

Gauss's flux theorem does not hold in time-varying electric fields

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Abstract: In classical electromagnetism, Gauss's flux theorem in integral form states that the electric flux through a closed surface is proportional to the net electric charge enclosed by that surface. Gauss's flux theorem in integral form is originally derived from Coulomb's law of electrostatics. However, electrodynamics wrongly extends Gauss's flux theorem to time-varying electric fields without additional theoretical and experimental justification, while confining Coulomb's law to the scope of electrostatics. Through concise and rigorous mathematical derivations, this paper reveals that the integral form of Gauss's flux theorem in classical electromagnetism does not hold in time-varying electric fields. Furthermore, it is demonstrated that Gauss's flux theorem in differential form, which is expressed in terms of the flux-density divergence or the nabla-operator divergence, also does not hold in time-varying electric fields. Gauss's flux theorem does not hold in time-varying electric fields, which poses a serious challenge to Maxwell's equations and theoretical physics. The inapplicability of divergence to time-varying vectors also presents a significant challenge to fundamental mathematics.

Keywords: Gauss's flux theorem, The flux-density divergence, The nabla-operator divergence, Coulomb's law.

1. Introduction

By 1850, experiments and observations had led to the discovery of Coulomb's law, Gauss's flux theorem, Ampere's force theorem, Biot-Savart theorem, Ampere's circuital theorem, and Faraday's law of electromagnetic induction. Classical electromagnetism had been largely established. Based on divergence, curl, and the nabla-operator, Maxwell's equations are an integration of partial differential equations and classical electromagnetism, they are not part of classical electromagnetism.

Coulomb's law describes the electrostatic force acting between two point charges in a vacuum. For two point charges, q_1 and q_2 , separated in space, the electric force exerted by q_1 on q_2 can be expressed as follows:

$$\mathbf{F} = \frac{q_1 q_2}{4\pi \epsilon_0 r^2} \mathbf{r} \quad (1-1)$$

where r is the distance between q_1 and q_2 , $\mathbf{r}$ is the unit vector directed from q_1 to q_2 , and ϵ_0 is the vacuum permittivity.

The electric field intensity $\mathbf{E}$ is defined as the electric force per unit test charge. For a point charge Q at a distance r , the electric field intensity is expressed as follows:

$$\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2} \mathbf{r} \quad (1-2)$$

Gauss's flux theorem follows directly from Coulomb's law of electrostatics. As shown in

$$d\Phi_E = \mathbf{E}(x, y, z) \cdot d\mathbf{S} = E dS \cos \theta$$

Figure 1.1, for an infinitesimal surface element dS in three-dimensional space, the infinitesimal electric flux $d\Phi_E$ is determined by the product of the electric field intensity $\mathbf{E}(x, y, z)$ and the effective area of the infinitesimal surface element dS :

where θ is the angle between the electric field $\mathbf{E}(x, y, z)$ and the outward normal to the infinitesimal surface element dS .

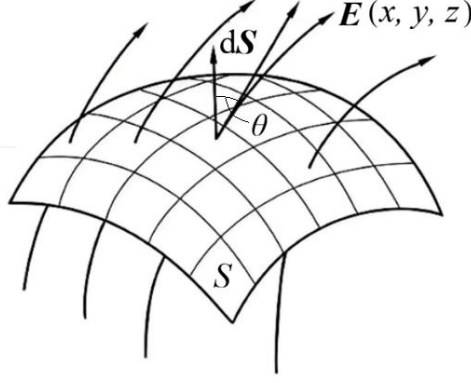


Fig. 1.1 Electric flux through a surface

In three-dimensional space, the electric flux Φ_E through a general surface S is given by the surface integral:

$$\Phi_E = \oiint_S \mathbf{E}(x, y, z) \cdot d\mathbf{S} \quad (1-3)$$

Gauss's flux theorem, derived from Coulomb's law of electrostatics, states that the electric flux through a closed surface S is proportional to the net charge Q enclosed by that surface. Mathematically, this relation is expressed as follows:

$$\oiint_S \mathbf{E}(x, y, z) \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \quad (1-4)$$

Equation (1-4) is Gauss's flux theorem in integral form, serves as a cornerstone of classical electromagnetism.

In electrodynamics, based on the nabla operator (∇) and the divergence, the integral form of Gauss's flux theorem given by equation (1-4) is converted into differential form.

For a three-dimensional vector

$$\mathbf{F}(x, y, z) = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$$

where F_x , F_y , and F_z are the projections of the vector $\mathbf{F}(x, y, z)$ on the x , y , and z axes, respectively.

Definition of the divergence of a vector $\mathbf{F}(x, y, z)$: Let a closed surface S enclose a three-dimensional solid volume ΔV . As $\Delta V \rightarrow 0$, the divergence of the vector $\mathbf{F}(x, y, z)$ is defined as

$$\text{div } \mathbf{F}(x, y, z) = \lim_{\Delta V \rightarrow 0} \frac{\oiint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}}{\Delta V} \quad (1-5)$$

the limit of the ratio of the flux of $\mathbf{F}(x, y, z)$ through S to the volume ΔV :

Equation (1-5) defines vector divergence in terms of the flux volume density, which is referred to as the **flux-density divergence**.

