

A highly-evolving recursive formula reproducing the sieve of Eratosthenes and generating all prime numbers sequentially

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ABSTRACT

This paper presents an alternative recursive piecewise formula that generates all prime numbers only once and in increasing order using only elementary arithmetic operations, square roots and the floor function. The formula does not require previously known prime numbers as input or a precomputed list of primes. Its operation can be interpreted as an algebraic representation of the sieving mechanism underlying the sieve of Eratosthenes through the use of sets of nested equations.

The formula is not claimed to improve upon the computational complexity of the sieve of Eratosthenes. Rather, its purpose is theoretical: it provides an explicit and elementary recursive formulation of the sieve mechanism without factorials, trigonometric functions (like the famous 1964 formula of Willans [1], [2] that tautologically defines the prime numbers), or constants whose construction depends on prior knowledge of the primes. The formula was discovered empirically, and no proof of its correctness is provided in this paper.

At the end of the article, a comparison is made between prime numbers and lucky numbers, with a similar set of nested equations generating naturally all lucky numbers.

SYMBOLS

p_n is the n th prime number. $p_1 = 2$.

$\lfloor \]$ is the floor function. $\lfloor 3 \rfloor = 3$. $\lfloor 3.999 \rfloor = 3$.

METHODOLOGY

Two concrete examples are given, followed by the theory behind the construction of the sets of nested equations. Lucky numbers are discussed at the end of the article.

FIRST EXAMPLE

Using a cascading effect, the following set of nested mathematical equations produces the first 30 prime numbers up to $p_{30} = 113$.

for $n = 1$, $p_n = 2$

for $n = 2$ to 4 , $p_n = p_2(n) = 2n - 1$

for $n = 5$ to 9 , $p_n = p_{2,3}(n) = p_2\left(n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor\right)$

for $n = 10$ to 15 , $p_n = p_{2,3,5}(n) = p_{2,3}\left(n - 1 + \left\lfloor \frac{n+4}{8} \right\rfloor + \left\lfloor \frac{n-2}{8} \right\rfloor\right)$

for $n = 16$ to 30 ,

$$p_n = p_{2,3,5,7}(n) = p_{2,3,5}\left(n - 1 + \left\lfloor \frac{n+43}{48} \right\rfloor + \left\lfloor \frac{n+32}{48} \right\rfloor + \left\lfloor \frac{n+26}{48} \right\rfloor + \left\lfloor \frac{n+23}{48} \right\rfloor\right)$$

SECOND EXAMPLE

In order to find a larger number of prime numbers than in the first example, this time all primes smaller than chosen number $N = 1000$, we will take the set of nested equations used in the first example and expand it both horizontally and vertically. The functions $p_2(n)$, $p_{2,3}(n)$ and $p_{2,3,5}(n)$ remain exactly the same since their respective periodicities of 2, 6 ($2*3$) and 30 ($2*3*5$) are smaller than 113, the highest prime number found in the first example. However, since the periodicity of $p_{2,3,5,7}(n)$ is $2*3*5*7 = 210$, we will have to proceed to a horizontal expansion by adding the following four floor functions $\left\lfloor \frac{n+17}{48} \right\rfloor + \left\lfloor \frac{n+14}{48} \right\rfloor + \left\lfloor \frac{n+8}{48} \right\rfloor + \left\lfloor \frac{n-3}{48} \right\rfloor$ inside the parentheses of the function

$$p_{2,3,5}\left(n - 1 + \left\lfloor \frac{n+43}{48} \right\rfloor + \left\lfloor \frac{n+32}{48} \right\rfloor + \left\lfloor \frac{n+26}{48} \right\rfloor + \left\lfloor \frac{n+23}{48} \right\rfloor\right)$$

A simplification can then be made by merging $\left\lfloor \frac{n+32}{48} \right\rfloor$ and $\left\lfloor \frac{n+8}{48} \right\rfloor$ together to obtain the equivalent $\left\lfloor \frac{n+8}{24} \right\rfloor$.

There is also a vertical expansion to implement and we will need to compute $p_{2,3} \dots p_x$ where p_x is the greatest prime number smaller than or equal to $\lfloor \sqrt{N} \rfloor$. In this second example, $\lfloor \sqrt{1000} \rfloor = 31$, and 31 is the greatest prime smaller than or equal to 31, hence the necessary use of $p_2(n)$, $p_{2,3}(n)$... up to $p_{2,3,5,7,11,13,17,19,23,29,31}(n)$.

