

Uniformly Accelerated Motion Observed from a Perpendicularly Moving Inertial Frame: Kinematics and Energy–Momentum Transformation

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Abstract

We analyze the motion of a point particle executing uniformly accelerated motion (constant proper acceleration) from rest in an inertial frame S' , as observed from a second inertial frame S whose velocity relative to S' is perpendicular to the acceleration axis. Starting from the Lorentz transformation for a perpendicular boost, we derive the complete kinematics of the particle in S : the x -coordinate obeys a hyperbolic law of motion with reduced acceleration parameter $a_0 = a'_0/\gamma$, the transverse coordinates grow linearly in time, the velocity parallel to the acceleration is suppressed by the Lorentz factor ($V_x = V'_x/\gamma$), and the coordinate acceleration at the instant of rest in S' is reduced by γ^2 . We then extend the analysis to dynamics and derive the transformation of the four-momentum. In the present perpendicular configuration the transformation laws reduce to $E = \gamma E'$, $p_x = p'_x$, $p_y = \gamma v_{0y} E'/c^2$, and $p_z = \gamma v_{0z} E'/c^2$, each of which admits a transparent physical reading. We give the explicit time dependence of E and $\mathbf{p}$ for hyperbolic motion, verify the invariance of the rest mass, and show that all results are mutually consistent and reduce to the Galilean limit for $v_0 \ll c$.

1 Introduction

Uniformly accelerated motion in special relativity—often called *hyperbolic motion*—is the motion of a particle subject to a constant proper acceleration. It was studied by Born [1], Rindler [2, 3], and plays a central role in relativistic kinematics, in the analysis of the Unruh effect, and in the theory of rigid motion.

We consider a particle that executes hyperbolic motion along the x' -axis in its instantaneous rest frame S' , viewed from another inertial frame S that moves with constant velocity *perpendicular* to the acceleration axis. Because the boost is perpendicular to the acceleration, transverse spatial coordinates are unaffected by Lorentz contraction, while the time coordinate mixes with the boost direction. The resulting motion in S is a superposition of the hyperbolic law along x and a uniform drift along the boost direction.

The purpose of this paper is twofold. First, we derive the complete kinematics of the particle in S (trajectory, velocity, acceleration). Second, we derive the transformation of the four-momentum in the same perpendicular configuration and show that the transformation laws take a remarkably simple form, whose physical content can be read off directly. In particular, we show that the relation between energy and momentum is precisely what is required to leave the rest mass invariant, and we

verify that the velocity obtained from $\mathbf{p}/(\gamma_p m)$ agrees with the velocity obtained from the coordinate transformation.

Section 2 fixes notation. Section 3 recalls the hyperbolic-motion formulas in S' . Section 4 applies the perpendicular Lorentz boost to the worldline. Section 5 derives the trajectory, velocity, and acceleration in S . Section 6 presents the energy–momentum transformation, which is the central new result of this paper. Section 7 collects consistency checks and limiting cases. Sections 8 and 9 discuss the physical interpretation and conclude.

2 Setup and notation

Let S and S' be inertial frames related by a Lorentz boost, with the boost velocity $\mathbf{v}_0$ of S' relative to S lying in the yz -plane:

$$\mathbf{v}_0 = (0, v_{0y}, v_{0z}), \quad v_0^2 = v_{0y}^2 + v_{0z}^2. \quad (1)$$

Define the standard Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \beta_0^2}}, \quad \beta_0 = \frac{v_0}{c}. \quad (2)$$

Throughout, γ denotes the Lorentz factor (2), consistent with the time-dilation relation $dt = \gamma dt'$ for a clock at rest in S' . The particle's own Lorentz factor will be denoted γ_p and γ'_p in S and S' , respectively, to avoid confusion.

