

Reproducing Kernels, Theta Functions and Fundamental Inequalities

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(At first, I will introduce simply the analytical and classical reproducing kernels on the complex plane, based on [24], in particular, Section 3, Examples of Reproducing Kernels).

Here, firstly, we recall the representations of reproducing kernels in terms of the theta functions that were obtained by D. A. Hejhal, J. D. Fay and A. Yamada. Indeed, the materials stated here are in Yamada [30, 31]. Then, we will consider their meanings from the viewpoints of reproducing kernel theory, in particular, from the viewpoint of inequalities. We will find many important open problems.

(The Second Lecture at Peking University on 3 October, 2026)

1. INTRODUCTION

Let R be a planar regular region with n boundary components. Its Schottky double $\hat{R}$ is a compact Riemann surface of genus $g = n - 1$ admitting an anti-conformal involution ϕ fixing the boundary ∂R of R . For simplicity we use the notation that $\bar{z} = \phi(z)$ for $z \in \hat{R}$ and $\bar{R} = \phi(R)$. Note that $\bar{R}$ is not the closure of R . Let C denote a general (not necessarily symmetric) compact Riemann surface of genus g .

For theta functions $\theta(z)$ and prime-forms $E(x, y)$, we cite the excellent lecture note by Fay [4].

Simply:

The first order theta function with characteristic $\begin{bmatrix} \delta \\ \varepsilon \end{bmatrix}$ ($\delta, \varepsilon \in \mathbb{R}^g$) is defined by

$$\theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z) = \sum_{m \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}(m + \delta)\tau(m + \delta)^t + (z + 2\pi i\varepsilon)(m + \delta)^t\right\}, \quad z \in \mathbf{C}^g.$$

For $j = 1, \dots, g$, $\theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z)$ satisfies the identities

$$(1.1) \quad \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z_1, \dots, z_j + 2\pi i, \dots, z_g) = e^{2\pi i \delta_j} \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z)$$

and

$$(1.2) \quad \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z_1 + \tau_{j1}, \dots, z_g + \tau_{jg}) = e^{-\frac{1}{2}\tau_{jj} - z_j - 2\pi i \varepsilon_j} \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z).$$

Riemann's theta function is denoted by $\theta(z) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z)$.

The prime-form is given by

$$E(x, y) = \frac{\theta[\alpha](y - x)}{h_\alpha(x)h_\alpha(y)} \quad x, y \in \hat{R}$$

where α is a non-singular odd half-period and h_α is a “half-order” differential on $\hat{R}$ satisfying $h_\alpha^2(x) = \sum_{j=1}^g \frac{\partial \theta[\alpha]}{\partial z_j}(0) u_j(x)$. Here, $\{u_i(x)\}_{i=1}^g$ are the normalized differentials of the first kind with some canonical homology basis. For fixed $x \in R$, $E(x, y)$ is a multiplicative $-\frac{1}{2}$ order differential in y with multipliers 1 and $\exp(-\frac{\tau_{ji}}{2} - \int_x^y u_j)$ along the A_j and B_j cycles respectively.

Let $C_B(a)$ and $C_\beta(a)$ be the **analytic capacity** and the **logarithmic capacity** of R at $a \in R$ respectively. More explicitly these constants are defined by $C_B(a) = \sup\{|f'(a)| \mid f \in H^\infty(R) \text{ and } \|f\|_\infty \leq 1\}$ (*The extremal function is the Ahlfors function*) and $C_\beta(a) = \lim_{z \rightarrow a} \exp(-g(z, a))/|z - a|$, where $H^\infty(R)$ is the space of bounded holomorphic functions on R .

In 1972 by using theta functions D. Hejhal [12] solved an open problem mentioned in the book of Sario and Oikawa [27]: If $g > 0$ then, for all $z \in R$,

$$(1.3) \quad C_B(z)^2 < \pi K(z, \bar{z})$$

where $C_B(z)$ is the analytic capacity and $K(z, \bar{w})$ is the Bergman kernel. – Meanwhile, N. Suita gave a simple and direct proof.

To prove this he found the following identity: for all $x, y \in C$ and $e \in \mathbf{C}^g$ with $\theta(e) \neq 0$

$$(1.4) \quad \frac{\theta(y - x - e)\theta(y - x + e)}{\theta^2(e)E(x, y)^2} = \omega(x, y) + \sum_{i,j=1}^g \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(e) u_i(x) u_j(y),$$

where $\omega(x, y) = \frac{d^2}{dx dy} \log E(x, y) dx dy$ is the bilinear normalized differential of the second kind on a compact Riemann surface C of genus g . Specializing $x = z$, $y = \bar{z}$ and $e = 0$ in (1.4) we have

$$(1.5) \quad \left(\frac{\theta(z - \bar{z})}{\theta(0)E(z, \bar{z})} \right)^2 = -\omega(z, \bar{z}) - \sum_{i,j=1}^g \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(0) u_i(z) \overline{u_j(z)}.$$

