

IS THE RIEMANN HYPOTHESIS TRUE ?

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Abstract

In 1859, Georg Friedrich Bernhard Riemann announced the following conjecture, called the Riemann hypothesis : *The nontrivial roots (zeros) of the zeta function, have real part $\sigma = 1/2$.* In this paper, I will give the proof using an equivalent statement of the Riemann hypothesis: the Dirichlet eta function.

Keywords: Zeta function, non-trivial zeros of eta function, Riemann hypothesis, equivalence statements, the definition of limits of real sequences.

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*To the memory of my Father who taught me arithmetic, To my wife Wahida,
my daughter Sinda, my son Mohamed Mazen and my granddaughter Rayhane
To the memory of the Prof. Mohamed AMARA, former Dean of the Faculty
of Sciences of Tunis*

1 Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1] known Riemann hypothesis:

Conjecture 1.1. *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we use the Dirichlet η function so we obtain a contradiction and we arrive to give the proof that $\sigma = \frac{1}{2}$.

1.1 The function zeta(s)

We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s . Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 1.2. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple ;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

So, instead of using one detailed formula of the zeta function, we will use for our proof the function presented by G.H. Hardy [3] namely the Dirichlet eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2]. It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [3]).

1.2 A Equivalent statement to the Riemann Hypothesis

Among the equivalent statements to the Riemann hypothesis is that of the Dirichlet eta function which is stated as follows [2]:

Equivalence 1.3. *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :*

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \quad (1)$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (1) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \quad (2)$$

From the equivalent statement above, we take the critical strip as the region defined by $0 < \Re(s) = \sigma < 1$. We suppose the case $t = 0$, but we known the following proposition:

Proposition 1.4. *For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.*

Then, we consider in the following $s = \sigma + it$ with $0 < \sigma < 1$ and $t > 0$. $\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \implies \rho^2 = \eta(s).\overline{\eta(s)} \quad (3)$$

and $\eta(s) = 0 \iff \rho = 0$.

2 Preliminaries of the proof that the zeros of the function eta(s) are on the critical line

Proof. We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \iff (1 - 2^{1-s})\zeta(s) = 0$. We verify easily the two propositions:

s , is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of
$\zeta(s)$ in the critical strip

(4)

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \implies \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

s , is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of
$\eta(s)$ in the critical strip

(5)

Let us write the function η :

$$\begin{aligned}\eta(s) &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-s \operatorname{Log} n} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it) \operatorname{Log} n} = \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \operatorname{Log} n} \cdot e^{-it \operatorname{Log} n} \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \operatorname{Log} n} (\cos(t \operatorname{Log} n) - i \sin(t \operatorname{Log} n))\end{aligned}$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. We define the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^n \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} - i \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma} = \rho_n \cdot e^{i \cdot \alpha_n}$$

with $s = \sigma + it$ and $t \neq 0$.

Let $s = \sigma + it$ with $0 < \sigma < 1$ be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^N \frac{(-1)^{n-1}}{n^s} \right| < \epsilon' \implies 0 \leq |\eta_N(s)|^2 < \epsilon'^2$$

We obtain:

$$0 \leq \Re^2(\eta(s)_N) + \Im^2(\eta(s)_N) < \epsilon'^2 \quad (6)$$

Then:

$$\begin{aligned}0 &\leq \left(\sum_{k=1}^N \frac{\cos^2(t \operatorname{Log} k)}{k^{2\sigma}} + \sum_{k, k'=1; k \neq k'}^N \frac{(-1)^{k+k'} \cos(t \operatorname{Log} k) \cdot \cos(t \operatorname{Log} k')}{k^\sigma k'^\sigma} \right) + \\ &\left(\sum_{k=1}^N \frac{\sin^2(t \operatorname{Log} k)}{k^{2\sigma}} + \sum_{k, k'=1; k \neq k'}^N \frac{(-1)^{k+k'} \sin(t \operatorname{Log} k) \cdot \sin(t \operatorname{Log} k')}{k^\sigma k'^\sigma} \right) < \epsilon'^2 \implies \\ 0 &\leq \sum_{k=1}^N \frac{1}{k^{2\sigma}} + \sum_{k, k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < \epsilon'^2 \quad (7)\end{aligned}$$

We can write the above equation as :

$$0 \leq \rho_N^2 < \epsilon'^2 \quad (8)$$

or $\rho(s) = 0$.