A point particle of rest mass m moves in S' along the x' -axis with constant proper acceleration $a'_0 > 0$, starting from rest at the origin at $t' = 0$:

$$x'(0) = 0, \quad \dot{x}'(0) = 0, \quad y'(t') = z'(t') = 0. \quad (3)$$

The proper acceleration a'_0 is a Lorentz scalar; its magnitude is the same in every inertial frame.

3 Hyperbolic motion in the rest frame S'

The worldline of a particle with constant proper acceleration a'_0 along the x' -axis, starting from rest at the origin, is hyperbolic motion:

$$x'(t') = \frac{c^2}{a'_0} \left(\sqrt{1 + \frac{a'^2_0 t'^2}{c^2}} - 1 \right), \quad y'(t') = z'(t') = 0. \quad (4)$$

The velocity and coordinate acceleration are

$$V'_x(t') = \frac{a'_0 t'}{\sqrt{1 + a'^2_0 t'^2/c^2}}, \quad (5)$$

$$a'_x(t') = \frac{a'_0}{(1 + a'^2_0 t'^2/c^2)^{3/2}}. \quad (6)$$

At $t' = 0$: $V'_x = 0$ and $a'_x = a'_0$; as $t' \rightarrow \infty$, $V'_x \rightarrow c$. The particle's Lorentz factor in S' and its momentum-energy are

$$\gamma'_p(t') = \sqrt{1 + \frac{a'^2_0 t'^2}{c^2}}, \quad p'_x(t') = \gamma'_p m V'_x = m a'_0 t', \quad E'(t') = \gamma'_p m c^2. \quad (7)$$

The simple linear form $p'_x = ma'_0 t'$ is a special feature of one-dimensional hyperbolic motion: the rate of change of momentum in S' , $dp'_x/dt' = ma'_0$, is constant, because the coordinate force equals the proper force in the instantaneous rest frame. The proper time satisfies $d\tau = dt'/\gamma'_p$, giving $\tau = (c/a'_0) \operatorname{arcsinh}(a'_0 t'/c)$.

4 Perpendicular Lorentz boost to S

Decompose spatial vectors into components parallel and perpendicular to $\mathbf{v}_0$: $\mathbf{r}'_{\parallel} = (\mathbf{r}' \cdot \mathbf{v}_0) \mathbf{v}_0 / v_0^2$, $\mathbf{r}'_{\perp} = \mathbf{r}' - \mathbf{r}'_{\parallel}$. Since $\mathbf{v}_0$ lies in the yz -plane and $\mathbf{r}' = (x', 0, 0)$, we have $\mathbf{r}'_{\parallel} = 0$ and $\mathbf{r}'_{\perp} = (x', 0, 0)$. The Lorentz transformation from S' to S reads

$$t = \gamma \left(t' + \frac{\mathbf{v}_0 \cdot \mathbf{r}'}{c^2} \right), \quad (8)$$

$$\mathbf{r}_{\parallel} = \gamma \left(\mathbf{r}'_{\parallel} + \mathbf{v}_0 t' \right), \quad (9)$$

$$\mathbf{r}_{\perp} = \mathbf{r}'_{\perp}. \quad (10)$$

For the particle, $\mathbf{v}_0 \cdot \mathbf{r}' = 0$, hence

$$t = \gamma t', \quad (11)$$

$$x = x'(t'), \quad (12)$$

$$y = \gamma v_{0y} t' = v_{0y} t, \quad (13)$$

$$z = \gamma v_{0z} t' = v_{0z} t. \quad (14)$$

Thus: (i) $t = \gamma t'$ (time dilation); (ii) $x = x'$ (the boost is perpendicular to x); (iii) y, z grow linearly with constant velocities v_{0y}, v_{0z} .