Definition of the nabla-operator (∇): In a rectangular coordinate system, the nabla-operator is defined as follows:

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \quad (1-6)$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors along the respective coordinate axes.

Using the nabla-operator, the divergence of the vector $\mathbf{F}(x, y, z)$ can be expressed as follows:

$$\nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (1-7)$$

Equation (1-7) defines vector divergence in terms of the partial differential nabla-operator, which is referred to as the **nabla-operator divergence**.

According to Equations (1-5) and (1-7), the definitions of divergence of vector $\mathbf{F}(x, y, z)$ can be expressed as follows:

$$\nabla \cdot \mathbf{F}(x, y, z) = \lim_{\Delta V \rightarrow 0} \frac{\oiint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}}{\Delta V} \quad (1-8)$$

In the two definitions of divergence presented in Equation (1-8), the right-hand side denotes the **flux-density divergence**, and the left-hand side denotes the **nabla-operator divergence**. The flux-density divergence offers the physical definition of divergence, whereas the nabla-operator divergence functions as a mathematical tool for computation and derivation. **The flux-density divergence and the nabla-operator divergence are equivalent [1-3].**

In Maxwell's equations, using the nabla operator (∇), the integral form of Gauss's flux theorem can be converted into differential form, which is expressed as follows:

$$\nabla \cdot \mathbf{E}(x, y, z) = \frac{\rho(x, y, z)}{\epsilon_0} \quad (1-9)$$

where $\rho(x, y, z)$ denotes the charge density at a specific point in space. Gauss's flux theorem in differential form is also referred to as Gauss's divergence theorem,

In classical electromagnetism, Gauss's flux theorem in integral form is originally derived from Coulomb's law of electrostatics [4-7]. However, electrodynamics, without additional theoretical and experimental justification, wrongly extends Gauss's flux theorem to time-varying electric fields while confining Coulomb's law to the scope of electrostatics.

Recent research indicates that Coulomb's law in electrostatics can be extended to the electrodynamics of time-varying charge [8]. For a time-varying point charge $q(t) = q_{\max} \sin(\omega t)$ in a vacuum, the electric field at any point P located at a distance r from the charge can be expressed as:

$$\mathbf{E}(r, t) = \frac{q_{\max}}{4\pi\epsilon_0 r^2} \sin\left(\omega t - \frac{\omega}{c} r\right) \mathbf{r} \quad (1-10)$$

Here, ω denotes the angular frequency, c is the propagation speed of the electric field, ω/c is the phase-shift coefficient, $(\omega/c) r$ is the phase difference, and $\mathbf{r}$ is the unit vector from $q(t)$ to point P.

Equation (1-10) is valid in both the near-field and far-field (radiation field). The direction of the time-varying electric field is from the point charge $q(t)$ to point P, that is, the direction of the time-varying electric field is identical to its propagation direction, which is a longitudinal wave. However, according to the “electromagnetic waves” theory, its time-varying electric field component should be a transverse wave.

Recent research on “electromagnetic waves” shows that Hertz’s verification experiment in 1887, which was based on high-voltage discharge sparks, did not prove the existence of “electromagnetic waves”; but rather proved that wireless communication was achieved by independent “electric field waves” [9]. More than 150 years after the formulation of “electromagnetic wave” theory, no single experiment has definitively confirmed the existence of “electromagnetic waves” through quantitative measurements, representing a significant oversight in the history of science and technology. In 2024, based on quantitative measurements, an experiment revealed that, in the radiation field of an antenna, the electric-field energy density was 137.6 times greater than the magnetic-field energy density [10]. Therefore, the experimental result differed by a factor of 137.6 from the value predicted by the conventional “electromagnetic waves” theory.

2. Gauss's flux theorem in differential form holds in static electric fields

According to Equation (1-4), Gauss's flux theorem in integral form holds for static electric fields in classical electromagnetism. In this section, based on two definitions of divergence, we demonstrate the validity of Gauss's flux theorem in differential form for static electric fields in a vacuum.

To simplify the discussion without losing generality. Consider a static point charge q in a vacuum. Choose an arbitrary point P_1 at distance x from q . Taking the point charge q as the origin O and the line passing through q and P_1 as the x -axis, the electric field at any point on the x -axis has only an x -component. As illustrated in Figure 2.1, define an O -xyz coordinate system with q at the origin and consider a cubic infinitesimal element of volume $\Delta x \Delta y \Delta z$ ($\Delta x = \Delta y = \Delta z$). The face normal to the x -axis at the left is ΔS_1 , centered at P_1 ; the corresponding right face is ΔS_2 , centered at P_2 . The four faces parallel to the x -axis are denoted ΔS_3 , centered at P_3 . The field at P_1 has only an x -component $E(x)$, and at P_2 only $E(x + \Delta x)$. The field at P_3 is approximately E_3 and forms an angle of $\Delta\Omega/2$ with the x -axis.