E1 (Equation 1) for $n = 1$, $p_n = 2$

E2 for $n = 2$ to 4, $p_n = p_2(n) = 2n - 1$

E3 for $n = 5$ to 9, $p_n = p_{2,3}(n) = p_2\left(n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor\right)$

E4 for $n = 10$ to 15, $p_n = p_{2,3,5}(n) = p_{2,3}\left(n - 1 + \left\lfloor \frac{n+4}{8} \right\rfloor + \left\lfloor \frac{n-2}{8} \right\rfloor\right)$

E5 for $n = 16$ to 30,

$$p_n = p_{2,3,5,7}(n) = p_{2,3,5}\left(n - 1 + \left\lfloor \frac{n+8}{24} \right\rfloor + \left\lfloor \frac{n+43}{48} \right\rfloor + \left\lfloor \frac{n+26}{48} \right\rfloor + \left\lfloor \frac{n+23}{48} \right\rfloor + \left\lfloor \frac{n+17}{48} \right\rfloor + \left\lfloor \frac{n+14}{48} \right\rfloor + \left\lfloor \frac{n-3}{48} \right\rfloor\right)$$

E6 for $n = 31$ to 39,

$$p_n = p_{2,3,5,7,11}(n) = p_{2,3,5,7}\left(n - 1 + \left\lfloor \frac{n+474}{480} \right\rfloor + \left\lfloor \frac{n+449}{480} \right\rfloor + \left\lfloor \frac{n+445}{480} \right\rfloor + \left\lfloor \frac{n+436}{480} \right\rfloor + \left\lfloor \frac{n+432}{480} \right\rfloor + \left\lfloor \frac{n+422}{480} \right\rfloor + \left\lfloor \frac{n+408}{480} \right\rfloor + \left\lfloor \frac{n+405}{480} \right\rfloor + \left\lfloor \frac{n+390}{480} \right\rfloor + \left\lfloor \frac{n+381}{480} \right\rfloor + \left\lfloor \frac{n+377}{480} \right\rfloor + \left\lfloor \frac{n+369}{480} \right\rfloor + \left\lfloor \frac{n+355}{480} \right\rfloor + \left\lfloor \frac{n+340}{480} \right\rfloor + \left\lfloor \frac{n+336}{480} \right\rfloor + \left\lfloor \frac{n+322}{480} \right\rfloor + \left\lfloor \frac{n+313}{480} \right\rfloor + \left\lfloor \frac{n+309}{480} \right\rfloor + \left\lfloor \frac{n+295}{480} \right\rfloor + \left\lfloor \frac{n+285}{480} \right\rfloor + \left\lfloor \frac{n+272}{480} \right\rfloor\right)$$

E7 for $n = 40$ to 61,

$$p_n = p_{2,3,5,7,11,13}(n) = p_{2,3,5,7,11}\left(n + \left\lfloor \frac{n+960}{1000} \right\rfloor + \left\lfloor \frac{n+952}{1000} \right\rfloor + \left\lfloor \frac{n+946}{1000} \right\rfloor + \left\lfloor \frac{n+936}{1000} \right\rfloor + \left\lfloor \frac{n+922}{1000} \right\rfloor + \left\lfloor \frac{n+916}{1000} \right\rfloor + \left\lfloor \frac{n+902}{1000} \right\rfloor + \left\lfloor \frac{n+892}{1000} \right\rfloor + \left\lfloor \frac{n+888}{1000} \right\rfloor + \left\lfloor \frac{n+878}{1000} \right\rfloor + \left\lfloor \frac{n+863}{1000} \right\rfloor + \left\lfloor \frac{n+848}{1000} \right\rfloor + \left\lfloor \frac{n+844}{1000} \right\rfloor + \left\lfloor \frac{n+828}{1000} \right\rfloor + \left\lfloor \frac{n+818}{1000} \right\rfloor + \left\lfloor \frac{n+813}{1000} \right\rfloor\right)$$