This identity gives the basic relation of the Bergman kernel and Szegő kernel on the Schottky double $\hat{R}$:

Indeed, let $K(x, \bar{y})$ and $\hat{K}(x, \bar{y})$ be the Bergman kernel and the Szegő kernel on R . Then we have

$$K(x, \bar{y}) = -\frac{1}{\pi} \omega(x, \bar{y}), \quad \hat{K}(x, \bar{y}) = \frac{1}{2\pi i} \frac{\theta(x - \bar{y})}{\theta(0)E(x, \bar{y})}.$$

By direct calculation using the definition of theta functions, the matrix $(\frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(0))$ is easily seen to be positive definite. Noting that $\frac{\theta(z - \bar{z})}{\theta(0)E(z, \bar{z})} = C_B(z)$, $-\omega(z, \bar{z}) = \pi K(z, \bar{z})$, Hejhal obtained (1.3) from (1.5) immediately.

In 1973 J. D. Fay obtained a formula more powerful than (1.4) in his lecture note [4] (triscant formula):

for $x, y, a, b \in C$ and $e \in \mathbf{C}^g$ with $\theta(e) \neq 0$

$$(1.6) \quad \frac{\theta(x-a-e)\theta(y-b-e)}{\theta^2(e)E(x,a)E(y,b)} - \frac{\theta(x-b-e)\theta(y-a-e)}{\theta^2(e)E(x,b)E(y,a)} \\ = \frac{\theta(x+y-a-b-e)E(x,y)E(b,a)}{\theta(e)E(x,a)E(x,b)E(y,a)E(y,b)}.$$

Hejhal's formula (1.4) is obtained from (1.6) as follows. Dividing both sides of (1.6) by $\theta(x-a-e)/E(x,a)$ and then letting $y \rightarrow b$ using L'Hospital's rule, we have, after suitably permuting the variables,

$$(1.7) \quad \frac{\theta(x-a+e)\theta(x-b-e)E(a,b)}{\theta(e)\theta(e+b-a)E(x,a)E(x,b)} \\ = \omega_{b-a}(x) + \sum_{j=1}^g \left[\frac{\partial \log \theta}{\partial z_j}(e+b-a) - \frac{\partial \log \theta}{\partial z_j}(e) \right] u_j(x),$$

where $\omega_{b-a}(x)$ is the normalized differential of the third kind with simple poles of residue $-1, 1$ at a, b respectively. Again differentiating on b at $b = a$, we have Hejhal's formula (1.4). Thus, (1.6) $\implies$ (1.7) $\implies$ (1.4). In fact, Hejhal deduced (1.4) by first proving (1.7). See Yamada [31] for the details.

In conclusion we can say that Fay has found a formula stronger than Hejhal's but Hejhal has found the meaning of the formula. We emphasize that the key point of the proof was the positive definiteness of the matrix $(\frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(0))$.

The basic identity (1.5) was not considered to consider the Oikawa-Sario open problem, but to consider the Szegő kernel on the Schottky double $\hat{R}$. Indeed, the Szegő kernel is not a differential as in the Bergman kernel, but half order differential and its structure is very complicated, indeed, it is not determined uniquely and we will need a long historical results as in

G. F. B. Riemann (1826–1866); F. Klein (1849–1925); S. Bergman;
G. Szegő; Z. Nehari; M. M. Schiffer; P. R. Garabedian; D. A. Hejhal
(1972, thesis).

Hejhal's thesis [12] is a profound result which establishes the fundamental relationship between the Bergman and the Szegő reproducing kernels on Riemann surfaces [12].

In particular, note the very interesting results that the Bergman and the Szegő reproducing kernels are extended analytically (with poles) onto the Schottky double $\hat{R}$.

The basic domain functions (differentials) may be represented in terms of the Krein prime forms and the theta functions; they seem to be the atom decomposition.

2. RELATION BETWEEN CONFORMAL INVARIANTS AND THETA FUNCTIONS

We recall some symmetry relations because of its importance [4, Chap. VI].

Proposition 2.1 (Fay). *On $\hat{R}$ the following symmetry holds: for all $x, y \in \hat{R}$ and $e \in \mathbf{C}^g$*

$$(2.1) \quad \overline{\theta(x-y+e)} = \theta(\bar{x}-\bar{y}+\bar{e}), \quad \overline{E(x,y)} = E(\bar{x},\bar{y}), \quad \overline{\omega(x,y)} = \omega(\bar{x},\bar{y}),$$

where $\bar{e}$ is defined by $(\bar{e}_1, \dots, \bar{e}_g)$ if $e = (e_1, \dots, e_g)$.