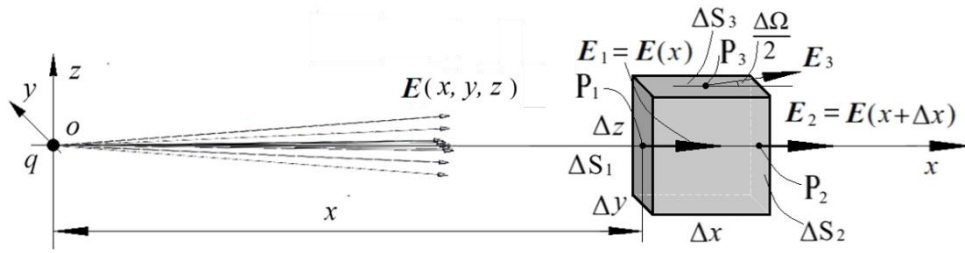


Figure 2.1 Divergence of a static electric field

In Figure 2.1, according to Coulomb's law, the electric field intensity at point P_1 is

$$E(x) = \frac{q}{4\pi\epsilon_0 x^2}$$

The electric flux through infinitesimal surface element ΔS_1 is as follows:

$$E_1 \cdot \Delta S_1 = \frac{q \Delta x^2}{4\pi \epsilon_0 x^2}$$

The electric flux through infinitesimal surface element ΔS_2 is as follows:

$$E_2 \cdot \Delta S_2 = \frac{q \Delta x^2}{4\pi \epsilon_0 (x + \Delta x)^2} = \frac{q \Delta x^2}{4\pi \epsilon_0 (x^2 + 2x\Delta x + \Delta x^2)}$$

The field at P_3 is E_3 and forms an angle of $\Delta\Omega/2$ with the x -axis. Let $\sin(\Delta\Omega/2) = \Delta x/2x$. Then the electric flux through the four infinitesimal surface elements ΔS_3 is as follows:

$$4 \mathbf{E}_3 \cdot \Delta \mathbf{S}_3 = \frac{4 q \Delta x^2 \sin(\Delta \Omega / 2)}{4 \pi \varepsilon_0 ((x + \Delta x / 2)^2 + (\Delta x / 2)^2)} = \frac{q \Delta x^3}{2 \pi \varepsilon_0 x (x^2 + x \Delta x + \Delta x^2 / 2)}$$

Then the net electric flux through the closed surface of the cubic infinitesimal element $\Delta x \Delta y \Delta z$ is given by

$$\begin{aligned} \oiint_S \mathbf{E}(x, y, z) \cdot d\mathbf{S} &= \mathbf{E}_2 \cdot \Delta \mathbf{S}_2 + 4 \mathbf{E}_3 \cdot \Delta \mathbf{S}_3 - \mathbf{E}_1 \cdot \Delta \mathbf{S}_1 \\ &= \frac{q \Delta x^2}{4 \pi \varepsilon_0 (x^2 + 2 x \Delta x + \Delta x^2)} + \frac{q \Delta x^3}{2 \pi \varepsilon_0 x (x^2 + x \Delta x + \Delta x^2 / 2)} - \frac{q \Delta x^2}{4 \pi \varepsilon_0 x^2} \\ &= \frac{q \Delta x^2}{4 \pi \varepsilon_0} \left(\frac{1}{(x^2 + 2 x \Delta x + \Delta x^2)} + \frac{2 \Delta x}{x (x^2 + x \Delta x + \Delta x^2 / 2)} - \frac{1}{x^2} \right) \\ &= \frac{q \Delta x^2}{4 \pi \varepsilon_0} \left(\frac{x^2 \Delta x^2 - \Delta x^4 / 2}{x^2 (x^2 + 2 x \Delta x + \Delta x^2) (x^2 + x \Delta x + \Delta x^2 / 2)} \right) \\ &= \frac{q \Delta x^2}{4 \pi \varepsilon_0} \left(\frac{x^2 \Delta x^2}{x^2 (x^2) (x^2)} \right) \\ \oiint_S \mathbf{E}(x, y, z) \cdot d\mathbf{S} &= \frac{q \Delta x^4}{4 \pi \varepsilon_0 x^4} \quad (2-1) \end{aligned}$$

Substituting $\Delta V = \Delta x^3$ and the electric flux from Equation (2-1) into Equation (1-5), the flux-density divergence at point P_1 in a vacuum is expressed as:

$$\begin{aligned} \lim_{\Delta V \rightarrow 0} \frac{\oiint_S \mathbf{E}(x, y, z) \cdot d\mathbf{S}}{\Delta V} &= \lim_{\Delta x \rightarrow 0} \frac{q \Delta x^4}{4 \pi \varepsilon_0 x^4 \Delta x^3} = \lim_{\Delta x \rightarrow 0} \frac{q \Delta x}{4 \pi \varepsilon_0 x^4} \\ \lim_{\Delta V \rightarrow 0} \frac{\oiint_S \mathbf{E}(x, y, z) \cdot d\mathbf{S}}{\Delta V} &= 0 \quad (2-2) \end{aligned}$$

According to Equation (2-2), the flux-density divergence at point P_1 is zero. In a vacuum, the charge density ρ at P_1 is zero. Therefore, expressed in terms of the flux-density divergence, Gauss's flux theorem in differential form holds true for static electric fields in a vacuum.

In Maxwell's equations, Gauss's flux theorem in differential form is expressed via the nabla-operator divergence. Below, when expressed in terms of the nabla-operator divergence, we demonstrate the validity of Gauss's flux theorem for static electric fields in a vacuum.

In Figure 2.1, the magnitude of the electric field intensity produced by a point charge q at an arbitrary point $P(x, y, z)$ in three-dimensional space is given by

$$E(x, y, z) = \frac{q}{4 \pi \varepsilon_0 r^2}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$.