E8 for $n = 62$ to 72 ,

$$p_n = p_{2,3,5,7,11,13,17}(n) = p_{2,3,5,7,11,13} \left(n + \left\lfloor \frac{n+938}{1000} \right\rfloor + \left\lfloor \frac{n+933}{1000} \right\rfloor + \left\lfloor \frac{n+921}{1000} \right\rfloor + \left\lfloor \frac{n+903}{1000} \right\rfloor + \left\lfloor \frac{n+898}{1000} \right\rfloor + \left\lfloor \frac{n+880}{1000} \right\rfloor + \left\lfloor \frac{n+868}{1000} \right\rfloor + \left\lfloor \frac{n+862}{1000} \right\rfloor + \left\lfloor \frac{n+851}{1000} \right\rfloor + \left\lfloor \frac{n+831}{1000} \right\rfloor + \left\lfloor \frac{n+814}{1000} \right\rfloor \right)$$

E9 for $n = 73$ to 99 ,

$$p_n = p_{2,3,5,7,11,13,17,19}(n) = p_{2,3,5,7,11,13,17} \left(n + \left\lfloor \frac{n+927}{1000} \right\rfloor + \left\lfloor \frac{n+915}{1000} \right\rfloor + \left\lfloor \frac{n+897}{1000} \right\rfloor + \left\lfloor \frac{n+891}{1000} \right\rfloor + \left\lfloor \frac{n+871}{1000} \right\rfloor + \left\lfloor \frac{n+859}{1000} \right\rfloor + \left\lfloor \frac{n+855}{1000} \right\rfloor + \left\lfloor \frac{n+840}{1000} \right\rfloor + \left\lfloor \frac{n+822}{1000} \right\rfloor \right)$$

E10 for $n = 100$ to 146 ,

$$p_n = p_{2,3,5,7,11,13,17,19,23}(n) = p_{2,3,5,7,11,13,17,19} \left(n + \left\lfloor \frac{n+900}{1000} \right\rfloor + \left\lfloor \frac{n+878}{1000} \right\rfloor + \left\lfloor \frac{n+872}{1000} \right\rfloor + \left\lfloor \frac{n+852}{1000} \right\rfloor + \left\lfloor \frac{n+837}{1000} \right\rfloor + \left\lfloor \frac{n+830}{1000} \right\rfloor \right)$$

E11 for $n = 147$ to 162 ,

$$p_n = p_{2,3,5,7,11,13,17,19,23,29}(n) = p_{2,3,5,7,11,13,17,19,23} \left(n + \left\lfloor \frac{n+853}{1000} \right\rfloor + \left\lfloor \frac{n+845}{1000} \right\rfloor \right)$$

E12 for $n = 163$ to 168 with $p_{168} = 997$,

$$p_n = p_{2,3,5,7,11,13,17,19,23,29,31}(n) = p_{2,3,5,7,11,13,17,19,23,29} \left(n + \left\lfloor \frac{n+837}{1000} \right\rfloor \right)$$

The formula works also for $n = 169$ to 172 with $p_{172} = 1021$.

After this, if we would like to expand and produce all prime numbers below 1100, we would have to modify horizontally the above set of nested equations by adding floor functions to some equations this way :

Add $\left\lfloor \frac{n+253}{480} \right\rfloor$ to E6 (Equation 6), $\left\lfloor \frac{n+798}{1000} \right\rfloor + \left\lfloor \frac{n+788}{1000} \right\rfloor$ to E7, $\left\lfloor \frac{n+807}{1000} \right\rfloor$ to E8, $\left\lfloor \frac{n+815}{1000} \right\rfloor$ to E10, and $\left\lfloor \frac{n+818}{1000} \right\rfloor$ to E11.

For example, E11 (Equation 11) would be transformed into

$$p_n = p_{2, 3, 5, 7, 11, 13, 17, 19, 23, 29}(n) = p_{2, 3, 5, 7, 11, 13, 17, 19, 23} \left(n + \left\lfloor \frac{n+853}{1000} \right\rfloor + \left\lfloor \frac{n+845}{1000} \right\rfloor + \left\lfloor \frac{n+818}{1000} \right\rfloor \right)$$

Since $\lfloor \sqrt{1100} \rfloor = 33$, 31 remains the highest prime number equal to or smaller than 33, so there is no need to expand vertically the set of nested equations.

The following pages will show how to construct those sets of nested equations.

