

A Fourier view of the Riemann hypothesis

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Abstract

An elementary proof of the Riemann hypothesis is provided using Fourier analysis on the alternating zeta function $\eta(s)$. The real series $|\eta(s)|^2$, whose zeros are equivalent to the non-trivial zeros, increases along $\Im(s)$ in the left half of the critical strip $0 < \Re(s) < 1/2$.

1. Introduction

The Riemann zeta function is intimately connected to the distribution of prime numbers through the Euler product:

$$(1) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

where p runs over all prime numbers and s is a complex variable. The zeta function is extended to the complex plane by analytical continuation, except for its simple pole at $s=1$. Its logarithmic derivative can be expressed as the Dirichlet series:

$$(2) \quad \varphi(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n)$ denotes the von Mangoldt function, which is nonzero precisely when n is a prime power. In contrast, the prime zeta function $P(s)$ is defined directly in terms of the primes:

$$(3) \quad P(s) = \sum_{\text{prime } p} \frac{1}{p^s}$$

For $0 < \Re(s) < 1$, the Zeta-function can be evaluated through the alternating zeta function $\eta(s)$, according to:

$$(4) \quad \eta(s) = (1-2^{1-s})\zeta(s)$$

This representation provides a convergent formulation in the critical strip. The zeta function is also related to $\zeta(1-s)$ through its functional equation:

$$(5) \quad \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

The non-trivial zeros of the zeta-function, which lie in the critical strip, $0 < \Re(s) < 1$, are believed to be deeply connected to the distribution of prime numbers. The Riemann hypothesis, first formulated by Bernhard Riemann in 1859[1], states that all non-trivial zeros lie on the critical line $\Re(s) = 1/2$. Despite extensive efforts to prove or disprove the hypothesis, it remains one of the most important open problems in mathematics [2–6].

2. Theorem

Lemma 2.1 The Riemann zeta function represents the Fourier transform of a one-dimensional weighted logarithmic series of all natural numbers.

Proof:

Let $f(x)$ represent a series of point mass at position set $\{x_i = \ln(n_i)\}$ with weight set $\{w_i = n_i^{-\beta}\}$ in one dimensional (1D) direct space $\mathbf{x}$ (β is a tuning parameter to adjust the weight):

$$(6) \quad f(x) = \sum_{n=1}^{\infty} n^{-\beta} \delta(x - \ln(n))$$

where δ is the Dirac delta function.

The Fourier transform of $f(x)$ is:

$$\begin{aligned} \zeta(k) &= \int_{-\infty}^{+\infty} \sum_{n=1}^{\infty} n^{-\beta} \delta(x - \ln(n)) e^{-2\pi i k x} dx = \sum_{n=1}^{\infty} n^{-\beta} \int_{-\infty}^{+\infty} \delta(x - \ln(n)) e^{-2\pi i k x} dx \\ &= \sum_{n=1}^{\infty} n^{-\beta} e^{-2\pi i k \ln(n)} = \sum_{n=1}^{\infty} n^{-\beta} n^{-2\pi i k} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \end{aligned}$$

where $s = \beta + 2\pi ki$ is a complex number as in (1), and k is the frequency variable in 1D k -space.

Property 2.1 Similarly, the alternating zeta function $\eta(k)$ represents the same series as $\zeta(k)$ but with different weight:

$$(7) \quad \eta(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\beta}} e^{-2\pi i k \ln(n)}$$

Also the Dirichlet series $\varphi(s)$ (2) represents the Fourier transform of the logarithmic series of the prime powers $\{x_i = \ln(q_i)\}$ with weight $\{w_i = \Lambda(q_i)q_i^{-\beta}\}$ and the prime zeta function $P(k)$ (3) represents the Fourier transform of the logarithmic series of all primes $\{x_i = \ln(p_i)\}$ with weight $\{w_i = p_i^{-\beta}\}$. Since $\eta(k)$ represents $\zeta(k)$ in the critical strip, we'll mainly use it in the following discussion.