Because the direction of the electric field intensity $\mathbf{E}(x, y, z)$ at point $P(x, y, z)$ is from the origin O , where the point charge q is located, to point $P(x, y, z)$, the projections of the electric field intensity $\mathbf{E}(x, y, z)$ on the x -, y -, and z -axes are respectively given by

$$E_x = \frac{q x}{4\pi\epsilon_0 r^3}; \quad E_y = \frac{q y}{4\pi\epsilon_0 r^3}; \quad E_z = \frac{q z}{4\pi\epsilon_0 r^3}$$

Then the partial derivative of E_x with respect to x is given by

$$\frac{\partial E_x}{\partial x} = \frac{\partial(\frac{q x}{4\pi\epsilon_0 r^3})}{\partial x} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r^3} + x(-3r^{-4}) \frac{\partial r}{\partial x} \right) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r^3} - x(3r^{-4}) \frac{x}{r} \right)$$

$$\frac{\partial E_x}{\partial x} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r^3} - \frac{3x^2}{r^5} \right)$$

Based on the symmetry, the partial derivatives of E_y and E_z with respect to y and z are respectively given by

$$\frac{\partial E_y}{\partial y} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r^3} - \frac{3y^2}{r^5} \right)$$

$$\frac{\partial E_z}{\partial z} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right)$$

According to Equation (1-7), at an arbitrary point $P(x, y, z)$ in the three-dimensional space, the nabla-operator divergence is expressed as:

$$\begin{aligned} \nabla \cdot \mathbf{E}(x, y, z) &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\ &= \frac{q}{4\pi\epsilon_0} \left(\frac{3}{r^3} - \frac{3(x^2 + y^2 + z^2)}{r^5} \right) = \frac{q}{4\pi\epsilon_0} \left(\frac{3}{r^3} - \frac{3r^2}{r^5} \right) \end{aligned}$$

$$\nabla \cdot \mathbf{E}(x, y, z) = 0 \quad (2-3)$$

According to Equations (2-2) and (2-3), at point P_1 in a vacuum, the flux-density divergence equals the nabla-operator divergence, and both are zero. Therefore, in a vacuum, Gauss's flux theorem in differential form, which is expressed in terms of the flux-density divergence or the nabla-operator divergence, holds true in static electric fields.

3. Gauss's flux theorem in integral form does not hold for time-varying electric fields

Because the propagation speed of an electric field is 2.998×10^8 m/s, Gauss's flux theorem does not hold for time-varying electric fields. Below, we demonstrate with a specific example that the integral form of Gauss's flux law in classical electromagnetism, as given in Equation (1-4), fails for time-varying electric fields.

As shown in Figure 3.1, consider a point charge $q(t)$ in space and a closed spherical surface S with diameter $D = 299.8$ m. The point charge $q(t)$ lies inside S , near point A on one side, and the distance from $q(t)$ to point B on the opposite side of S is 299.8 m. The electric field generated by $q(t)$ requires 1 microsecond to propagate from $q(t)$ to point B.

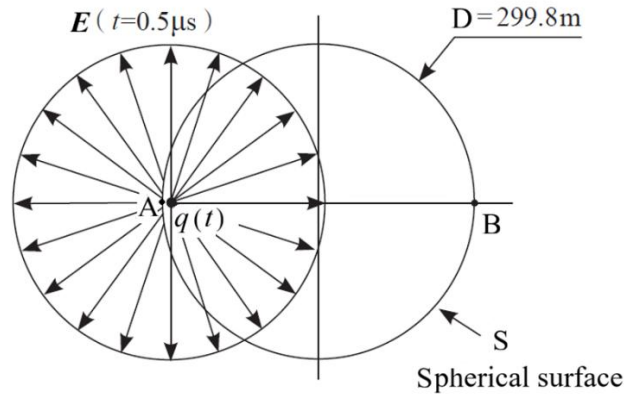


Figure 3.1 A time-varying point charge within a closed spherical surface

Assume the point charge $q(t) = 0$ when time $t < 0$ μ s, and $q(t) = q_0$ when $t \geq 0$ μ s. When $t \geq 0$ μ s, the right-hand side of Equation (1-4) is q_0/ϵ_0 , which is a constant value.

1) At $t = 0$ μ s, the right-hand side of Equation (1-4) equals q_0/ϵ_0 . However, the electric field generated by the point charge q_0 has not yet reached the spherical surface S with a diameter of D , so the electric field E at every point on S is equal to zero, and the left-hand side of Equation (1-4) equals 0. The left-hand side and the right-hand side of Equation (1-4) are not equal. Therefore, at $t = 0$ μ s, Equation (1-4) does not hold; in other words, Gauss's flux theorem in integral form is not valid.

2) At $t = 0.5$ μ s, the right-hand side of Equation (1-4) equals q_0/ϵ_0 . However, the electric field generated by the point charge q_0 only reaches part of the left side of the spherical surface S , so the left-hand side of Equation (1-4) is smaller than its right-hand side, indicating the left-hand side and the right-hand side of Equation (1-4) are not equal. Therefore, at $t = 0.5$ μ s, Equation (1-4) does not hold; in other words, Gauss's flux theorem in integral form is not valid.