BASIC FORMULAS

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If p is a positive integer greater than 1, we define the function

$$LB_p(n) = n + \left\lfloor \frac{n-1}{p-1} \right\rfloor$$

which gives all the integers not divisible by p . For example, if $p = 5$,

$$LB_5(n) = n + \left\lfloor \frac{n-1}{4} \right\rfloor$$

with $LB_5(n) = 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16...$ starting with $n = 1$.

We then define the reverse function of $LB_p(n)$ as

$$LB_p^{-1}(n) = KM_p(n) = n - \left\lfloor \frac{n}{p} \right\rfloor$$

With $p = 5$,

$$KM_5(n) = n - \left\lfloor \frac{n}{5} \right\rfloor$$

If q is another positive integer greater than 1, and if q is coprime with p , then the numbers which are not divisible either by p or q is given by the function

$$LB_{p, q}(n) = LB_p \left(n + \sum_{m=1}^{p-1} \left\lfloor \frac{n+(p-1)(q-1)+m-1-KM_p(q*LB_p(m))}{(p-1)(q-1)} \right\rfloor \right)$$

With $p = 5$ and $q = 7$, we get

$$LB_{5,7}(n) = LB_5\left(n + \left\lfloor \frac{n+18}{24} \right\rfloor + \left\lfloor \frac{n+13}{24} \right\rfloor + \left\lfloor \frac{n+9}{24} \right\rfloor + \left\lfloor \frac{n+4}{24} \right\rfloor\right)$$

which gives $LB_{5,7}(n) = 1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 16, 17, 18, 19, 22, 23, 24, 26, 27, 29, 31, 32, 33, 34, 36, 37, 38, 39, 41, 43, 44...$ starting with $n = 1$.

The reverse function of $LB_{p,q}(n)$ is

$$LB_{p,q}^{-1}(n) = KM_{p,q}(n) = KM_p(n) - \sum_{m=1}^{p-1} \left\lfloor \frac{KM_p(n) + (p-1)q - KM_p(q * LB_p(m))}{(p-1)q} \right\rfloor$$

The term $KM_p(q * LB_p(m))$ is found inside the brackets of the floor functions of both equations $LB_{p,q}(n)$ and $KM_{p,q}(n)$, so once it's calculated first for $LB_{p,q}(n)$, it can serve afterwards to calculate $KM_{p,q}(n)$.

It is not necessary that $p < q$. If $p > q$ however, since the value of m goes from 1 to $(p-1)$, $LB_{p,q}(n)$ and $KM_{p,q}(n)$ will have more floor functions to calculate.

p and q can both be composite (like $p = 4$ and $q = 25$) as long as they are coprime.

More generally, if the positive integers greater than 1 $a_1, a_2 \dots a_{x-1}, a_x$ are all coprime between themselves, then

$$LB_{a_1, a_2 \dots a_{(x-1)}, a_x}(n) = \left(n + \sum_{m=1}^{(a_1-1)(a_2-1) \dots (a_{(x-1)}-1)} \left\lfloor \frac{LB_{a_1, a_2 \dots a_{(x-1)}}(n) + (a_1-1)(a_2-1) \dots (a_{(x-1)}-1)(a_x-1) - KM_{a_1, a_2 \dots a_{(x-1)}}(a_x * LB_{a_1, a_2 \dots a_{(x-1)}}(m))}{(a_1-1)(a_2-1) \dots (a_{(x-1)}-1)(a_x-1)} \right\rfloor \right)$$

and

$$KM_{a_1, a_2 \dots a_{(x-1)}, a_x}(n) = KM_{a_1, a_2 \dots a_{(x-1)}}(n) - \sum_{m=1}^{(a_1-1)(a_2-1) \dots (a_{(x-1)}-1)} \left\lfloor \frac{KM_{a_1, a_2 \dots a_{(x-1)}}(n) + (a_1-1)(a_2-1) \dots (a_{(x-1)}-1)(a_x) - KM_{a_1, a_2 \dots a_{(x-1)}}(a_x * LB_{a_1, a_2 \dots a_{(x-1)}}(m))}{(a_1-1)(a_2-1) \dots (a_{(x-1)}-1)(a_x)} \right\rfloor$$

Here also, the term $KM_{a_1, a_2 \dots a_{(x-1)}}(a_x * LB_{a_1, a_2 \dots a_{(x-1)}}(m))$ is found inside the brackets of the floor functions of both equations $LB_{a_1, a_2 \dots a_{(x-1)}, a_x}(n)$ and $KM_{a_1, a_2 \dots a_{(x-1)}, a_x}(n)$, so once it's calculated first for $LB_{a_1, a_2 \dots a_{(x-1)}, a_x}(n)$, it can serve afterwards to calculate $KM_{a_1, a_2 \dots a_{(x-1)}, a_x}(n)$.

