

ON THE UNIQUENESS OF SOLUTIONS TO THE CAUCHY PROBLEM FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH PARTIAL DERIVATIVES.

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Despite the efforts of many
outstanding mathematicians,
this question is still
not completely resolved
I.G.Petrovsky [1.p. 49]

This article presents examples of linear partial differential equations with analytic coefficients and system of such equations for which the solutions to the Cauchy problem in the class of continuously differentiable functions are non-unique. More precisely, the time during which the analytical solution to the Cauchy problem remains unique is finite for such equations and system, after which this solution may even transform into one of many possible solutions. These solutions may have unusual properties.

Key words: systems of linear partial differential equations; analytical coefficients; Cauchy problem; Holmgren's theorem; uniqueness of solution.

I. We will further consider the following system of linear partial differential equations (and its special cases):

$$\frac{\partial \mathbf{u}}{\partial t} = \sum_{i=1}^m A_i(t, \mathbf{x}) \frac{\partial \mathbf{u}}{\partial x_i}, \quad (1)$$

here $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{x} = (x_1, \dots, x_m)$; $A_i(t, \mathbf{x})$ - are square matrices of size $n \times n$, whose elements are analytic functions in t and $\mathbf{x}$. For system (1) the Cauchy problem is solved with the initial condition

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in R^m, \quad (2)$$

If the function $\mathbf{u}_0(\mathbf{x}) = (u_{01}(\mathbf{x}), \dots, u_{0n}(\mathbf{x}))$ from the initial condition are analytic, then, as Kovalevskaya's theorem states, the Cauchy problem (1)-(2) has a unique analytic solution ([1],[2],[3]).

However, system (1) may have other, non-analytic solutions, and therefore the question arises about the uniqueness of such solutions in the class of continuously functions. The classical result that is believed to prove the uniqueness of such solutions is Holmgren's theorem ([1],[2],[3]), which proves that if two solutions $\mathbf{u}_1$

and $\mathbf{u}_2$ of system (1) of class C^1 satisfy the same initial condition at $t = 0$, then $\mathbf{u}_1 = \mathbf{u}_2$ in some neighborhood of the point $t = 0, \mathbf{x} = 0$.

The examples given below show that this does not mean that these two solutions coincide on a common domain of existans, which indicates the locality of the statement Holmgren's theorem. Some sufficient conditions for the non-uniqueness of solutions to the Cauchy problem (1)-(2) in the class of continuously differentiable functions are formulated. For such problems, if the initial condition (2) is analytic, then the solution may turn out to be analytic only on some finite time interval, and as t increases, it may cease to be so.

Example 1. Consider a simple equation of one unknown function with two independent variables and analytical coefficients

$$\frac{\partial u}{\partial t} = (1 + x^2) \frac{\partial u}{\partial x}, \quad (3).$$

its solution of class C^1 is the function

$$u(t, x) = \varphi(t + \operatorname{arctg} x) \quad (4),$$

where $\varphi(\xi)$, $\xi \in R^1$ - is an arbitrary continuously differentiable function.

If the initial condition is given $u(0, x) = \varphi(\operatorname{arctg} x) = u_0(x)$, then for $\xi = \operatorname{arctg} x$ we have $\varphi(\xi) = u_0(\operatorname{tg} \xi)$, i.e., the initial condition determines the function $\varphi(\xi)$ only for $|\xi| \leq \pi/2$, for other ξ , it remain arbitrary. Therefore the solution to the Cauchy problem is unique if $-\pi/2 \leq t + \operatorname{arctg} x \leq \pi/2$. For values $t > 0$, this defines a region of x , for which the solution is unique: $x \in (-\infty, \operatorname{tg}(\pi/2 - t))$. As t increases, this region of uniqueness of the solution decreases and disappears at $t = \pi$, after which the solution can transform into any of the solutions determined by the function $\varphi(\xi)$. Thus, the property of a solution to remain unique to some extent is preserved only for $t < \pi$, although the solution itself can be defined for any t and any x .

For example, the analytical solution $u(t, x) = t + \operatorname{arctg} x$, satisfies the initial condition $u(0, x) = \operatorname{arctg} x$, and is defined for all t and x . However, the same initial condition is satisfied by a different solution $u(t, x) = t + \operatorname{arctg} x + \Phi_0(t + \operatorname{arctg} x)$, where $\Phi_0(\xi)$ is an arbitrary function of class C^1 , that satisfies the condition $\Phi_0(\xi) = 0$ for $|\xi| \leq \pi/2$ and is non zero for the remaining ξ .

Note, that this result is consistent with Holmgren's theorem, with asserts the uniqueness of the solution of the Cauchy problem in some neighborhood of the point $t = 0, x = 0$.