There is a simple relation between the prime-forms and Green's function $g(x, y)$ on R .

Lemma 2.1. $\left| \frac{E(x, y)}{E(x, \bar{y})} \right| = \exp(-g(x, y))$, for all $x, y \in R$.

Proof. See Lemma 2.2 of [30]. $\square$

Proposition 2.2. Let $K(x, \bar{y})$ and $\hat{K}(x, \bar{y})$ be the Bergman kernel and the Szegő kernel on R . Then we have

$$K(x, \bar{y}) = -\frac{1}{\pi} \omega(x, \bar{y}), \quad \hat{K}(x, \bar{y}) = \frac{1}{2\pi i} \frac{\theta(x - \bar{y})}{\theta(0)E(x, \bar{y})}.$$

Proof. See [12] or [4]. However, by using Schiffer's identity $K(x, \bar{y}) = -\frac{2}{\pi} \frac{\partial^2 g(x, y)}{\partial x \partial \bar{y}}$ [23, p.21], the first identity is an easy consequence of Lemma 2.1, since we have $\omega(x, y) = \frac{d^2}{dx dy} \log E(x, y)$ [4, p.20]. $\square$

As seen in Introduction, many conformal invariants are expressed by theta functions and prime-forms.

Proposition 2.3. $C_B(x) = \frac{\theta(x - \bar{x})}{\theta(0)iE(x, \bar{x})}$, $C_\beta(x) = \frac{1}{iE(x, \bar{x})}$.

Proof. Since $C_B(x) = 2\pi\hat{K}(x, \bar{x})$ [1, p.118], the first identity is clear from Proposition 2.2. By noting that, near $x = y$, the expansion $E(x, y) = y - x +$ higher order terms holds [4, p.19], the second identity follows from Lemma 2.1. $\square$

Remark 2.1. Our choice of a canonical homology basis permits us to assume that the vector $x - \bar{x} \in \mathbf{C}^g$ is purely imaginary for all $x \in R$. Then from the definition of theta function it is easy to see that $\theta(x - \bar{x}) < \theta(0)$ for all $x \in R$ when $g > 0$. Thus Proposition 2.3 gives immediately the trivial inequality $C_B(x) < C_\beta(x)$.

Remark 2.2. Now Hejhal's identity (1.5) is rewritten as

$$C_B(x)^2 = \pi K(x, \bar{x}) - \sum_{i,j=1}^g \frac{\partial^2 \log \theta}{\partial z_i \partial \bar{z}_j}(0) u_i(x) \overline{u_j(x)}.$$

Hence the inequality $C_B(x)^2 < \pi K(x, \bar{x})$ holds at once as stated in the Introduction.

In 1972 N. Suita [28] conjectured, when $g > 0$, the inequality

$$(2.2) \quad C_\beta(x)^2 < \pi K(x, \bar{x}), \quad \forall x \in R,$$

which is called the *Suita conjecture*. This is solved by Z. Blocki [2]. Furthermore, Professor Q. Guan proved this conjecture completely with other many inequalities by an entirely different method.

I considered an analogous problem for the Hardy H^2 kernel and posed an open problem [23, p.37], in our context, to prove that the matrix

$$(2.3) \quad \left(\frac{\partial^2 \log \theta}{\partial z_i \partial \bar{z}_j}(e_0) \right) \text{ is negative definite.}$$

This is called the *Saitoh conjecture* by Professor Guan's group. The constant e_0 is determined from the critical points of Green's function of R whose definition will be given in the next section.

3. WEIGHTED HARDY $\mathbf{H}^2$ KERNELS AND THETA FUNCTIONS

We collect here the main results of the paper of Yamada [30]. For proofs of these results see [30].

Lemma 3.1. *The Hilbert spaces $H_e^2(R)$ and $H_e^{2,1}(R)$ possess the reproducing kernels $R_e(x, \bar{y})$ and $\hat{R}_e(x, \bar{y})$ respectively given by:*

$$(3.1) \quad R_e(x, \bar{y}) = \frac{\theta(a - \bar{a} + e)\theta(x - \bar{y} + e)}{\theta(x - \bar{a} + e)\theta(a - \bar{y} + e)} \frac{E(x, \bar{a})E(\bar{y}, a)}{E(\bar{a}, a)E(x, \bar{y})}, \quad x, y \in R$$

$$(3.2) \quad \hat{R}_e(x, \bar{y}) = \frac{\theta(x - \bar{a} + e)\theta(a - \bar{y} + e)\theta(x - \bar{y} - e)E(a, \bar{a})}{\theta^2(e)\theta(a - \bar{a} + e)E(x, \bar{y})E(x, \bar{a})E(\bar{y}, a)}, \quad x, y \in R.$$