Property 2.2 The frequency component set $\{k_i\}$ of $\zeta(s)$ in k-space is determined by the position set $\{x_i\}$ and is independent of the weighting parameter β . It is expected that $\zeta(k)$, $\eta(k)$, $P(k)$ and $\varphi(k)$ all share the same component set $\{k_i\}$ because they are all logarithmic series of the natural numbers and the logarithm of every natural number is a finite linear combination of the logarithms of the primes. Figure 1 shows 4 calculated Fourier transforms. It can be seen that $\zeta(k)$ and $\eta(k)$ have minima at the non-trivial zero positions marked by green dotted lines with $\{\gamma_i = 2\pi k_i = 14.13, 21.02, 25.01, \dots\}$, where $\varphi(k)$ and $P(k)$ have maxima at these positions. Therefore, the imaginary parts of the non-trivial zeros are the frequency components of $\zeta(k)$ and their positions can be determined from the minima of $\zeta(k)$ and $\eta(k)$, or the maxima of $P(k)$ and $\varphi(k)$ with different accuracy.

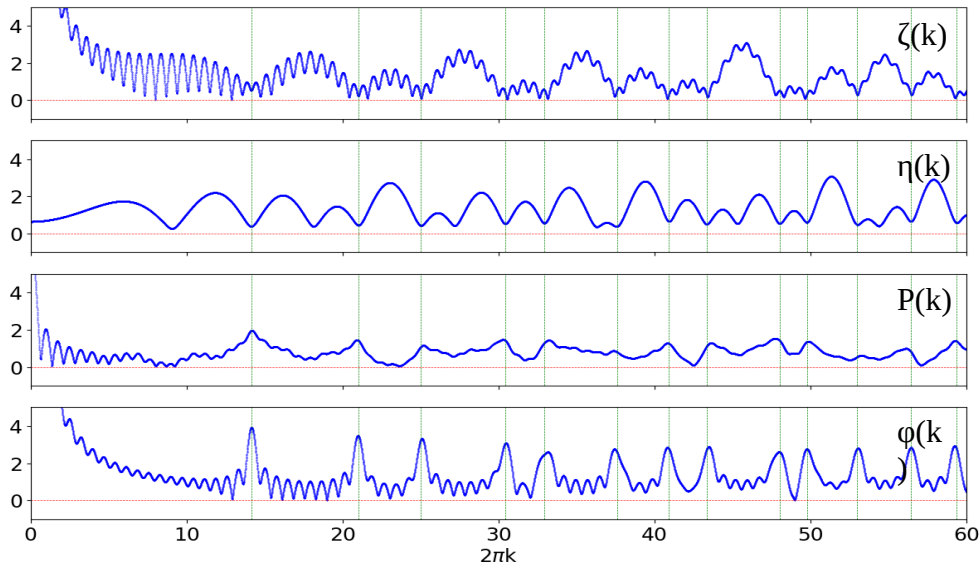


Figure 1. Fourier transforms of four logarithmic series of numbers with $\beta=0.75$. The frequency positions of the non-trivial zeros are marked with green lines. Vertical axis: amplitude of the spectrum.

Property 2.3 The weight set $\{w_i\}$ modulates the appearance or amplitudes of $\{k_i\}$ in k -space through the tuning parameter β . $\eta(k)$ at a different β represents a different view of the same component set $\{k_i\}$, therefore, the Fourier transform can be expressed as $\eta(k)|_\beta$ to reflect the role of β . For convenience, we use $\eta(s)$ in what follows. The variation of β spans the 1D k -space into a complex plane with k running along the vertical imaginary axis and β on the real axis.

Theorem 2.1 Real series $|\eta(s)|^2$ increases with $\Im(s)$ in the left half critical strip $0 < \Re(s) < 1/2$.

Proof:

The intensity of $\eta(s=\beta+i\gamma)$ in (7) can be expressed as:

$$(8) \quad I(s) = |\eta(s)|^2 = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{(-1)^{n_1+n_2}}{(n_1 n_2)^\beta} \cos(\gamma \ln(\frac{n_2}{n_1}))$$

$$(9) \quad = \sum_{n=1}^{\infty} \frac{1}{n^{2\beta}} + 2 \sum_{n_1=1}^{\infty} \sum_{n_2=n_1+1}^{\infty} \frac{(-1)^{n_1+n_2}}{(n_1 n_2)^\beta} \cos(\gamma \ln(\frac{n_2}{n_1}))$$

$$(10) \quad = I_e - I_o$$

where I_o is the sum of odd terms (n_1+n_2 is odd) and I_e is the sum of even terms including the diagonal sum in (9). Since $I(s)$ is non-negative, $I_e \geq I_o$.