On the other hand, it also implies the locality of the conclusion of this theorem (as well as other uniqueness theorems), which does not guarantee the uniqueness

of the solution in the entire domain of its existence. Thus an analytic solution may either remain analytic or cease to be so as t increases. This requires clarification for same statements about the obligatory analyticity of solutions ([1], [2], [3]).

It can also be said that a monosolution (unique and determined by the initial condition) ultimately transforms for $t > \pi$ into a polysolution (one of a set of possible solutions, determined by the choice of the function $\varphi(\xi)$).

Remark 1. Equation (4) and similar ones can describe systems or processes whose behavior for sufficiently large t is determined not by their state at the initial moment of time, but by some hidden parameters (in this case , the function $\varphi(\xi)$).

II. The properties of the solutions described in example 1 can also be encountered when considering more general problems. For example, for a first-order linear equation with analytic coefficients

$$\frac{\partial u}{\partial t} = \sum_{i=1}^m a_i(t, \mathbf{x}) \frac{\partial u}{\partial x_i}, \quad (6)$$

the following result holds.

Proposition 1. If the characteristic system of equation(6)

$$\frac{dx_i}{dt} = -a_i(t, x_1, \dots, x_m)$$

has a first integral $u(t, \mathbf{x})$ defined for any t and $\mathbf{x}$ and satisfying the following two conditions:

1. For some $d > 0$ and for any $\mathbf{x}$, $u(0, \mathbf{x}) \leq d$,
2. There exist values t and $\mathbf{x}$ such that $u(t, \mathbf{x}) > d$, than the solution to the Cauchy problem with the initial condition $u(0, \mathbf{x}) = u_0(\mathbf{x})$, $\mathbf{x} \in R^m$ for sufficiently large t in the class of function C^1 is not unique.

Proof. The solution to equation (6) is

$$\tilde{u}(t, \mathbf{x}) = \varphi(u(t, \mathbf{x})),$$

where $\varphi(\xi)$, $\xi \in R^1$ - is an arbitrary continuously differentiable function. If $\varphi(\xi) = 0$ for $\xi \leq d$ and is nonzero for other ξ , then $\tilde{u}(0, \mathbf{x}) = 0$, but the solution $\tilde{u}(t, \mathbf{x})$ is nonzero, i.e. solution to the Cauchy problem for equation (6) is not unique.

For a more general equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^m a_i(t, \mathbf{x}) \frac{\partial u}{\partial x_i} + b(t, \mathbf{x})u, \quad (7)$$

the following result is true.

Corollary. If the solution to the Cauchy problem for equation (6) is non-unique, then it is also non-unique for equation (7).

Proof. We will look for a solution to equation (7) in the form $u(t, \mathbf{x}) = v(t, \mathbf{x})w(t, \mathbf{x})$. Substituting into (7), we obtain

$$\frac{\partial v}{\partial t}w + v\frac{\partial w}{\partial t} = \sum_{i=1}^m a_i(t, \mathbf{x})\frac{\partial v}{\partial x_i}w + v\sum_{i=1}^m a_i(t, \mathbf{x})\frac{\partial w}{\partial x_i} + b(t, \mathbf{x})vw,$$

that is, $v(t, x)$ can be taken to be the solution $\tilde{u}(t, x)$ of equation (6), and $w(t, \mathbf{x})$ can be an arbitrary nonzero solution of equation (7). Since $v(t, \mathbf{x})$ is a nonzero solution satisfying the zero initial condition, then $u(t, x)$ is a nonzero solution of equation (7) satisfying the zero initial condition.

Sometimes the case of the general Cauchy problem (1)-(2) can be investigated similarly.

Definition. A solution $\mathbf{u}(t, \mathbf{x})$ is called a scalar-type solution if

$$u_1(t, \mathbf{x}) = u_2(t, \mathbf{x}) = \dots = u_n(t, \mathbf{x}) = v(t, \mathbf{x}),$$

those $\mathbf{u}(t, \mathbf{x}) = (v(t, \mathbf{x}), \dots, v(t, \mathbf{x}))$.

Proposition 2. If the system (1) has a solution $\mathbf{u}(t, \mathbf{x})$ of scalar type, defined for any t и $\mathbf{x}$, and such that $v(t, x)$ satisfies conditions 1 and 2 of proposition 1, than the solution to the Cauchy problem to the class of functions C^1 is not unique.

Proof.