In particular,

$$(3.3) \quad \hat{R}_e(x, \bar{y}) = -R_e(\bar{y}, x)\omega_e(x)\omega_e(\bar{y}).$$

We shall call $R_e(x, \bar{y})$ the *Hardy H^2 kernel* and $\hat{R}_e(x, \bar{y})$ the *conjugate Hardy H^2 kernel* associated with $e \in T_{++}$ (c.f. [23, Chapter III]). Note that $R_{e_0}(x, \bar{y}) = K_a(x, \bar{y})$ and $\hat{R}_{e_0}(x, \bar{y}) = \hat{K}_a(x, \bar{y})$ where $K_a(x, \bar{y})$ and $\hat{K}_a(x, \bar{y})$ are the classical Hardy H^2 kernel and conjugate Hardy H^2 kernel respectively [23, p. 30].

Surprisingly enough the representation (3.1) is given by Fay. His basic idea is that the adjoint L kernel $L_t(z, u)$ with my notation of the Hardy H^2 kernel is the **Cauchy kernel**; that is, $L_t(z, u)dz$ is an analytic differential with a simple pole, meanwhile, it is a meromorphic function in u , with a simple pole. Therefore, it is very fundamental and so he considered its representation in terms of the theta function and the Krein prime form. From its representation, we can obtain the representation of other related kernels, by using the elementary properties of the both functions.

4. A REMARK ON MY CONJECTURE

From (3.1) and (3.2) we have, for all $x, y \in R$,

$$(4.1) \quad R_e(x, \bar{y})\hat{R}_e(x, \bar{y}) = \frac{\theta(x - \bar{y} + e)\theta(x - \bar{y} - e)}{\theta^2(e)(iE(x, \bar{y}))^2},$$

and so

$$(4.2) \quad R_e(x, \bar{y})\hat{R}_e(x, \bar{y}) = \pi K(x, \bar{y}) - \sum_{i,j=1}^g \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(e) u_i(x) \overline{u_j(y)}.$$

In particular, the Saitoh conjecture (2.3) implies

$$(4.3) \quad \pi K(a, \bar{a}) < \hat{R}_{e_0}(a, \bar{a}),$$

since $R_{e_0}(a, \bar{a}) = 1$. We cannot prove the inequality (4.3) (the weak Saitoh conjecture) except for the case $g = 1$ [29, 23].

Proposition 4.1. *If $g > 0$ then for all $a \in R$*

$$C_\beta(a)^2 < \hat{R}_{e_0}(a, \bar{a}).$$

Proof. From (4.1) and Proposition 2.3 it suffices to show that

$$\frac{\theta(a - \bar{a} + e_0)\theta(a - \bar{a} - e_0)}{\theta^2(e_0)} > 1,$$

or equivalently, $F(e_0)/F(e_0 - a + \bar{a}) > 1$ with $F(e) = \theta(a - \bar{a} + e)/\theta(e)$, $e \in \hat{T}_0$. $\square$

Of course, if both the Suita and the Saitoh conjectures are proved then the above inequality becomes trivial.

5. MY CONJECTURE AND A VERY SURPRISINGLY DEEP RESULT BY Q. GUAN

My conjecture on the conjugate Hardy H_2 kernel was

$$R_e(x, \bar{y})\hat{R}_e(x, \bar{y}) \gg \pi K(x, \bar{y})$$

that means the contrary direction

$$\pi K(x, \bar{y}) \gg \hat{K}(x, \bar{y})^2$$

for the Szego kernel $\hat{K}(x, \bar{y})$ and this was proved by D. A. Hajnal by using the Riemann theta function as in stated in the above. The simple case is

$$R_t(x, \bar{y})\hat{R}_t(x, \bar{y}) \gg \pi K(x, \bar{y}).$$

Since,

$$R_t(x, \bar{t}) \equiv 1,$$

my conjecture, in particular, means that

$$\hat{R}_t(t, \bar{t}) \geq \pi K(t, \bar{t}).$$

I can not find any papers that discussed the relations of reproducing kernels and the Riemann theta functions, except A. Yamada's papers. Indeed, it seems that the relations are too complicated and too deep.

Some new proof of the positive definiteness of Hejhal that is not used the Riemann theta function is still an open problem. The proof of Hejhal is not elementary. However, the result is fundamental and very important, as we will see it.

Surprisingly enough, Professor Q. Guan proved completely this inequality with an entirely different and new idea. We were very and very surprised at his results and very extension results with his colleagues, see the cited references.

