

When atmost one runner prevents loneliness

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Abstract: - Consider the problem of finding the time T at which none of the $n-1$ moving runners are in the arc Z of length $2/n$ centered at the stationary runner 0 in an instance of the Shifted Lonely Runner Conjecture (SLRC). We show two important results for prime n and for large n . Firstly, there exist atleast 3 time intervals during which atmost 1 moving runner is in Z for the SLRC, and, there exist atleast $4(n-1)$ time intervals during which atmost 1 moving runner is in Z for the LRC. Secondly, for both the SLRC and the LRC considered separately, there are atleast 2 different moving runners which prevent the loneliness of runner 0 by being the “atmost one runner in Z ” at some time.

1. Introduction

The Lonely Runner Conjecture (LRC) and the Shifted LRC (SLRC) with n runners, both ask whether there exists a time at which there are no moving runners in an arc of length $2/n$ centered at the stationary runner 0 . While the SLRC has already been proved to be false [1], the LRC is still open. The best known minimal gaps for the LRC tend to $1/(2n)$ as n tends to infinity [2][3]. There is also a well-known configuration of moving runners with equal speeds and positioned equidistant from each other, that always ensures the presence of atleast one moving runner in Z , with an upper bound of $1/(2n)$ on the minimal gap. However, the LRC and SLRC offer hope since each runner has a unique speed. For both the LRC and SLRC, it is still not known whether it is possible to improve the minimal gap to f/n where $0.5 < f < 1$, and f is rational. One approach to improving this minimal gap is to study variants of the SLRC and LRC, in the hope that some clue might be obtained to improve the minimal gap.

In 2020, we developed a general approach for all n , to maximize the fraction of runners not in Z [5]. In May 2026, we developed an approach for prime n and large n , that minimized the average gap to $1/n$, simultaneously ensuring a minimum gap of $1/(2n)$, for the stationary runner 0 from the $n-1$ moving runners in an instance of the SLRC [4].

In this paper, we focus on defining the set of time-intervals for the SLRC and LRC at which the number of moving runners in the arc Z of length $2/n$ centered at the stationary runner 0 , is minimized to atmost 1. This appears promising to LRC and SLRC research, especially if it can be proven that the gap between runner 0 and that remaining moving runner in Z , is atmost f/n where $0.5 < f < 1$, and f is rational.

2. Notations

The notations in this paper are similar to those used in [4], except that P is now a prime fixed to n (so we assume that the total number of runners is prime), and we divide the circle into sectors in a slightly different way.

1. $P=n$, which is assumed to be prime.
2. The circular running track is divided into n sectors numbered from 0 to $(n-1)$, where sector j is bounded by the arc in $[(j/n), (j+1)/n]$ for each integer j in $[0, (n-1)]$. So the length of the arc bounding each sector is $1/n$.
3. Let Z be the arc of length $2/n$, such that the fixed Runner 0 is the mid-point of Z .
4. The speeds of the $n-1$ runners, and the target time T , is expressed in the base of n . So, the speed of runner i is defined as: $S_i = s_{i,F,i} n^{F,i} + s_{i,F,i-1} n^{F,i-1} + \dots + s_{i,2} n^2 + s_{i,1} n^1 + s_{i,0}$, and any target time T is defined as: $T = t_L n^L + t_{L-1} n^{L-1} + \dots + t_2 n^2 + t_1 n^1$, where L is a very large positive integer. L is loosely defined to be very large, much greater than $\max(F_i, \text{ over runners } i \text{ in } [1, n-1])$, to give a more accurate definition of T .
5. $k_P = \max(E_i, \text{ over runners } i \text{ in } [1, n-1])$ without any constraint on H_{k_P} . Note that in Theorem 4 of [4], we assumed that $k_P = \max(E_i, \text{ over runners } i \text{ in } [1, n-1])$ with the constraint that $H_{k_P} \geq (P/2)$.
6. $S_{T,Z}$ is the set of moving runners in Z at time T .

3. The approach

Theorems 1 and 2 below arise from the application of Lemma 5.1 of the paper [4] to the SLRC and the LRC respectively.

Theorem 1: Let n be prime. There exist atleast 3 time intervals in the SLRC, when $\text{cardinality}(S_{T,Z}) \leq 1$. The union of $S_{T,Z}$ over these time intervals has atleast 2 different moving runners.