A non-trivial zero occurs if and only if $I(s)=0$, so the even sum should equal to the odd sum:

$$(11) \quad \lim_{N \rightarrow \infty} \sum_{k=1}^{2k < N} \sum_{n=1}^{N-2k} \frac{\cos(\gamma \ln(\frac{n+2k}{n}))}{(n(n+2k))^\beta} + \frac{1}{2} \sum_{n=1}^N \frac{1}{n^{2\beta}} = \lim_{N \rightarrow \infty} \sum_{k=1}^{2k-1 < N} \sum_{n=1}^{N-2k+1} \frac{\cos(\gamma \ln(\frac{n+2k-1}{n}))}{(n(n+2k-1))^\beta}$$

Since the frequency components $\{\gamma_i\}$ exist for all β values in the critical strip $(0, 1/2]$ and they can be determined independent of β , constrain (11) is mainly for β and it determines the optimal β value that turns the normal minima at positions $\{\gamma_i\}$ to zeros.

Figure 2 shows 6 $I(s)$ spectra at $\beta=0.0, 0.1, 0.2, 0.3, 0.4$ and 0.5 . It is apparent that for $\beta < 1/2$ the gap $\Delta = I_e - I_o$ increases with γ and attenuates with β , and it is oscillating in between two envelopes.

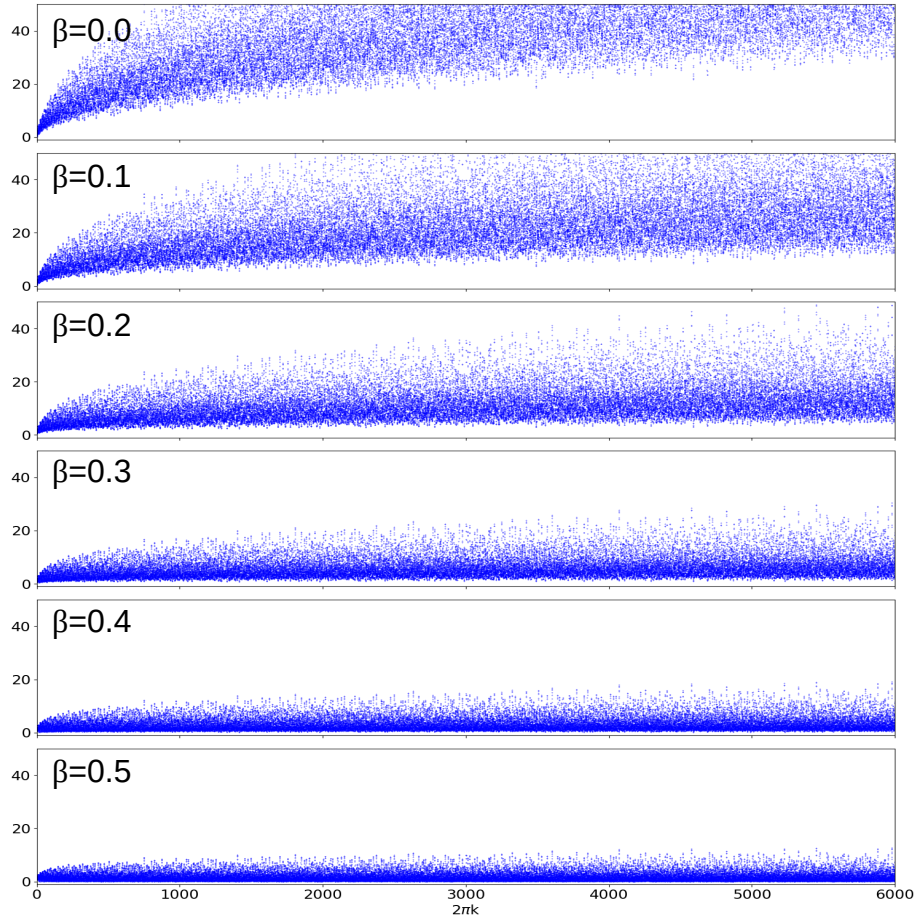


Figure 2. Amplitude spectra of $\eta(s=\beta+i\gamma)$ show the increase of I_e-I_o along imaginary axis ($\gamma=2\pi k$) and attenuation with β . Vertical axis: amplitude $\sqrt{I(s)}$. The upper and lower envelopes of the spectra are both linear in $I(s)$.