6. Q. GUAN'S GREAT EXTENSIONS

The great application of the Hejhal's positive definiteness is the following:

For two functions φ and ψ of $H_2(D)$ for any regular domain D , the analytic Hardy 2 space (Szegő space), we obtain the generalized isoperimetric inequality

$$(6.1) \quad \frac{1}{\pi} \iint_D |\varphi(z)\psi(z)|^2 dx dy \leq \frac{1}{2\pi} \int_{\partial D} |\varphi(z)|^2 |dz| \frac{1}{2\pi} \int_{\partial D} |\psi(z)|^2 |dz|,$$

and so, we obtain the bounded linear operator from the Szegő space to the Bergman space

$$(6.2) \quad \iint_D |f(z)|^2 dx dy \leq \frac{l(\partial D)}{4\pi} \int_{\partial D} |f(z)|^2 |dz|,$$

for the length $l(\partial D)$ of the boundary ∂D ([23]).

The meanings and applications of this norm inequality was discussed in [3] from the viewpoint of operator theory :

Indeed, for any fixed $\psi(z)$ member of the Szegő space, the operator

$$\varphi(z)|_\psi = \varphi(z)\psi(z)$$

is a bounded linear operator on the Szegő space into the Bergman space. Therefore, we can consider the general fractional, for any Bergman space member F ,

$$\frac{F}{\psi}.$$

Note that the inequality (6.1) is a curious one in the sense that the Bergman norm is for analytic differentials, but the Szegő norm for half order differentials. In connection with this inequality, we recall the interesting best possible norm inequality:

$$(6.3) \quad \begin{aligned} \iint_D |f'(z)|^2 dx dy &\leq \frac{1}{2} \int_{\partial D} \frac{|f'(z)dz|^2}{idW(z,t)} \\ &= \frac{1}{2\pi} \int_{\partial D} |f'(z)|^2 \left(\frac{\partial G(z,t)}{\partial \nu_z} \right)^{-1} |dz|, \end{aligned}$$

that means the relation between the Bergman norms and the weighted Szegő norm for (exact) differentials. Here, for the conjugate harmonic function $G^*(z,t)$ of the Green function $G(z,t)$ of D , $W(z,t) = G(z,t) + iG^*(z,t)$, and $idW(z,t)$ is a single-valued meromorphic differential and positive along the boundary ∂D and $\partial/\partial \nu_z$ is the inner normal derivative with respect to D . $\partial G(z,t)/\partial \nu_z$ is a positive continuous function on ∂D .

This surprising result was derived from very difficult problems on some comparison of the magnitudes of the Bergman and Rudin (Hardy H^2) reproducing kernels, however, here, we would like to consider some interesting and direct application of the identity. For the great and deep background for the problems, see the original paper ([7]). Its source was given by [22] and then the topics were cited in the book [23] in details.

The conjugate Hardy space

$$(6.4) \quad \begin{aligned} &\frac{1}{2} \int_{\partial D} \frac{|f(z)dz|^2}{idW(z,t)} \\ &= \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \left(\frac{\partial G(z,t)}{\partial \nu_z} \right)^{-1} |dz|, \end{aligned}$$

is defined for analytic differentials $f(z)dz$ and this space was very surprisingly generalized as the Ohsawa-Saitoh-Hardy space on n -dimensional complex manifolds by Q. Guan and Z. Yuan ([10]) through ([7, 8, 9]) with many concrete and deep results. Furthermore, surprisingly enough, further extensions were given recently in [11].

In particular, note that the curious conjugate Hardy space norm (6.4) is conformally invariant and the norm may be considered on the compact bordered Riemann surface, meanwhile, the Szegő norm is not and is not considered on the compact bordered Riemann surface.

7. FROM THE IDENTITY OF G. GUAN

Q. Guan ([7]) derived the surprising identity of the conjugate analytic Hardy norm and the Bergman norm. Let $H^2(D)$ denote the analytic Hardy class on D defined as the set of all analytic functions $f(z)$ on D such that the subharmonic functions $|f(z)|^2$ have harmonic majorants $U(z)$, i.e. $|f(z)|^2 \leq U(z)$ on D .

Here, for simplicity, we assume that the domain D is a regular domain surrounded by a finite number of analytic Jordan curves. All the results in this paper are valid on any compact bordered Riemann surface.

Then, Q. Guan ([7]) derived the **surprising identity**:

Lemma 2.3([7], page 3, Lemma 2.3): *For any given $t \in D$ and for any analytic function f on D which is continuous on $D \cup \partial D$, the identity, for $z = x + iy$*

$$\begin{aligned} & \lim_{r \rightarrow 1-0} \frac{1}{1-r} \int \int_{\{e^{2G(z,t)} \geq r\}} |f(z)|^2 dx dy \\ &= \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 (\partial G(z,t)/\partial \nu)^{-1} |dz| \end{aligned}$$

holds.

For the very surprising generalization of the conjugate Hardy H^2 space was given as follows by Professors Q. Guan and Z. Yuan and they obtained the corresponding results in a very general situation:

Let M be an n -dimensional complex manifold, and let K_M be the canonical line bundle on M . Let $\psi < 0$ be a plurisubharmonic function on M . Let $\rho \geq 0$ be a weight, which is a Lebesgue measurable function on M .