Proof: Denote any runner i whose $E_i = k_P$ as runner X . Clearly, since runner X has a real starting point, there exists a vector of values $\langle t_L, t_{L-1}, \dots, t_{k_P+1} \rangle$ such that runner X comes close to an integer multiple of $1/n$. So we have atleast 1 runner placed at an integer multiple of $1/n$, and $(n-2)$ remaining runners. If runner i is at an integer multiple of $1/n$ after choosing $\langle t_L, t_{L-1}, \dots, t_{E_i+1} \rangle$ then $n-1$ out of n choices of t_{E_i} keep runner i out of Z . If runner i is not at an integer multiple of $1/n$ after choosing $\langle t_L, t_{L-1}, \dots, t_{E_i+1} \rangle$ then $n-2$ out of n choices of t_{E_i} keep runner i out of Z . Applying the pigeon hole principle (PHP) as we did in Lemma 5.1 of [4], the expected number of runners not in Z is:

$$\begin{aligned} &= ((n-2)(n-2)/n) + 1(n-1)/n \\ &= (n^2 - 3n + 3)/n \\ &= n - 3 + 3/n. \end{aligned}$$

This means that there are atleast 3 values of t_{E_i} when the number of runners not in Z is atleast $n-3+1 = n-2$. Note that if our answer for the expected number of runners not in Z is $(n - a + b/n)$, where a and b are positive integers, it would mean that there are atleast b time intervals when the number of runners not in Z is atleast $(n - a + 1)$. This means that there are atleast 3 values of t_{E_i} when the number of moving runners in Z is atmost 1, i.e. when $\text{cardinality}(S_{T,Z}) < 1$. Since it takes atmost 2 values of t_{E_i} to keep the same runner inside Z , this means atleast 2 different runners have the honor of being the “atmost one runner in Z at some time”.

Hence Proved

Theorem 2: Let n be prime. There exist atleast $4(n-1)$ time intervals in the LRC, when $\text{cardinality}(S_{T,Z}) \leq 1$. Also, the union of $S_{T,Z}$ over these time intervals has atleast 2 different moving runners.

Proof: Since all runners start from Runner 0 at time $T=0$, set $t_L = t_{L-1} = \dots = t_{k_P+1} = 0$. Setting t_{k_P} to any integer in $[1, n-1]$ ensures that all runners i with their $E_i = k_P$, are positioned at integer multiples of $1/n$. So there are basically $(n-1)$ choices of t_{k_P} , choosing any of which leaves us with atmost $(n-2)$ remaining moving runners. It is obvious that if all runners i have their speeds in the LRC with same E_i , then the LRC becomes trivial with a separation gap of $1/n$ guaranteed as per Theorem 3 of [4]. Applying the PHP as we did earlier, the expected number of runners not in Z is:

$$\begin{aligned} &= (n-2)(n-2)/n \\ &= (n^2 - 4n + 4)/n \\ &= n - 4 + 4/n. \end{aligned}$$

This means that there are totally atleast 4 values of t_{E_i} when the number of runners not in Z is atleast $n-4+1 = n-3$, for each value of t_{k_P} . This means that there are atleast 4 values of t_{E_i} when the number of moving runners in Z is atmost 1, i.e. when $\text{cardinality}(S_{T,Z}) < 1$, for each of the $n-1$ values of t_{k_P} . So we have a total of $4(n-1)$ time intervals when $\text{cardinality}(S_{T,Z}) < 1$. Since it takes atmost 2 values of t_{E_i} to keep the same runner inside Z , this means atleast 2 different runners have the honor of being the “atmost one runner in Z at some time”.

Hence Proved

Theorem 3: There exists a positive integer λ , such that Theorems 1 and 2 hold for all $n > \lambda$.

Proof: As followed in Theorem 7 of [4], when n is very large and not prime, there exists a positive integer C_n so that $(n+C_n)$ is prime and C_n/n tends to 0. So simply increase our target minimal loneliness gap from $1/n$ to $1/(n+C_n)$, and add C_n dummy runners, each having the same speed and starting point as any original runner that is not in $S_{T,Z}$.

Hence Proved

4. Conclusion and future work

We described an approach to obtain the set of time-intervals during which loneliness for any runner in the LRC or SLRC, is prevented by atmost 1 other runner. We found that the number of such time intervals for the LRC is considerably more than that of the SLRC. For both the LRC and the SLRC, we also found that for every runner i , there are atleast two other runners j and k , which at separate times, prevent runner i 's loneliness by being the “atmost one other runner in the $1/n$ neighborhood of runner i ”. Studying these time-intervals could hopefully improve the minimal separation gap for the LRC and the SLRC.

References

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About the author

I, Deepak Ponvel Chermakani, am a citizen of India and I wrote this paper, which is original to the best of my knowledge, out of my own interest and initiative during my spare time. I completed a fulltime two-year Master of Science Degree in Electrical Engineering from University of Hawaii at Manoa USA (www.hawaii.edu) in Aug 2015, a fulltime one-year Master of Science Degree in Operations Research with Computational Optimization from University of Edinburgh UK (www.ed.ac.uk) in Sep 2010, a fulltime four-year Bachelor of Engineering Degree in Electrical and Electronic Engineering, from Nanyang Technological University Singapore (www.ntu.edu.sg) in Jul 2003, fulltime high schooling from National Public School Indiranagar in Bangalore India in Jul 1999, and fulltime middle schooling from Bishop Cotton Boys School in Bangalore India in Jul 1997. I am most grateful to my parents (my mother Mrs. Kanaga Rathinam Chermakani and my father Mr. T. Chermakani) for their sacrifices in educating me and bringing me up.