3) When $t \geq 1$ μ s, the right-hand side of Equation (1-4) equals q_0/ϵ_0 . At this time, the electric field generated by the point charge q_0 reaches the entire closed spherical surface S , so the left-

hand side of Equation (1-4) equals its right-hand side. Therefore, Equation (1-4) holds when $t \geq 1 \mu s$.

According to the above analysis, Equation (1-4) does not hold when $t \geq 0 \mu s$ and $t < 1 \mu s$. Therefore, when a time-varying charge is enclosed by a closed surface, Gauss's flux theorem in integral form of classical electromagnetism does not hold.

Below, we analyze the case in which the point charge lies outside the closed surface. As shown in Figure 3.2, the point charge $q(t)$ is located outside the spherical surface S , near point A on one side of S , and its distance to point B on the opposite side of S is 299.8 m.

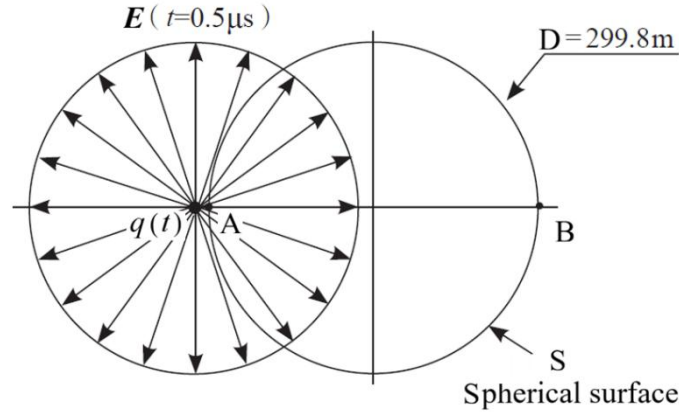


Figure 3.2 A time-varying point charge outside a closed spherical surface

Let the point charge $q(t) = 0$ when time $t < 0 \mu s$, and $q(t) = q_0$ when $t \geq 0 \mu s$. Because the point charge $q(t)$ lies outside the closed surface, the right-hand side of Equation (1-4) is identically zero.

1) At $t = 0 \mu s$, the right-hand side of Equation (1-4) equals 0, and the electric field generated by the point charge q_0 has not yet reached the spherical surface S with a diameter of D , so the electric field E at every point on S is equal to zero, and the left-hand side of Equation (1-4) equals 0. The left-hand side and the right-hand side of Equation (1-4) are equal. Therefore, at $t = 0 \mu s$, Equation (1-4) holds.

2) At $t = 0.5 \mu s$, the right-hand side of Equation (1-4) equals 0. However, the electric field generated by the point charge q_0 only reaches part of the left side of the spherical surface S , so the left-hand side of Equation (1-4) is not equal to zero, indicating the left-hand side and the right-hand side of Equation (1-4) are not equal. Therefore, Equation (1-4) does not hold when $t = 0.5 \mu s$.

3) When $t \geq 1 \mu s$, the right-hand side of Equation (1-4) equals 0. At this time, the electric field generated by the point charge q_0 reaches the entire closed spherical surface S , so the left-hand side of Equation (1-4) equals its right-hand side. Therefore, Equation (1-4) holds when $t \geq 1 \mu s$.

According to the above analysis, Equation (1-4) does not hold when $t = 0.5 \mu\text{s}$. Therefore, when a time-varying charge lies outside a closed surface, Gauss's flux theorem in integral form in classical electromagnetism does not hold.

In summary, Gauss's flux theorem in integral form fails when a time-varying point charge is present either inside or outside a closed surface. By the superposition principle, this conclusion extends to charged bodies of any shape. Therefore, Gauss's flux theorem in integral form in classical electromagnetism does not hold for time-varying electric fields.

In the analysis above, the point charge q_0 is generated instantaneously at $t = 0 \mu\text{s}$. According to classical electrodynamics, the instantaneous generation of electric charge contradicts the principle of charge conservation [7].

The instantaneous generation of electric charge at a specific point in space is not difficult in engineering practice. Figure 3.3 illustrates one such technical implementation: a charge q_0 is produced on a small metal sphere, while an equal and opposite charge $-q_0$ is generated inside the power supply and confined within an electromagnetic shielding layer. For high-frequency fields, wave-absorbing materials are commonly used for shielding, effectively isolating the internal and external electromagnetic fields.

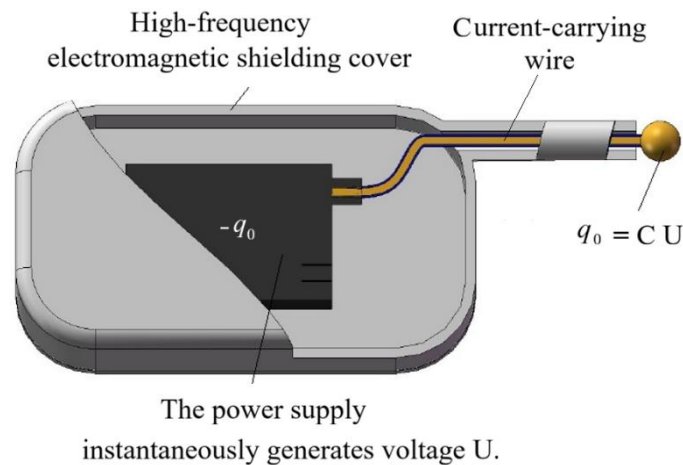


Figure 3.3 Technical implementation for instantaneous generation of charge

Modern communication has entered the terahertz era, wherein a charge q_0 can be generated instantaneously in less than 1 picosecond. Therefore, the instantaneous generation of charge on the small metal sphere described above does not violate the principle of charge conservation.