3 Case $0 < \Re(s) < 1/2$

Suppose there exists $s = \beta + i\gamma$ one zero of $\eta(s) \implies \eta(s) = 0 \implies \rho^2(s) = 0$ with $0 < \beta < \frac{1}{2} \implies s$ lies inside the critical band. We write the equation (7):

$$0 \leq \sum_{k=1}^N \frac{1}{k^{2\beta}} + \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(\gamma \text{Log}(k/k'))}{(kk')^\beta} < \epsilon'^2 \quad (9)$$

or:

$$-\sum_{k=1}^N \frac{1}{k^{2\beta}} \leq \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(\gamma \text{Log}(k/k'))}{(kk')^\beta} < \epsilon'^2 - \sum_{k=1}^N \frac{1}{k^{2\beta}} \quad (10)$$

The middle term of the above inequality is a double sum, but as $\beta < \frac{1}{2}$, it follows that:

$$\lim_{N \rightarrow +\infty} \sum_{k=1}^N \frac{1}{k^{2\beta}} = +\infty$$

then, we obtain necessarily from the equation (10):

$$s = \beta + i\gamma, \quad \eta(s) = 0 \implies \lim_{N \rightarrow +\infty} \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(\gamma \text{Log}(k/k'))}{(kk')^\beta} = -\infty \quad (11)$$

4 Case $1/2 < \Re(s) < 1$

Let $s = \beta + i\gamma$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the section 3. From the proposition (4), $\zeta(s) = 0$. According to point 4 of theorem 1.2, the complex number $s = 1 - \beta + i\gamma$ is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (5) that $\eta(s) = 0 \implies \rho(s) = 0$. By applying (7), we get:

$$0 \leq \sum_{k=1}^N \frac{1}{k^{2(1-\beta)}} + \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(\gamma \text{Log}(k/k'))}{(kk')^{(1-\beta)}} < \epsilon'^2$$

$$-\sum_{k=1}^N \frac{1}{k^{2(1-\beta)}} \leq \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(\gamma \log(k/k'))}{(kk')^{(1-\beta)}} < \epsilon'^2 - \sum_{k=1}^N \frac{1}{k^{2(1-\beta)}} \quad (12)$$

As $0 < \beta < \frac{1}{2} \implies 2 > 2(1-\beta) > 1$, then the series $\sum_{k=1}^N \frac{1}{k^{2(1-\beta)}}$ is convergent to a positive constant not null $C(1-\beta)$. As $1/k^2 < 1/k^{2(1-\beta)}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2(1-\beta)}} = C(1-\beta) = \zeta_1(2(1-\beta)) = \zeta(2(1-\beta))$$

From the equation (12), it follows that :

$$s = 1 - \beta + i\gamma, \eta(s) = 0 \implies \lim_{N \rightarrow +\infty} \sum_{k,k'=1; k \neq k'}^N (-1)^{k+k'} \frac{\cos(\gamma \log(k/k'))}{(kk')^{(1-\beta)}} = -\zeta(2(1-\beta)) > -\infty \quad (13)$$

The second line of the above equation is the key of the proof. Its left-hand side contains the parameters γ and β , its right-hand side depends only on β . It follows that $\gamma = 0$ or $\gamma \rightarrow +\infty$ account for the absence of parameter γ from the right-hand side of the equation (13), but :

- if $\gamma = 0 \implies \eta(s) = \eta(1-\beta)$, following the proposition (1.4), $\eta(s) > 0$ and the contradiction with $\eta(s) = 0$,

- if $\gamma \rightarrow +\infty$, the zero $s = 1 - \beta + i\gamma$ will not be considered as a zero of the function eta because $\gamma, \beta \in \mathbb{R}^*$.

The two cases above implies that $\eta(s)$ does not vanish for $\Re(s) \in]1/2, 1[$ and $\Re(s) \in]0, 1/2[$. Then, the function $\eta(s)$ has all its zeros on the critical line $\Re(s) = \frac{1}{2}$. From the equivalent statement (1.3), it follows that **the Riemann hypothesis is verified**.

Q.E.D.

□

We therefore announce the important theorem as follows:

Theorem 4.1. *The Riemann hypothesis is true :*

All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

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References

- [1] E. Bombieri, The Riemann Hypothesis, In The millennium prize problems, edited by J. Carlson, A. Jaffe and A. Wiles, (2006), 107-124, Clay Math. Institute, Amer. Math. Soc., Providence, RI. <https://www.claymath.org/wp-content/uploads/2022/05/riemann.pdf>
- [2] P. Borwein, S. Choi, B. Rooney and A. Weirathmueller, The Riemann hypothesis - a resource for the aficionado and virtuoso alike, 1st Ed. CMS Books in Mathematics, Springer-Verlag, New-York, 2008. <https://doi.org/10.1007/978-0-387-72126-2>
- [3] E.C. Titchmarsh and D.R. Heath-Brown, The theory of the Riemann zeta-function. 2nd Ed. revised by D.R. Heath-Brown, Oxford University Press, New-York, 1986.