5 Kinematics in S : trajectory, velocity, and acceleration

5.1 Trajectory

Substituting $t' = t/\gamma$ into (4) and using $x = x'$,

$$x(t) = \frac{c^2}{a'_0} \left(\sqrt{1 + \frac{a'^2_0 t^2}{\gamma^2 c^2}} - 1 \right). \quad (15)$$

Introducing the *reduced acceleration parameter*

$$a_0 := \frac{a'_0}{\gamma} = a'_0 \sqrt{1 - \beta_0^2}, \quad (16)$$

equation (15) becomes

$$x(t) = \frac{c^2}{\gamma a_0} \left(\sqrt{1 + \frac{a_0^2 t^2}{c^2}} - 1 \right). \quad (17)$$

Together with (13)–(14), the worldline in S is

$$\boxed{x(t) = \frac{c^2}{\gamma a_0} \left(\sqrt{1 + \frac{a_0^2 t^2}{c^2}} - 1 \right), \quad y(t) = v_{0y} t, \quad z(t) = v_{0z} t.} \quad (18)$$

5.2 Velocity

Differentiating (17),

$$V_x(t) = \frac{a_0 t / \gamma}{\sqrt{1 + a_0^2 t^2 / c^2}} = \frac{a'_0 t / \gamma^2}{\sqrt{1 + a_0'^2 t^2 / (\gamma^2 c^2)}}. \quad (19)$$

Comparing with (5) and using $t' = t/\gamma$,

$$V_x(t) = \frac{V'_x(t')}{\gamma}. \quad (20)$$

This is the standard rule for a velocity component *perpendicular* to the boost. The transverse components are constant:

$$V_y = v_{0y}, \quad V_z = v_{0z}. \quad (21)$$

Hence $\mathbf{V} = (V'_x/\gamma, v_{0y}, v_{0z})$, and one checks $V^2 = V_x'^2/\gamma^2 + v_0^2 < c^2$. The particle's Lorentz factor in S is

$$\gamma_p = \frac{1}{\sqrt{1 - V^2/c^2}} = \gamma \gamma'_p. \quad (22)$$

5.3 Acceleration

Differentiating (19),

$$a_x(t) = \frac{a_0/\gamma}{(1 + a_0^2 t^2 / c^2)^{3/2}} = \frac{a'_0}{\gamma^2 (1 + a_0'^2 t^2 / (\gamma^2 c^2))^{3/2}}. \quad (23)$$

At $t = 0$ (equivalently $t' = 0$),

$$a_x(0) = \frac{a_0}{\gamma} = \frac{a'_0}{\gamma^2}. \quad (24)$$

Thus the coordinate acceleration in S at the instant of rest in S' is reduced by γ^2 relative to the proper acceleration. Transverse accelerations vanish: $a_y = a_z = 0$.

5.4 Remark on the notation of the problem statement

The equations quoted in the problem statement correspond to (4), (17), (13)–(14), (20), (21), and (24), provided that: (i) γ is the standard Lorentz factor (2); (ii) the symbol a_{0x} denotes the reduced parameter $a_0 = a'_0/\gamma$, not the coordinate acceleration $a_x(0) = a'_0/\gamma^2$; (iii) the relation “ $dy = dy' = dz = dz' = 0$ ” is interpreted as $dy' = dz' = 0$ along the worldline in S' ; in S one has $dy = v_{0y} dt \neq 0$ and $dz = v_{0z} dt \neq 0$.

6 Energy–momentum transformation

We now turn to the dynamics. The four-momentum of a particle of rest mass m is

$$P^\mu = (E/c, \mathbf{p}), \quad E = \gamma_p m c^2, \quad \mathbf{p} = \gamma_p m \mathbf{V}, \quad (25)$$

so that $P^\mu P_\mu = m^2 c^2$. Since P^μ is a four-vector, it transforms under the boost from S' to S in the same way as the position four-vector $(ct, \mathbf{r})$.

6.1 General transformation of the four-momentum

Decomposing $\mathbf{p}'$ into components parallel and perpendicular to $\mathbf{v}_0$,

$$E = \gamma(E' + \mathbf{v}_0 \cdot \mathbf{p}'), \quad (26)$$

$$\mathbf{p}_{\parallel} = \gamma\left(\mathbf{p}'_{\parallel} + \mathbf{v}_0 \frac{E'}{c^2}\right), \quad (27)$$

$$\mathbf{p}_{\perp} = \mathbf{p}'_{\perp}. \quad (28)$$

Here γ is the boost Lorentz factor (2); it must be carefully distinguished from the particle's γ_p .