We also calculated spectra with β close to 0.5. The mean density of the spectra (partial sum for $N=500,000$) for β (0, 0.6) in three intervals of γ are listed in the following table.

$\gamma \setminus \beta$	0.0	0.1	0.2	0.3	0.4	0.49	0.5	0.51	0.6
[2000, 2100]	812.43	277.71	99.01	37.75	15.96	8.31	7.79	7.31	4.49
[4000, 4100]	1623.24	482.45	149.77	50.01	18.8	9.02	8.4	7.84	4.6
[6000, 6100]	2457.72	683.01	199.29	63	22.62	10.49	9.73	9.04	5.16

Both envelopes appear linear in the intensity spectra $I(s)$. The lower envelope can be estimated from the minima of the intensity spectra in each interval $\Delta\gamma=10$ of γ from 10. Assuming linearity and zero intercept, the estimated slopes are listed below and the lowest is $8.3405e^{-7}$ at $\beta=0.5$. For every γ ,

β	0.0	0.1	0.2	0.3	0.4	0.49	0.5	0.51	0.6
slope	0.1298	0.021	$2.7e^{-3}$	$2.47e^{-4}$	$1.01e^{-5}$	$8.59e^{-7}$	$8.34e^{-7}$	$8.56e^{-7}$	$2.81e^{-6}$

the intensity at $\beta=0.5$ is the lowest, so there is no zero detected for $\beta < 1/2$ with γ up to 6100.

The rising background of $I(s)$ mainly originates from the odd-even pairs $(2k-1)^{-s}$, $(2k)^{-s}$ of $\eta(s)$ for $2k \gg \gamma > 1$:

$$\begin{aligned}
 d_k &= (2k-1)^{-s} - (2k)^{-s} = -(2k)^{-s} \left(1 - \left(\frac{2k-1}{2k}\right)^{-s}\right) = -(2k)^{-s} \left(1 - \left(\frac{2k}{2k-1}\right)^\beta e^{-i\gamma \ln(\frac{2k-1}{2k})}\right) \\
 &\approx -\frac{e^{-i\gamma \ln(2k)}}{(2k)^\beta} \left(1 - e^{-i\gamma \ln(\frac{2k-1}{2k})}\right) \approx -\frac{e^{-i\gamma \ln(2k)}}{(2k)^\beta} \left(1 - e^{\frac{i\gamma}{2k}}\right) \\
 &\approx -\frac{e^{-i\gamma \ln(2k)}}{(2k)^\beta} \left(1 - \left(1 + \frac{i\gamma}{2k}\right)\right) \\
 (12) \quad &= \frac{\gamma}{(2k)^{\beta+1}} e^{i(\pi/2 - \gamma \ln(2k))}
 \end{aligned}$$

Since the phase differences between the k^{th} pair and the $(k+1)^{\text{th}}$ pair, the $(k+2)^{\text{th}}$ pair, ..., are very small,

$$\gamma \ln(2(k+1)) - \gamma \ln(2k) = \gamma \ln(1 + 1/k) \approx \gamma/k$$

d_k, d_{k+1}, d_{k+2} are all in phase, d_k in (12) can be multiplied by a number m . Assume the partial sum $\eta_N(s) = I_N^{1/2} e^{i\varphi}$ for $N=2k-2$, then the new partial sum is

$$(13) \quad \eta_{N+} = I_N^{1/2} e^{i\varphi} + m \frac{\gamma}{(2k)^{\beta+1}} e^{i(\pi/2 - \gamma \ln(2k))}$$

and the new intensity is:

$$(14) \quad I_{N+} = I_N + 2mI_N^{1/2} \frac{\gamma}{(2k)^{\beta+1}} \sin(\varphi + \gamma \ln(2k)) + m^2 \frac{\gamma^2}{(2k)^{2\beta+2}}$$

Since the phase difference between the $(k-1)^{\text{th}}$ pair and the k^{th} pair is very small, the phase φ built from previous pairs $(k-1)^{\text{th}}, (k-2)^{\text{th}}, \dots$, should often be close to the phase of the k^{th} pair (12), and much more often $I_{N+} - I_N > 0$, and it accumulates, which leads to a generally increasing background.