Within the framework of electromagnetism, electric charge is neither generated nor destroyed. As illustrated in Figure 3.3, the charge on the small metal sphere does not generate from nothing; instead, only its spatial distribution changes instantaneously. A more accurate phrasing substitutes “the instantaneous generation of electric charge” with “the instantaneous change in the distribution of electric charge.”

4. Gauss's flux theorem in differential form does not hold in time-varying electric fields

Gauss's flux theorem in integral form fails in time-varying electric fields. Below, we demonstrate that Gauss's flux theorem in differential form also does not hold in time-varying electric fields.

As noted above, Coulomb's law in electrostatics can be extended to electrodynamics when the electric charge varies in time. To simplify the discussion without losing generality. Consider a stationary time-varying point charge $q(t) = q_{\max} \sin(\omega t)$ in a vacuum. Choose an arbitrary point P_1 at distance x from $q(t)$. Taking the point charge $q(t)$ as the origin O and the line passing through $q(t)$ and P_1 as the x -axis, the electric field at any point on the x -axis has only an x -component. As illustrated in Figure 4.1, define an O - xyz coordinate system with $q(t)$ at the origin and consider a cubic infinitesimal element of volume $\Delta x \Delta y \Delta z$ ($\Delta x = \Delta y = \Delta z$). The face normal to the x -axis at the left is ΔS_1 , centered at P_1 ; the corresponding right face is ΔS_2 , centered at P_2 . The four faces parallel to the x -axis are denoted ΔS_3 , centered at P_3 . The field at P_1 has only an x -component E_1 , and at P_2 only E_2 . The field at P_3 is approximately E_3 and forms an angle of $\Delta\Omega/2$ with the x -axis.

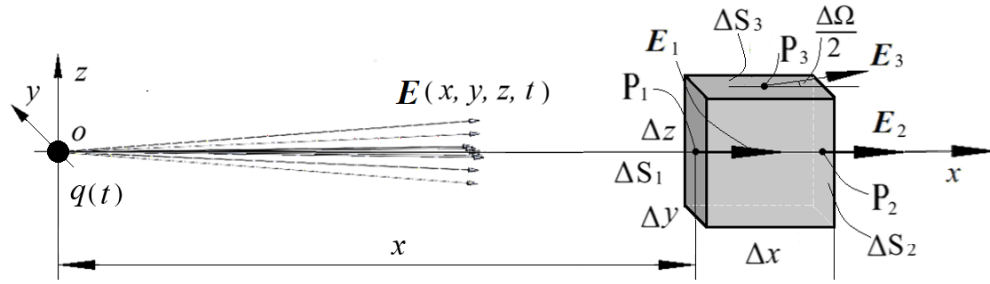


Figure 4.1 Divergence of a time-varying electric field

In Figure 4.1, according to Coulomb's law in Equation (1-10), the time-varying electric field intensity at point P_1 is

$$E_1 = E(x, t) = \frac{q_{\max} \sin(\omega t - \frac{\omega}{c} x)}{4\pi\epsilon_0 x^2}$$

The time-varying electric field intensity at point P_2 is

$$E_2 = E(x + \Delta x, t) = \frac{q_{\max} \sin(\omega t - \frac{\omega}{c} (x + \Delta x))}{4\pi\epsilon_0 (x + \Delta x)^2}$$

The time-varying electric field intensity at point P_3 is

$$\mathbf{E}_3 = \mathbf{E}(x + \Delta x / 2, t) = \frac{q_{max} \sin(\omega t - \frac{\omega}{c}(x + \Delta x / 2))}{4\pi\epsilon_0 (x + \Delta x / 2)^2}$$

In Figure 4.1, the electric flux through infinitesimal surface element ΔS_1 is as follows:

$$\mathbf{E}_1 \cdot \Delta S_1 = \frac{q_{max} \sin(\omega t - \frac{\omega}{c}x)}{4\pi\epsilon_0 x^2} \Delta x^2$$

The electric flux through infinitesimal surface element ΔS_2 is as follows:

$$\mathbf{E}_2 \cdot \Delta S_2 = \frac{q_{max} \sin(\omega t - \frac{\omega}{c}(x + \Delta x))}{4\pi\epsilon_0 (x + \Delta x)^2} \Delta x^2$$

The field at P_3 is approximately $\mathbf{E}_3$ and forms an angle of $\Delta\Omega/2$ with the x -axis. Let $\sin(\Delta\Omega/2) = \Delta x/2x$, the electric flux through the four infinitesimal surface elements ΔS_3 is as follows:

$$4 \mathbf{E}_3 \cdot \Delta S_3 = \frac{4 q_{max} \sin(\omega t - \frac{\omega}{c}(x + \Delta x / 2))}{4\pi\epsilon_0 (x + \Delta x / 2)^2} \Delta x^2 \sin(\Delta\Omega / 2)$$

$$4 \mathbf{E}_3 \cdot \Delta S_3 = \frac{q_{max} \sin(\omega t - \frac{\omega}{c}(x + \Delta x / 2))}{2\pi\epsilon_0 x(x + \Delta x / 2)^2} \Delta x^3$$