6.2 Application to the perpendicular configuration

In the present problem the particle moves along the x' -axis in S' , so that $\mathbf{p}' = (p'_x, 0, 0)$. Since $\mathbf{v}_0$ lies in the yz -plane,

$$\mathbf{p}'_{\parallel} = 0, \quad \mathbf{v}_0 \cdot \mathbf{p}' = 0, \quad (29)$$

and equations (26)–(28) reduce to the remarkably simple form

$$\boxed{E = \gamma E', \quad p_x = p'_x, \quad p_y = \gamma v_{0y} \frac{E'}{c^2}, \quad p_z = \gamma v_{0z} \frac{E'}{c^2}.} \quad (30)$$

Each relation has a direct physical reading:

- $E = \gamma E'$: because the particle's momentum in S' is perpendicular to the boost, only the time-dilation factor enters the energy transformation.
- $p_x = p'_x$: the x -component is perpendicular to the boost and is therefore invariant.
- p_y and p_z : the transverse momenta acquire a term proportional to the transformed energy E' , reflecting the relativistic inertia of the particle's energy content (the “energy has mass” aspect of $E = mc^2$).

6.3 Explicit time dependence via hyperbolic motion

Combining (7) with (30) and using $t' = t/\gamma$ together with the reduced parameter $a_0 = a'_0/\gamma$ of (16), we obtain the explicit time dependence in S :

$$E(t) = \gamma m c^2 \sqrt{1 + \frac{a_0^2 t^2}{c^2}} = \gamma_p(t) m c^2, \quad (31)$$

$$p_x(t) = m a_0 t = p'_x(t'), \quad (32)$$

$$p_y(t) = \gamma v_{0y} m \sqrt{1 + \frac{a_0^2 t^2}{c^2}}, \quad (33)$$

$$p_z(t) = \gamma v_{0z} m \sqrt{1 + \frac{a_0^2 t^2}{c^2}}, \quad (34)$$

where $\gamma_p = \gamma \gamma'_p$ is the particle's Lorentz factor in S (22). Three structural features are worth emphasizing:

1. $p_x(t) = m a_0 t$ has the *same linear form* as in S' , but with the reduced acceleration $a_0 = a'_0/\gamma$; the momentum grows linearly in t because dp_x/dt is constant.

2. $E(t) = \gamma_p mc^2$ grows asymptotically as $ma'_0 tc$, while $p_x(t)c = ma_0 tc = ma'_0 tc/\gamma$; hence $E/(p_x c) \rightarrow \gamma$ as $t \rightarrow \infty$, so the particle in S is *not* asymptotically lightlike but retains a finite invariant mass, as it must.
3. $p_y(t) = \gamma_p mv_{0y}$ and $p_z(t) = \gamma_p mv_{0z}$: since the transverse velocities $V_y = v_{0y}$, $V_z = v_{0z}$ are constant, the transverse momenta grow simply because the particle's relativistic mass $\gamma_p m$ grows with time.

6.4 Consistency with the velocity transformation

Dividing the momenta in (32)–(34) by $\gamma_p m$ and using $\gamma_p = \gamma \gamma'_p$ and $\gamma'_p = \sqrt{1 + a_0'^2 t^2 / c^2}$,

$$V_x = \frac{p_x}{\gamma_p m} = \frac{ma_0 t}{\gamma \gamma'_p m} = \frac{a'_0 t' / \gamma}{\gamma'_p} = \frac{V'_x}{\gamma}, \quad (35)$$

$$V_y = \frac{p_y}{\gamma_p m} = \frac{\gamma v_{0y} m \gamma'_p}{\gamma \gamma'_p m} = v_{0y}, \quad V_z = \frac{p_z}{\gamma_p m} = v_{0z}, \quad (36)$$

in agreement with (20)–(21). The energy–momentum transformation (30) and the velocity transformation (20)–(21) are therefore consistent.