The rising background of $I(s)$ can also be derived directly from (11). Start from the diagonal line with $n_1 = n_2$ or $\delta = n_2 - n_1 = 0$, all terms are even and the sum is a constant:

$$(15) \quad I_d = \sum_{n=1}^N \frac{1}{n^{2\beta}}$$

For the first two near diagonal lines with $\delta=|n_2-n_1|=1$, all terms are odd and the next with $\delta=|n_2-n_1|=2$ all even and so on. For each term with $(n-1)/n \sim 1$, it can be approximated as:

$$\cos(\gamma \ln(1-\frac{1}{n})) \approx \cos(\frac{\gamma}{n}) \approx 1 - \frac{\gamma^2}{2n^2}$$

Generally, there are more odd terms ($\sim N$) than non-diagonal even terms close to (N, N) , and they contribute to the gap $\Delta=I_e-I_o$ as:

$$(16) \quad I_d - \sum_{n=m}^N \frac{1 - \frac{\gamma^2}{2n^2}}{n^{2\beta}} = \frac{1}{2}\gamma^2 \sum_{n=m}^N \frac{1}{n^{2\beta+2}} + \sum_{n=N-m}^N \frac{1}{n^{2\beta}}$$

where m is close to N . The effectiveness of (16) can be confirmed: if these cosine terms are replaced by 1, the background will fall quickly to negative even for $\beta=0$. Figure 3 shows an intensity spectrum with $N=10000$ and the cosine terms of $|1-n_1/n_2|<0.001$ replaced by 1.0.

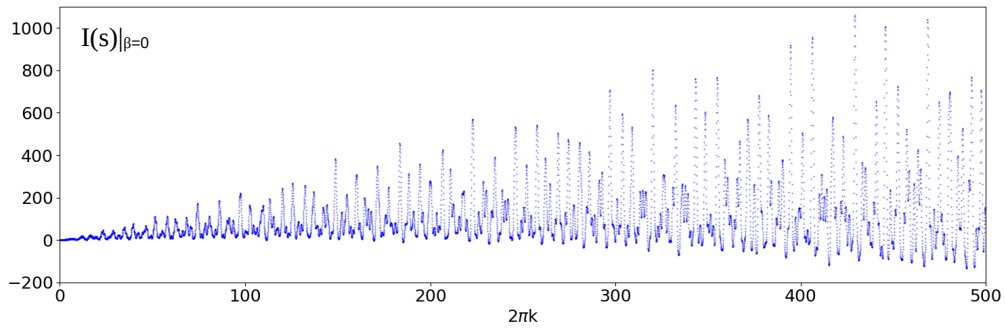


Figure 3. Intensity spectrum $I(s)$ with $\beta=0$ and some near-diagonal terms are replaced by 1.

Similarly, for a pair of far-off diagonal terms such as $(n-1,1)$ and $(n,1)$, they can be approximated as:

$$(17) \quad \begin{aligned} \frac{1}{(n-1)^\beta} \cos(\gamma \ln(n-1)) - \frac{1}{n^\beta} \cos(\gamma \ln(n)) &\approx -\frac{2}{n^\beta} \sin\left(\frac{1}{2}\gamma \ln\left(\frac{n-1}{n}\right)\right) \sin\left(\frac{1}{2}\gamma \ln((n-1)n)\right) \\ &\approx \frac{\gamma}{n^{\beta+1}} \sin\left(\frac{1}{2}\gamma \ln((n-1)n)\right) \end{aligned}$$

These properties of $\eta(s)$ can be revealed from a small case with $N=4$. For convenience, let $\beta=0$ and denote $c_{ij} = \cos(\gamma \ln(i/j))$. Then

$$\begin{aligned}
I_{N=4} &= 4+2(c_{13}+c_{24}-c_{12}-c_{14}-c_{23}-c_{34}) = 4+2((c_{13}-c_{14})-c_{23}-c_{34}) \\
&\approx 2+\frac{\gamma}{2}\sin\left(\frac{\gamma}{2}\ln(12)\right)-2\cos\left(\gamma\ln\left(\frac{3}{2}\right)\right)+\frac{\gamma^2}{16} \\
(18) \quad &= 2+\frac{\gamma}{2}\sin(1.24\gamma)-2\cos(0.41\gamma)+\frac{\gamma^2}{16}
\end{aligned}$$

where term c_{24} is canceled by c_{12} . It is a good approximation for γ up to $\sim 2N=8$. The first minimum ($\gamma>0$) is at $1.24\gamma \sim 3\pi/2$, $\gamma\sim 3.8$ with intensity of ~ 1.0 , and the second minimum occurs when $1.24\gamma \sim 7\pi/2$, $\gamma\sim 8.87$ with intensity 4.2, which is higher than the first minimum 1.0.