Definition. *The generalization of the conjugate Hardy H^2 space is defined as the set of all $f \in H^0(M, \mathcal{O}(K_M))$ such that*

$$\liminf_{t \rightarrow 0+0} \frac{1}{t} \int_{\{-t < \psi < 0\}} |f|^2 \rho < +\infty,$$

which is called Ohsawa-Saitoh-Hardy space and denoted by $OSH_\rho^2(M, \psi)$. We define a norm on the space $OSH_\rho^2(M, \psi)$ as follows:

$$\|f\|_{OSH_\rho^2(M, \psi)} := \left(\liminf_{t \rightarrow 0+0} \frac{1}{t} \int_{\{-t < \psi < 0\}} |f|^2 \rho \right)^{\frac{1}{2}}$$

for any $f \in OSH_\rho^2(M, \psi)$, where $|f|^2 := \sqrt{-1}^{n^2} f \wedge \bar{f}$.

Meanwhile, for some generalization of the weight in the Hardy H^2 space, we can consider some delicate property in [32]. We wrote an essay on the great Guan group mathematics in [17].

MAIN RESULT

The following inequalities seem to be interesting on its own sense:

Theorem: *For any given $\epsilon > 0$ and for any analytic function $f(z)$ on $D \cup \partial D$, there exists r_0 such that for all r , $r_0 < r < 1$ the inequality*

$$\int \int_D |f'(z)|^2 dx dy - \epsilon \leq \frac{1}{1-r} \int \int_{\{e^{2G(z,t)} \geq r\}} |f'(z)|^2 dx dy$$

is valid.

This inequality may be looked as an isoperimetric inequality, because the Dirichlet integral on a domain is estimated (restricted) by the Dirichlet integral on some small boundary neighborhood of the domain. Here, the neighborhood size and estimation are stated by the level curve of the Green function, precisely.

Even the case of the identity function $f(z) = z$, we can enjoy the senses of the estimation and the result.

PROOF OF THE MAIN RESULT

The integral of the conjugate analytic Hardy $H^2(D)$ norm seems to be not popular, however, the norm will have a beautiful structure and the norm is conformally invariant. Furthermore, we obtain the inequality:

For any analytic function $f(z)$ on $D \cup \partial D$, we have the inequality

$$(7.1) \quad \int \int_D |f'(z)|^2 dx dy \leq \frac{1}{2\pi} \int_{\partial D} |f'(z)|^2 (\partial G(z, t) / \partial \nu)^{-1} |dz|.$$

This result was derived from some complicated theory of reproducing kernels in ([22]). The equality problem in the inequality was also established; that is, equality holds if and only if the domain is simply-connected and the function $f'(z)$ is expressible in the form $CK(z, \bar{t})$ for the exact Bergman reproducing kernel $K(z, \bar{u})$ on the domain D and for a constant C .

We can obtain the main result by combining of the Guan identity and this inequality.

REMARKS

We note the following interesting problems:

Problem 1 : *How will be some generalization of the Guan identity for a general weight for $(\partial G(z, t) / \partial \nu)^{-1}$? However their deep answers were give by the Guan group. See the cited references.*

Problem 2 : *We can consider similar inequalities for various function spaces. For example, how will be the case for the harmonic Dirichlet integrals?*

Problem 3 : *The structure and proof of the inequality (7.1) are very complicated (involved) and the Guan identity is very unique. So, we are interested in some direct proof of the theorem.*

How will be some direct proofs of the inequalities (6.1), (6.2) and (7.1)?

For the inequality (7.1), note the inequality

$$\begin{aligned} & \left(\frac{1}{\pi} \int \int_D |f'(z)|^2 dx dy \right)^2 = \left(\frac{1}{2\pi i} \int_{\partial D} \overline{f(z)} f'(z) dz \right)^2 \\ & \leq \frac{1}{2\pi} \int \int_D |f(z)|^2 \frac{\partial G(z, t)}{\partial \nu} |dz| \frac{1}{2\pi} \int \int_{\partial D} |f'(z)|^2 (\partial G(z, t)/\partial \nu)^{-1} |dz|. \end{aligned}$$

Note that the relation of the Bergman norm and the weighted H^2 norm is very delicate. See [23, 22, 23].