6.5 Invariance of the rest mass

Squaring (30) and using $1 - \beta_0^2 = \gamma^{-2}$, we verify the mass-shell condition $E^2 - \mathbf{p}^2 c^2 = m^2 c^4$:

$$\begin{aligned} E^2 - \mathbf{p}^2 c^2 &= \gamma^2 E'^2 - p_x'^2 c^2 - \gamma^2 (v_{0y}^2 + v_{0z}^2) \frac{E'^2}{c^2} \\ &= \gamma^2 (1 - \beta_0^2) E'^2 - p_x'^2 c^2 = E'^2 - p_x'^2 c^2 = m^2 c^4. \end{aligned} \quad (37)$$

This identity shows that the three nontrivial ingredients of (30)—the γ in the energy, the invariance of p_x , and the $\gamma v_{0i} E' / c^2$ in the transverse components—are not independent: they are precisely what is required to leave m invariant.

6.6 Summary of the energy–momentum transformation

For convenience we summarize the results in Table 1.

Table 1: Energy–momentum of a particle executing hyperbolic motion along the x' -axis in S' , viewed from the perpendicularly moving frame S . Here $\gamma'_p = \sqrt{1 + a_0'^2 t'^2 / c^2}$, $\gamma_p = \gamma \gamma'_p$, and $a_0 = a'_0 / \gamma$.

Quantity	In S'	In S
Energy	$E' = mc^2 \gamma'_p$	$E = \gamma E' = mc^2 \gamma_p$
Longitudinal momentum	$p'_x = ma'_0 t'$	$p_x = p'_x = ma_0 t$
Transverse momentum (y)	$p'_y = 0$	$p_y = \gamma v_{0y} E' / c^2 = \gamma_p m v_{0y}$
Transverse momentum (z)	$p'_z = 0$	$p_z = \gamma v_{0z} E' / c^2 = \gamma_p m v_{0z}$
Rest mass	m	m

7 Consistency checks and limiting cases

7.1 Acceleration transformation

For a boost perpendicular to the acceleration, at the instant of rest in S' , the acceleration-transformation law reduces to $a_\perp = a'_\perp/\gamma^2$, in agreement with (24). The factor γ^2 (rather than γ^3 for a parallel boost) reflects the absence of length contraction along the perpendicular direction.

7.2 Invariance of the proper acceleration

The proper acceleration is the invariant magnitude of the four-acceleration, $\alpha = \sqrt{-A^\mu A_\mu}$. In S' at $t' = 0$: $U'^\mu = (c, 0, 0, 0)$ and $A'^\mu = (0, a'_0, 0, 0)$, so $\alpha = a'_0$. Under a boost in the yz -plane, the x -component is perpendicular to the boost and is unchanged, $A^x = A'^x = a'_0$; the time and transverse components vanish because $A'^0 = 0$ and $A'_\parallel = 0$. Hence $A^\mu = (0, a'_0, 0, 0)$ in S as well, and $-A^\mu A_\mu = a_0'^2$. The coordinate acceleration $a_x(0) = a'_0/\gamma^2$ differs from A^x by the factor γ^2 , as expected from $A^x = \gamma^2 a_x$ when $\mathbf{V} \cdot \mathbf{a} = 0$.

7.3 Galilean limit

For $v_0 \ll c$: $\gamma \rightarrow 1$, $a_0 \rightarrow a'_0$, and (18) becomes $x \simeq \frac{1}{2}a'_0 t^2$, $y = v_{0y}t$, $z = v_{0z}t$, as in Newtonian mechanics. Correspondingly, in (30), $E \simeq E'$, $p_x \simeq p'_x$, and $p_y \simeq v_{0y}E'/c^2$, $p_z \simeq v_{0z}E'/c^2$.