A solution to system (1) is also $\mathbf{v}(t, \mathbf{x}) = (\varphi(v(t, \mathbf{x})), \dots, \varphi(v(t, \mathbf{x})))$, where $\varphi(\xi)$ is an arbitrary continuously differentiable function. Setting $\varphi(\xi) = 0$ if $\xi \leq d$ and nonzero for the remaining ξ , we obtain a nonzero solution to system (1) satisfying zero initial condition, i.e. the solution to the Cauchy problem is nonunique.

Example 2. Consider a system of two equation with two independent variables

$$\frac{\partial \mathbf{u}}{\partial t} = A(x)\frac{\partial \mathbf{u}}{\partial x}, \quad \mathbf{u} = (u_1, u_2), \quad A(x) = (a_{ij}(x)), \quad \mathbf{u}(0, x) = \mathbf{u}_0(x), \quad x \in R^1.$$

If the condition $a_{11} + a_{12} = a_{21} + a_{22}$ is satisfied, then the system has a solution of scalar type

$$u_1(t, x) = u_2(t, x) = t + \int_0^x \frac{dx}{a_{11}(x) + a_{12}(x)}.$$

If, in this case, for some $d > 0$ and any x the condition is satisfied

$$-d \leq \int_0^x \frac{dx}{a_{11}(x) + a_{12}(x)} \leq d,$$

then conditions 1 and 2 Proposition are satisfied 1 and the solution to the Cauchy problem for sufficiently large t is not unique.

III. For a system with two independent variables, having a quasidiagonal form

$$\frac{\partial u_k}{\partial t} = \lambda(x) \frac{\partial u_k}{\partial x} + \sum_{i=1}^n b_{ki}(x) u_i, \quad k = 1, \dots, n, \quad (8)$$

the following result is true.

Proposition 3. If the condition is satisfied: for some $d > 0$

$$-d \leq \int_0^x \frac{dx}{\lambda(x)} \leq d, \quad x \in R^1,$$

then the solution to the Cauchy problem for system (8) with initial condition $u_k(0, x) = u_{k0}(x), x \in R^1$ for sufficiently large t is not unique.

Proof. The system corresponding to the principal part in (8)

$$\frac{\partial v_k}{\partial t} = \lambda(x) \frac{\partial v_k}{\partial x}, \quad k = 1, 2, \dots, n, \quad (9)$$

has a solution

$$\tilde{v}_k(t, x) = \varphi \left(t + \int_0^x \frac{dx}{\lambda(x)} \right), \quad k = 1, 2, \dots, n.$$

which after choosing the function $\varphi(\xi) = 0$ if $|\xi| \leq d$ and nonzero for the remaining ξ , yields a nonzero solution for system (9) satisfying zero initial condition. For system (8) a similar solution can be taken: $u_k(t, x) = \tilde{v}_k(t, x) \tilde{u}_k(t, x)$, where $\tilde{v}_k(t, x)$ is the constructed solution of system (9), and $\tilde{u}_k(t, x)$ is an arbitrary nonzero solution of system (8).

Remark 2.

For $t > 2d$, the solution is non unique, and the properties of the solution determined by the function $\varphi(\xi)$ and can be quite unusual condition. For example if we require that $\varphi(\xi) = 0$ for $\xi \in [(4k-1)d, (4k+2)d], k = 1, 2, 3, \dots$ and is non-zero for other ξ , then the solution $v_k(t, x) = 0$ if $t \in [4kd, (4k+1)d]$ and is non-zero for other t .

This solution can be interpreted as a multitude of particles, each existing in its own time interval, or as a single particle appearing out of nowhere, disappearing, and reappearing.

Remark 3. If the characteristic system of a differential equation has several independent first integrals with the corresponding properties, then the solution of the Cauchy problem for it may also be non-unique.

Consider, for example, the equation

$$\frac{\partial u}{\partial t} = (1 + x_1^2) \frac{\partial u}{\partial x_1} + (1 + x_2^2) \frac{\partial u}{\partial x_2}.$$

Its solution is $u(t, x) = \varphi(t + \arctg x_1, t + \arctg x_2)$, where $\varphi(\xi_1, \xi_2)$ is an arbitrary continuously differentiable function. The solution to the Cauchy problem for this equation is not unique, in particular, at all points (t, x_1, x_2) , for which $t \geq \pi$. Moreover, by choosing the function $\varphi(\xi_1, \xi_2)$ the solution can be endowed with the most unexpected properties.

For example, if the function $\varphi(\xi_1, \xi_2)$ defines an image on the (ξ_1, ξ_2) plane, then the solution will be constructed using the values of this function lying in the square $(t + (-\pi/2, \pi/2), t + (-\pi/2, \pi/2))$, which moves along the diagonal $\xi_1 = \xi_2$ as t increases. When transitioning using the solution to the (x_1, x_2) plane, this yields a corrected, moving image.

LITERATURE

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