A three-dimensional conjecture on thin boundary layers and volume

Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞ -smooth boundary ∂D , and let $t \in D$ be fixed. Let $G(x, t)$ be the Green function on D with pole at t , that is,

$$\Delta_x G(x, t) = -\delta_t(x) \quad \text{in } D, \quad G(x, t) = 0 \quad \text{on } \partial D.$$

For $s > 0$ small, we define the thin Green collar

$$\Omega_s := \{x \in D : 0 < G(x, t) < s\}.$$

Conjecture 1 (Three-dimensional Guan type boundary collar limit). For each fixed $t \in D$, we have

$$\lim_{s \downarrow 0} \frac{1}{s} \text{Vol}(\Omega_s) = \int_{\partial D} \frac{1}{\partial_\nu G(x, t)} dS(x),$$

where $\partial_\nu G(x, t)$ denotes the inner normal derivative of $G(\cdot, t)$ on ∂D , and dS is the surface measure on ∂D .

Conjecture 2 (Thin boundary collar controls the whole volume). For any $\varepsilon > 0$, there exist $s_0 > 0$ and a constant $C(D, t) > 0$ such that for all s with $0 < s < s_0$,

$$\text{Vol}(D) - \varepsilon \leq C(D, t) \frac{1}{s} \text{Vol}(\Omega_s).$$

In particular, if Conjecture 1 holds, then we expect an isoperimetric-type inequality of the form

$$\text{Vol}(D) \leq C(D, t) \int_{\partial D} \frac{1}{\partial_\nu G(x, t)} dS(x),$$

showing that the volume of D is controlled by a suitable boundary quantity.

Example (Spherical shell). Let

$$A_{a,b} := \{x \in \mathbb{R}^3 : a < |x| < b\}, \quad 0 < a < b.$$

Then

$$|A_{a,b}| = \frac{4\pi}{3}(b^3 - a^3), \quad |\partial A_{a,b}| = 4\pi(a^2 + b^2).$$

For $\delta > 0$ small, consider the thin boundary layer

$$A_\delta := \{b - \delta < |x| < b\} \cup \{a < |x| < a + \delta\}.$$

A direct computation shows

$$|A_\delta| = 4\pi(a^2 + b^2)\delta + 4\pi(b - a)\delta^2 + \frac{8\pi}{3}\delta^3,$$

and hence

$$\lim_{\delta \downarrow 0} \frac{|A_\delta|}{\delta} = 4\pi(a^2 + b^2) = |\partial A_{a,b}|.$$

This is a Euclidean boundary-layer model, rather than a direct computation of the Green collar. Thus, in this model case, the thin boundary layer volume, normalized by its thickness, recovers the boundary area, illustrating the phenomenon

$$\frac{\text{thin boundary layer volume}}{\text{thickness}} \longrightarrow \text{boundary area}.$$

This motivates Conjectures 1 and 2 for general smooth domains.

For a further direction: from volume to function spaces, see [16].

A typical problem for the Bergman space

First, recall the fundamental:

Proposition: *Let E_0 be a subset of E . Then the Hilbert space that $K|_{E_0 \times E_0} : E_0 \times E_0 \rightarrow \mathbb{C}$ defines is given by:*

$$(7.2) \quad H_{K|_{E_0 \times E_0}}(E_0) = \{f \in \mathcal{F}(E_0) : f = \tilde{f}|_{E_0} \text{ for some } \tilde{f} \in H_K(E)\}.$$

Furthermore, the norm is expressed in terms of the one of $H_K(E)$:

$$(7.3) \quad \|f\|_{H_{K|_{E_0 \times E_0}}(E_0)} = \min\{\|\tilde{f}\|_{H_K(E)} : \tilde{f} \in H_K(E), f = \tilde{f}|_{E_0}\}.$$

The identity (7.3) creates the new problem and concept. We shall state the prototype problem:

On the unit disc $|z| < 1$, look for the identity

$$(7.4) \quad \iint_{\{|z| < 1\}} |f(z)|^2 dx dy = \iint_{\{|z| < a\}} |f(z)|^2 dx dy$$

+?

with some information of f on $|z| = a, 0 < a < 1$; that is, the integral

$$\iint_{\{a < |z| < 1\}} |f(z)|^2 dx dy$$

is represented in terms of f on $|z| = a, 0 < a < 1$.

8. ON THE BERGMAN SPACE

On the above line, we can obtain the following simple and general result:

Lemma 8.1: *For a general domain D existing the Green function $G(z, t)$ on the complex plane, an analytic function $f(z)$ of the Bergman space on D is represented by the Green function and the Bergman kernel $K(z, \bar{u})$ on a simply connected subdomain D_s of D with a piecewise smooth closed Jordan boundary curve in D in the following way, for $z \in D$*

$$(8.1) \quad f(z) = (f(\zeta), K(\zeta, \bar{z}))_{L_2(D_s)} + \frac{1}{\pi i} \int_{\partial D_s} f(\zeta) \frac{\partial G(\zeta, z)}{\partial z} d\zeta.$$