Then the net electric flux through the closed surface of the cubic infinitesimal element $\Delta x \Delta y \Delta z$ is given by

$$\begin{aligned} \oiint_S \mathbf{E}(x, y, z, t) \cdot d\mathbf{S} &= \mathbf{E}_2 \cdot \Delta S_2 + 4 \mathbf{E}_3 \cdot \Delta S_3 - \mathbf{E}_1 \cdot \Delta S_1 \\ &= \frac{q_{max} \sin(\omega t - \frac{\omega}{c}(x + \Delta x))}{4\pi\epsilon_0 (x + \Delta x)^2} \Delta x^2 + \frac{q_{max} \sin(\omega t - \frac{\omega}{c}(x + \Delta x / 2))}{2\pi\epsilon_0 x(x + \Delta x / 2)^2} \Delta x^3 \\ &\quad - \frac{q_{max} \sin(\omega t - \frac{\omega}{c}x)}{4\pi\epsilon_0 x^2} \Delta x^2 \\ \oiint_S \mathbf{E}(x, y, z, t) \cdot d\mathbf{S} &= -\frac{q_{max} \omega \Delta x^3}{4\pi\epsilon_0 c x^2} \cos(\omega t - \frac{\omega}{c}x) \quad (4-1) \end{aligned}$$

Substituting $\Delta V = \Delta x^3$ and the electric flux from Equation (4-1) into Equation (1-5), the flux-density divergence at point P_1 in a vacuum is expressed as:

$$\lim_{\Delta V \rightarrow 0} \frac{\oiint_S \mathbf{E}(x, y, z, t) \cdot d\mathbf{S}}{\Delta V} = - \lim_{\Delta x \rightarrow 0} \frac{q_{max} \omega \Delta x^3}{4\pi\epsilon_0 c x^2 \Delta x^3} \cos(\omega t - \frac{\omega}{c} x)$$

$$\lim_{\Delta V \rightarrow 0} \frac{\oiint_S \mathbf{E}(x, y, z, t) \cdot d\mathbf{S}}{\Delta V} = - \frac{q_{max} \omega}{4\pi\epsilon_0 c x^2} \cos(\omega t - \frac{\omega}{c} x) \quad (4-2)$$

According to Equation (4-2), within time-varying electric fields, the flux-density divergence at point P_1 is nonzero. Therefore, Gauss's flux theorem in differential form, which is expressed in terms of the flux-density divergence, does not hold for time-varying electric fields in a vacuum.

In Maxwell's equations, Gauss's flux theorem in differential form is expressed via the nabla-operator divergence. Below, when expressed in terms of the nabla-operator divergence, we demonstrate the validity of Gauss's flux theorem for time-varying electric fields in a vacuum.

In Figure 4.1, at an arbitrary point $P(x, y, z)$ in three-dimensional space, the magnitude of the time-varying electric field intensity produced by a time-varying point charge $q(t) = q_{max} \sin(\omega t)$ is given by

$$E(x, y, z, t) = \frac{q_{max}}{4\pi\epsilon_0 r^2} \sin(\omega t - \frac{\omega}{c} r)$$

where $r = (x^2 + y^2 + z^2)^{1/2}$.

Because the direction of the time-varying electric field intensity $\mathbf{E}(x, y, z, t)$ at point $P(x, y, z)$ is from the origin O to point $P(x, y, z)$, the projections of the time-varying electric field intensity $\mathbf{E}(x, y, z, t)$ on the x -, y -, and z -axes are respectively given by

$$E_x = \frac{q_{max} x}{4\pi\epsilon_0 r^3} \sin(\omega t - \frac{\omega}{c} r)$$

$$E_y = \frac{q_{max} y}{4\pi\epsilon_0 r^3} \sin(\omega t - \frac{\omega}{c} r)$$

$$E_z = \frac{q_{max} z}{4\pi\epsilon_0 r^3} \sin(\omega t - \frac{\omega}{c} r)$$

Then the partial derivative of E_x with respect to x is given by

$$\begin{aligned}
\frac{\partial E_x}{\partial x} &= \frac{\partial \left[\frac{q_{max} x}{4\pi\epsilon_0 r^3} \sin(\omega t - \frac{\omega}{c} r) \right]}{\partial x} = \frac{q_{max}}{4\pi\epsilon_0} \left[\sin(\omega t - \frac{\omega}{c} r) \frac{\partial(\frac{x}{r^3})}{\partial x} + \frac{x}{r^3} \frac{\partial(\sin(\omega t - \frac{\omega}{c} r))}{\partial x} \right] \\
&= \frac{q_{max}}{4\pi\epsilon_0} \left[\sin(\omega t - \frac{\omega}{c} r) \frac{\partial(x r^{-3})}{\partial x} + \frac{x}{r^3} (-\frac{\omega}{c}) \cos(\omega t - \frac{\omega}{c} r) \frac{\partial r}{\partial x} \right] \\
&= \frac{q_{max}}{4\pi\epsilon_0 r^3} \left[(1 - \frac{3x}{r} \frac{x}{r}) \sin(\omega t - \frac{\omega}{c} r) - \frac{\omega x}{c} \cos(\omega t - \frac{\omega}{c} r) \frac{x}{r} \right] \\
\frac{\partial E_x}{\partial x} &= \frac{q_{max}}{4\pi\epsilon_0 r^3} \left[\sin(\omega t - \frac{\omega}{c} r) - \frac{3x^2}{r^2} \sin(\omega t - \frac{\omega}{c} r) - \frac{x^2 \omega}{r c} \cos(\omega t - \frac{\omega}{c} r) \right]
\end{aligned}$$