CONSTRUCTION

We can now construct the prime-generating set of nested equations that will act like the sieve of Eratosthenes. We start with

$$p_1 = 2$$

In order to eliminate all numbers divisible by 2, we use

$$LB_2(n) = n + \left\lfloor \frac{n-1}{2-1} \right\rfloor = 2n - 1$$

The reverse function is

$$KM_2(n) = n - \left\lfloor \frac{n}{2} \right\rfloor$$

$LB_2(2) = 3$, $LB_2(3) = 5$, $LB_2(4) = 7$. Since $LB_2(5) = 9 = 3^2$ is not prime, we must now switch to $LB_{2,3}(n)$.

$$LB_{2,3}(n) = LB_2 \left(n + \left\lfloor \frac{n}{2} \right\rfloor \right)$$

Since $LB_{2,3}(n)$ eliminates the prime number 3, $LB_{2,3}(5)$ will produce a prime number one step ahead of where it should be : $LB_{2,3}(5) = 13$ instead of 11. So, we cannot use $LB_{2,3}(n)$ directly, and we need to correct this shifting phenomenon by replacing n with $(n-1)$, which gives the equation shown at the beginning of this article in the two concrete examples :

$$p_{2,3}(n) = p_2 \left(n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor \right)$$

And we get the reverse function

$$KM_{2,3}(n) = KM_2(n) - \left\lfloor \frac{KM_2(n)+1}{3} \right\rfloor$$

$LB_{2,3}(n)$ will produce prime numbers until we reach 5^2 , the square of the prime number immediately following 2 and 3. We can predict for which value of n this will happen with the function

$$KM_{2, 3 \dots p(x-1), p_x}((p_{x+1})^2)$$

In this case, the value is $n = \text{KM}_{2,3}(5^2) = 9$. Previously, it was $\text{KM}_2(3^2) = 5$.

We now switch to $\text{LB}_{2,3,5}(n)$.

$$\text{LB}_{2,3,5}(n) = \text{LB}_{2,3}\left(n + \left\lfloor \frac{n+6}{8} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor\right)$$

Since $\text{LB}_{2,3,5}(n)$ eliminates the prime number 5, just like previously with $\text{LB}_{2,3}(n)$, $\text{LB}_{2,3,5}(9)$ will produce a prime number one step ahead of where it should be : $\text{LB}_{2,3,5}(9) = 31$ instead of 29. So, like before, we cannot use $\text{LB}_{2,3,5}(n)$ directly. Again, in order to make the necessary correction, we must replace n with $(n-1)$, but this time only for the single “ n ” that is not inside the brackets of the floor functions; for the two “ n ”s inside each floor function, we have found empirically that they both need to be each replaced with $(n-2)$, which gives the equation shown at the beginning of this article in the two concrete examples :

$$p_{2,3,5}(n) = p_{2,3}\left(n - 1 + \left\lfloor \frac{n+4}{8} \right\rfloor + \left\lfloor \frac{n-2}{8} \right\rfloor\right)$$

For $\text{LB}_{2,3,5,7}(n)$, the single “ n ” outside the brackets of the floor functions will be replaced with $(n-1)$ and the “ n ”s inside all the brackets of the floor functions will be replaced with $(n-3)$ to create $p_{2,3,5,7}(n)$. For $\text{LB}_{2,3,5,7,11}(n)$, the single “ n ” outside the brackets of the floor functions will be replaced with $(n-1)$ and the “ n ”s inside all the brackets of the floor functions will be replaced with $(n-4)$ to create $p_{2,3,5,7,11}(n)$. And so on.

$$\text{KM}_{2,3,5}(n) = \text{KM}_{2,3}(n) - \left\lfloor \frac{\text{KM}_{2,3}(n)+8}{10} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3}(n)+1}{10} \right\rfloor$$

$\text{LB}_{2,3,5}(n)$ will produce prime numbers until we reach 7^2 , the square of the prime number immediately following 2, 3 and 5. This will happen at $\text{KM}_{2,3,5}(7^2) = 14$.