7.4 Differential relations

Along the worldline,

$$dt = \gamma dt', \quad dx = V_x dt = \frac{V'_x}{\gamma} dt' = V'_x dt' = dx', \quad dy = v_{0y} dt, \quad dz = v_{0z} dt. \quad (38)$$

The relation $dx = dx'$ reflects that the x -axis is perpendicular to the boost.

8 Discussion

The motion in S is a superposition of two effects:

1. A *reduced hyperbolic motion* along x , with parameter $a_0 = a'_0/\gamma$ and an overall $1/\gamma$ factor in the trajectory; the coordinate acceleration at $t = 0$ is reduced by γ^2 .
2. A *uniform drift* (v_{0y}, v_{0z}) along the boost direction.

The two effects decouple because the boost is perpendicular to the acceleration: the x -motion is affected only through $t = \gamma t'$, while the transverse motion is purely kinematic.

The relation $V_x = V'_x/\gamma$ is *not* due to length contraction along x (there is none), but to the relativity of simultaneity: the same displacement $dx = dx'$ occurs over $dt = \gamma dt'$, hence $V_x = dx/dt = V'_x/\gamma$. Each time derivative in $a_x = d^2x/dt^2$ brings a factor $1/\gamma$ from $dt = \gamma dt'$, giving $a_x = a'_x/\gamma^2$.

The energy-momentum transformation (30) completes this kinematic picture at the level of dynamics. Its three nontrivial features ($E = \gamma E'$, $p_x = p'_x$, $p_{y,z} = \gamma v_{0y,z} E'/c^2$) are jointly required by the invariance of the rest mass, as shown in (37). In the transverse direction, the particle's momentum is $p_{y,z} = \gamma_p m v_{0y,z}$, i.e. it equals the particle's relativistic mass $\gamma_p m$ times the boost velocity. Since γ_p grows without bound as the particle accelerates in S' , the transverse momenta grow

proportionally, even though the transverse velocities remain constant. This is a direct manifestation of the relativistic inertia of energy: the work done by the accelerating force in S' increases E' , which in S contributes to the transverse momentum through $p_{y,z} = \gamma v_{0y,z} E' / c^2$.

Finally, we note that in the limit $v_0 \rightarrow 0$ (so $\gamma \rightarrow 1$) the transformation laws (30) reduce to the identity $E = E'$, $\mathbf{p} = \mathbf{p}'$, as they must; in the opposite regime $\gamma \gg 1$, the energy and transverse momenta are strongly enhanced while p_x is unchanged, a characteristic signature of the perpendicular configuration.

9 Conclusion

The main results are:

1. Trajectory in S : Eq. (18).
2. Velocity: $V_x = V'_x / \gamma$, $V_y = v_{0y}$, $V_z = v_{0z}$.
3. Coordinate acceleration at the instant of rest in S' : $a_x(0) = a'_0 / \gamma^2$.
4. Energy–momentum transformation in the perpendicular configuration:

$$E = \gamma E', \quad p_x = p'_x, \quad p_y = \gamma v_{0y} \frac{E'}{c^2}, \quad p_z = \gamma v_{0z} \frac{E'}{c^2}.$$

5. Explicit time dependence for hyperbolic motion: $E = \gamma m c^2 \sqrt{1 + a_0^2 t^2 / c^2}$, $p_x = m a_0 t$, $p_{y,z} = \gamma v_{0y,z} m \sqrt{1 + a_0^2 t^2 / c^2}$.
6. Rest mass is invariant: $E^2 - \mathbf{p}^2 c^2 = m^2 c^4$ holds identically under the transformation laws.

These results provide a complete kinematical and dynamical description of uniformly accelerated motion as viewed from a perpendicularly moving frame, and illustrate the interplay between time dilation, the relativity of simultaneity, the relativistic inertia of energy, and the geometry of perpendicular boosts.

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