Based on the symmetry, the partial derivatives of E_y and E_z with respect to y and z are respectively given by

$$\begin{aligned}
\frac{\partial E_y}{\partial y} &= \frac{q_{max}}{4\pi\epsilon_0 r^3} \left[\sin(\omega t - \frac{\omega}{c} r) - \frac{3y^2}{r^2} \sin(\omega t - \frac{\omega}{c} r) - \frac{y^2 \omega}{r c} \cos(\omega t - \frac{\omega}{c} r) \right] \\
\frac{\partial E_z}{\partial z} &= \frac{q_{max}}{4\pi\epsilon_0 r^3} \left[\sin(\omega t - \frac{\omega}{c} r) - \frac{3z^2}{r^2} \sin(\omega t - \frac{\omega}{c} r) - \frac{z^2 \omega}{r c} \cos(\omega t - \frac{\omega}{c} r) \right]
\end{aligned}$$

According to Equation (1-7), within time-varying electric fields, at an arbitrary point P (x, y, z) in the three-dimensional space, the nabla-operator divergence is expressed as:

$$\begin{aligned}
\nabla \cdot \mathbf{E}(x, y, z, t) &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
&= \frac{q_{max}}{4\pi\epsilon_0 r^3} \left[3\sin(\omega t - \frac{\omega}{c} r) - \frac{3(x^2 + y^2 + z^2)}{r^2} \sin(\omega t - \frac{\omega}{c} r) - \frac{\omega(x^2 + y^2 + z^2)}{r c} \cos(\omega t - \frac{\omega}{c} r) \right] \\
&= \frac{q_{max}}{4\pi\epsilon_0 r^3} \left[3\sin(\omega t - \frac{\omega}{c} r) - \frac{3r^2}{r^2} \sin(\omega t - \frac{\omega}{c} r) - \frac{r^2 \omega}{r c} \cos(\omega t - \frac{\omega}{c} r) \right] \\
\nabla \cdot \mathbf{E}(x, y, z, t) &= -\frac{q_{max} \omega}{4\pi\epsilon_0 c r^2} \cos(\omega t - \frac{\omega}{c} r)
\end{aligned}$$

In Figure 4.1, point P₁ is on the x -axis, where $r = x$ in the above equation, so the nabla-operator divergence at point P₁ is expressed as:

$$\nabla \cdot \mathbf{E}(x, y, z, t) = -\frac{q_{max}\omega}{4\pi\epsilon_0 c x^2} \text{Cos}(\omega t - \frac{\omega}{c}x) \quad (4-3)$$

According to Equations (4-2) and (4-3), in a vacuum, the flux-density divergence equals the nabla-operator divergence; however, neither is zero. Therefore, expressed in terms of the flux-density divergence or the nabla-operator divergence, Gauss's flux theorem in differential form does not hold for time-varying electric fields in a vacuum.

In summary, the integral and differential forms of Gauss's flux theorem apply to static electric fields but do not hold for time-varying electric fields. Similarly, the integral and differential forms of the magnetic flux theorem are valid for static magnetic fields but fail in the case of time-varying magnetic fields.

5. Conclusion

By 1850, experiments and observations had led to the discovery of Coulomb's law, Gauss's flux theorem, Ampere's force theorem, Biot-Savart theorem, Ampere's circuital theorem, and Faraday's law of electromagnetic induction. Classical electromagnetism had been largely established. Based on divergence, curl, and the nabla-operator, Maxwell's equations are an integration of partial differential equations and classical electromagnetism, they are not part of classical electromagnetism.

In classical electromagnetism, Gauss's flux theorem in integral form states that the electric flux through a closed surface is proportional to the net charge enclosed by that surface. Gauss's flux theorem is originally derived from Coulomb's law of electrostatics. Maxwell's equations, however, incorrectly extend Gauss's flux theorem to time-varying electric fields without additional theoretical and experimental justification, while confining Coulomb's law to the scope of electrostatics.

Through concise and rigorous mathematical derivations, this paper reveals that the integral form of Gauss's flux theorem in classical electromagnetism does not hold in time-varying electric fields. Furthermore, it is demonstrated that Gauss's flux theorem in differential form, which is expressed in terms of the flux-density divergence or the nabla-operator divergence, also does not hold in time-varying electric fields.

Gauss's flux theorem and the magnetic flux theorem do not hold in time-varying electromagnetic, which poses a serious challenge to conventional Maxwell's equations and theoretical physics. Divergence, a significant tool in vector analysis, is widely used in electromagnetism, thermodynamics, and fluid mechanics. However, the divergence only applies to steady vector fields but does not hold in time-varying vectors, which also poses a serious challenge to fundamental mathematics.

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