Proof: First, recall the trivial identity

$$\begin{aligned} f(z) &= (f(\zeta), K(\zeta, \bar{z}))_{L_2(D)} \\ &= (f(\zeta), K(\zeta, \bar{z}))_{L_2(D_s)} + (f(\zeta), K(\zeta, \bar{z}))_{L_2(D \setminus D_s)}. \end{aligned}$$

We calculate the last integral. By using the identity

$$(8.2) \quad K(z, \bar{u}) = -\frac{2}{\pi} \frac{\partial^2 G(z, u)}{\partial z \partial \bar{u}}$$

and the Green formula, we have

$$(f(\zeta), K(\zeta, \bar{z}))_{L_2(D \setminus D_s)} = \frac{1}{\pi i} \int_{\partial D_s} f(\zeta) \frac{\partial G(\zeta, z)}{\partial z} d\zeta.$$

Note that the derivative of the Green function is zero on the boundary of the domain D . We thus obtain the desired result.

The inner product on $L_2(D \setminus D_s)$ will have the curious representation. We wish to represent it in some inner product form in the symmetric form on the boundary ∂D_s in order to look for the representation of the norm on the boundary.

For (7.4), we have

$$\iint_{\{|z| < a\}} f(z) \frac{1}{\pi} \frac{1}{(1 - \bar{u}z)^2} dx dy = a^2 f(a^2 u).$$

It will be a curious operator. For the Szegő kernel case, we have the correspondent result as in

$$\int_{\{|z|=a\}} f(z) \frac{1}{2\pi} \frac{1}{(1 - \bar{u}z)} |dz| = a f(au),$$

however, some correspondent result to Lemma 8.1 is unclear and curious situation.

Of course, we can apply the general theory for bounded linear operator equation on reproducing kernel spaces and the structure of the related key reproducing kernel Hilbert space H_k is involved in this situation, because the small kernel $k(z, \bar{u})$ is given in the present situation by

$$\begin{aligned} &k(z, \bar{u}) \\ &= \iint_{D_s} \left(\iint_{D_s} K(z_1, \bar{u}) \overline{K(z_1, \bar{z}_2)} dm(z_1) \right) \overline{K(z_2, \bar{z})} dm(z_2) \end{aligned}$$

with the L_2 surface measure. We can not use the isometric mapping L^*L from the Bergman space on D into the Bergman space on D for the realization of the space H_k .

However, with a numerical sense, we can obtain the reasonable result for the concrete case.

A remark for extension and restriction

In connection with restriction and extension of functions in reproducing kernel Hilbert spaces, recall the discretization principle; that is, the norms in reproducing kernels are represented in terms of a discrete countable uniqueness set ([25], page 127-139). Then, restriction and extension problems are reduced to those of reproducing kernels themselves.

9. FROM OHSAWA-TAKEGOSHI'S TYPE NORM INEQUALITIES

As well-known, the converse norm inequality

$$(9.1) \quad \|f\rho^{1/2}\|_{AL_2(D)}^2 \leq C \int_{D_0} |f(z)|^2 \rho(z) dm(z)$$

is known with some positive function ρ and some constant C on some complex domain D in a general dimension. Here, D_0 is a lower-dimensional complex submanifold. See for example, [19] and [20]. Surprisingly enough, the best constant C is even discovered, see [2, 5, 6]. The author wonders whether can we apply our idea in this paper to their great and deep results.

In this section, we assume this norm inequality

$$(9.2) \quad \|f\|_{AL_2(D)}^2 \leq C \int_{D_0} |f(z)|^2 \rho(z) |dz|$$

with a rectifiable curve D_0 as a typical case and we shall consider the bounded linear operator L from the space with finite norm

$$\int_{D_0} |f(z)|^2 \rho(z) |dz| < \infty$$

into the Bergman space $AL_2(D)$; that is for $Lf = f$, (9.2) is valid that means analytic extension operator in the framework of analytic functions. Here we write the same f as an analytic function and its restriction, because by the identity theorem we can assume that the correspondence is one-to-one. See [26] for the details.

We can enjoy the restriction and extension problems on many Sobolev spaces, see [26].

We wrote an essay on **Ohsawa-Takegoshi Theorem** from the viewpoint of structure and proposed an open problem in [15, 18].

Anyhow, it seems that the general theory for bounded linear operator equations on reproducing kernel Hilbert spaces in the first part in this note is a general and powerful theory. It creates many deep and concrete problems.

ACKNOWLEDGEMENTS

The author thanks the co-authors (containing the AI Mika members) of the cited papers in the references for their pleasant collaborations.

In particular, Professor A. Yamada kindly sent his many source files for this lecture sheets and gave many valuable suggestions for my lectures.

Furthermore, the author wishes to express his deep thanks to Professor Qi'an Guan and Dr. Zhijian Chen for their kind invitation and kind hospitality at Peking University.

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