) \Gamma(\frac{2}{3})}{(3 - \sqrt{3}) (\Gamma(\frac{1}{6}))^2 + 18 (\Gamma(\frac{2}{3}))^2}, \quad b = \frac{\sqrt{\pi} \sqrt{3(2 - \sqrt{3})} \sqrt[4]{27} \Gamma(\frac{1}{6}) \Gamma(\frac{2}{3})}{2((3 - \sqrt{3}) (\Gamma(\frac{1}{6}))^2 + 18 (\Gamma(\frac{2}{3}))^2)} \quad (103)$$

then the perimeter of the ellipse is $P = \pi$.



Entry 26.

$$\frac{2 \pi \sqrt{\pi}}{(\Gamma(\frac{1}{4}))^2} + \frac{(\Gamma(\frac{1}{4}))^2}{4 \sqrt{\pi}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{\frac{4}{n^2} + 2 \left(\sqrt{1 - \left(\frac{k}{n}\right)^2} - \sqrt{1 - \left(\frac{k-1}{n}\right)^2} \right)^2} \quad (104)$$

Remark 9: $\Gamma(x)$ is the Gamma function.

V. Some approximations and bounds for the perimeter of an ellipse

Entry 27. (Approximations)

Kepler (1609)

$$P \approx 2 \pi \sqrt{a b} \quad (105)$$

$$P \approx \pi (a + b) \quad (106)$$

Euler (1773)

$$P \approx 2 \pi \sqrt{\frac{a^2 + b^2}{2}} = \pi \sqrt{2(a^2 + b^2)} \quad (107)$$

Sipos (1792)

$$P \approx 2 \pi \left(\frac{a + b}{\sqrt{a} + \sqrt{b}} \right) \quad (108)$$

Cesaro (1859-1906)

$$P \approx \pi (a + b) + \frac{\pi (a - b)^2}{4 (a + b)} \quad (109)$$

Sir Thomas Muir (1883)

$$P \approx 2 \pi \left(\frac{a^{3/2} + b^{3/2}}{2} \right)^{2/3} \quad (110)$$

Peano, Boussinesq (1889)

$$P \approx \pi \left(\frac{3}{2} (a + b) - \sqrt{a b} \right) \quad (111)$$

Lindner (1904)

$$P \approx \pi (a + b) \left(1 + \frac{1}{8} \left(\frac{a - b}{a + b} \right)^2 \right) \quad (112)$$

Ramanujan (1914)

$$P \approx \pi (3 (a + b) - \sqrt{(3 a + b) (a + 3 b)}) \quad (113)$$

$$P \approx \pi \left((a + b) - \frac{3 (a - b)^2}{10 (a + b) + \sqrt{a^2 + 14 a b + b^2}} \right) \quad (114)$$

Almkvist (1988)

$$P \approx 2 \pi \frac{2 (a + b)^2 - (\sqrt{a} - \sqrt{b})^4}{(\sqrt{a} + \sqrt{b})^2 + 2 \sqrt{2 (a + b)} (a b)^{1/4}} \quad (115)$$

Cantrell (2004)

$$P \approx \pi \left((a+b) - \frac{3(a-b)^2}{10(a+b) + \sqrt{a^2 + 14ab + b^2}} + \left(\frac{4}{\pi} - \frac{14}{11} \right) (a+b) \left(\frac{a-b}{a+b} \right)^{24} \right) \quad (116)$$

Others approximations

$$P \approx \frac{\pi}{3} (a+b+2\sqrt{2}\sqrt{a^2+b^2}) \quad (117)$$

$$P \approx \frac{\pi}{4} \left(a+b + \frac{3}{2} (\sqrt{a^2+3b^2} + \sqrt{3a^2+b^2}) \right) \quad (118)$$

$$P \approx \frac{\pi}{20} (2a+2b+6\sqrt{2}\sqrt{a^2+b^2} + \sqrt{a^2+3b^2} + \sqrt{3a^2+b^2} + 5\sqrt{(2+\sqrt{3})a^2+(2-\sqrt{3})b^2} + 5\sqrt{(2-\sqrt{3})a^2+(2+\sqrt{3})b^2}) \quad (119)$$

$$P = P(1, b) \approx \frac{32\pi}{25} \left(\frac{1-b^{5/2}}{1-b^2} \right)^2, \quad 0 < b < 1, a = 1 \quad (120)$$

$$P = P(1, b) \approx \frac{14\pi}{11} \left(\frac{1-b^{11/4}}{1-b^{7/4}} \right), \quad 0 < b < 1, a = 1 \quad (121)$$

Entry 28. (Bounds)

$$\pi(a+b) \leq P \leq 2\pi \sqrt{\frac{a^2+b^2}{2}} \quad (122)$$

$$2\pi b \leq P \leq 2\pi a, \quad a \geq b \quad (123)$$

$$4\sqrt{a^2+b^2} \leq P \leq 4(a+b) \quad (124)$$

$$P \leq 4a + 2(\pi-2)b, \quad 0 \leq b \leq a \quad (125)$$

VI. Final formulas

Entry 29. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad 0 < b < a \quad (126)$$

then

$$P = 16a \left(\frac{1}{3} - \sum_{n=1}^{\infty} \frac{P_n \left(1 - \frac{2b^2}{a^2} \right)}{(4n^2-1)(2n+3)} \right) \quad (127)$$

where $P_n(z)$ are the Legendre polynomials.

Entry 30. Let us consider the following ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad 0 < b < a, \quad e^2 = 1 - \frac{b^2}{a^2} \quad (128)$$

then

$$P = 4a + 4a \int_1^\infty \left(1 - \sqrt{\frac{x^2 - 1}{x^2 - e^2}} \right) dx \quad (129)$$

$$P = 4a + 4a(1 - e^2) \int_0^1 \frac{1}{1 - e^2 x^2 + \sqrt{(1 - x^2)(1 - e^2 x^2)}} dx \quad (130)$$

$$P = 4a \int_0^1 \tanh^{-1} \left(\frac{1}{\sqrt{2}} \sqrt{1 + \frac{1}{e^2} - \frac{\sqrt{(1 - e^2)^2 + 4e^2 x^2}}{e^2}} \right) dx \quad (131)$$

$$P = \frac{2a\sqrt{\pi}}{(\Gamma(1/4))^2} \int_{x>0, y>0} \int_{x+y \leq 1} \frac{\sqrt{1 - (x+y)e^2}}{\sqrt{1-x-y} \sqrt{x^3 y^3}} dx dy \quad (132)$$

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