The attenuation of $I(s)$ with β from 0 to $\frac{1}{2}$ is reflected from (11)-(17). At $\beta=\frac{1}{2}$, the diagonal sum (15) equals $\zeta(s=1)$, which is a transition line from divergence to convergence.

Because of the increase of I_e-I_o along γ for $0<\beta<\frac{1}{2}$ and consider the fact that there is no zero ($I_e-I_o>0$) below $\gamma=(6000, 6100)$, we can conclude that there is no non-trivial zero forward for $\gamma>6100$ in the left half of the critical strip $0<\beta<1/2$; and according to the functional equation (5) no zero can exist in the other half of the critical strip $1/2<\Re(s)<1$, therefore, the non-trivial zeros have to lie on the critical line $\Re(s)=1/2$. Figure 4 shows how precisely the non-trivial zeros land on the critical line.

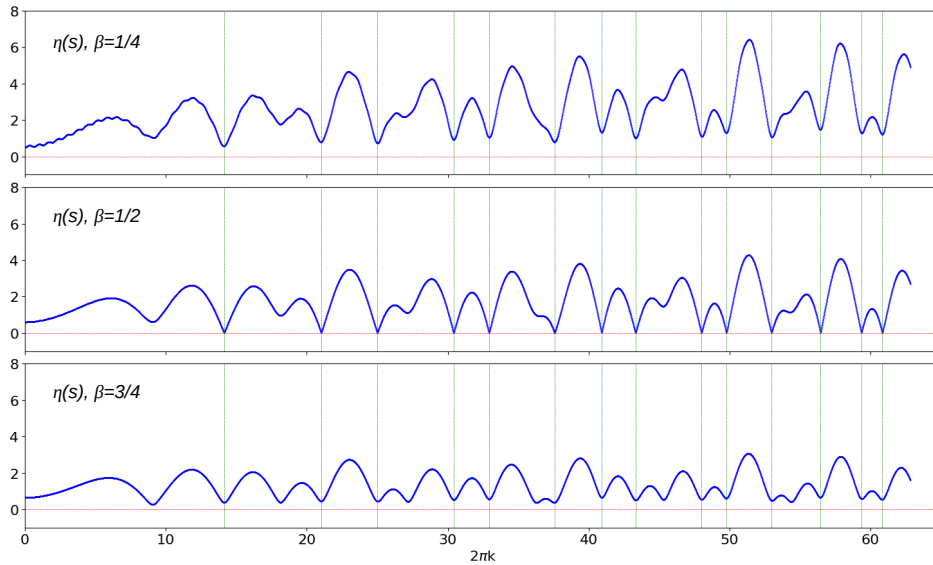


Figure 4. Amplitude spectra of $\eta(s)$ at $\beta = \frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$. All non-trivial zeros have a sharp ending on the critical line.

Remark 2.1 $\eta(\beta+i\gamma)$ is only convergent for limited γ in the critical strip $0<\beta<1$.

From (12), $\eta(\beta+i\gamma)$ becomes an even zeta series modulated by $\gamma/(2k)$ when $2k$ is sufficiently large. When γ is small it is close to $\zeta(\beta+1+i\gamma)$ and convergent for $0<\beta<1$; but when γ approaches to $\sim 2k$, it becomes $\zeta(\beta+i\gamma)$ and divergent for $0<\beta<1$.

Note 2.1 The non-trivial zero set $\{\gamma_i = 2\pi k_i\}$ represents the characteristic frequencies of the logarithmic series of the natural number system and $\{\lambda_i = 2\pi/\gamma_i\}$ are the wavelengths that represent the intrinsic gaps of the series. The first 5 wavelengths $\{0.44452122, 0.29888562, 0.25121831, 0.20651474, 0.19077497\}$ and the first 5 gaps of the prime logs $\{0.40546511, 0.51082562, 0.33647224, 0.45198512, 0.16705408\}$ are on the same scale, where the largest gap 0.51082562 is from the first twin primes 3 and 5.

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