We then obtain

$$\text{LB}_{2,3,5,7}(n) = \text{LB}_{2,3,5}\left(n + \left\lfloor \frac{n+46}{48} \right\rfloor + \left\lfloor \frac{n+35}{48} \right\rfloor + \left\lfloor \frac{n+29}{48} \right\rfloor + \left\lfloor \frac{n+26}{48} \right\rfloor + \left\lfloor \frac{n+20}{48} \right\rfloor + \left\lfloor \frac{n+17}{48} \right\rfloor + \left\lfloor \frac{n+11}{48} \right\rfloor + \left\lfloor \frac{n}{48} \right\rfloor\right)$$

Here, we can notice symmetry with the denominator number $48 - 2 = 46$. The first four numbers in the upper side of the brackets, 46, 35, 29 and 26, can produce the four following ones : $46-26 = 20$, $46-29 = 17$, $46-35 = 11$ and $46-46 = 0$. This

was also the case with the previous equation $LB_{2,3,5}(n)$ where $8 - 2 = 6$ which was the first number in the upper side of the brackets, and the second one was obtained with $6 - 6 = 0$. We have observed empirically that the same symmetrical phenomenon also appears in all the LB functions with $(2-1)(3-1)...(p-1) - 2$, which allows for an economy of calculations. $(2-1)(3-1)...(p-1) - 2$ is always the first number in the upper side of the brackets and 0 is always the last one.

A simplification can be made by merging $\left\lfloor \frac{n+35}{48} \right\rfloor$ and $\left\lfloor \frac{n+11}{48} \right\rfloor$ which are equivalent to $\left\lfloor \frac{n+11}{24} \right\rfloor$.

$$\begin{aligned} KM_{2,3,5,7}(n) = & KM_{2,3,5}(n) - \left\lfloor \frac{KM_{2,3,5}(n)+54}{56} \right\rfloor - \left\lfloor \frac{KM_{2,3,5}(n)+42}{56} \right\rfloor - \left\lfloor \frac{KM_{2,3,5}(n)+35}{56} \right\rfloor \\ & - \left\lfloor \frac{KM_{2,3,5}(n)+31}{56} \right\rfloor - \left\lfloor \frac{KM_{2,3,5}(n)+24}{56} \right\rfloor - \left\lfloor \frac{KM_{2,3,5}(n)+20}{56} \right\rfloor - \left\lfloor \frac{KM_{2,3,5}(n)+13}{56} \right\rfloor \\ & - \left\lfloor \frac{KM_{2,3,5}(n)+1}{56} \right\rfloor \end{aligned}$$

Through a cascading and trickle-down effect, the KM function acts like the well-known Legendre inclusion-exclusion formula.

$LB_{2,3,5,7}(n)$ will produce prime numbers until we reach 11^2 , the square of the prime number immediately following 2, 3, 5 and 7, at $n = KM_{2,3,5,7}(11^2) = 28$.

$$LB_{2,3,5,7,11}(n) = LB_{2,3,5,7}\left(n + \sum_{m=1}^{48} \left\lfloor \frac{n+480+m-1-KM_{2,3,5,7}(11*LB_{2,3,5,7}(m))}{480} \right\rfloor\right)$$

Here is a way to avoid useless calculations. The equation above has a periodicity of $2*3*5*7*11 = 2310$. Since, for the second concrete example, we decided to produce all prime numbers below $N = 1000$, we don't need practically to compute all the theoretical values of the equation above. We can stop at

$$m = KM_{2,3,5,7}\left(\left\lfloor \frac{1000}{11} \right\rfloor\right) = 21$$

instead of going all the way up to $m = 48$.

$$\begin{aligned}
LB_{2,3,5,7,11}(n) = & LB_{2,3,5,7}\left(n + \left\lfloor \frac{n+478}{480} \right\rfloor + \left\lfloor \frac{n+453}{480} \right\rfloor + \left\lfloor \frac{n+449}{480} \right\rfloor + \left\lfloor \frac{n+440}{480} \right\rfloor + \right. \\
& \left\lfloor \frac{n+436}{480} \right\rfloor + \left\lfloor \frac{n+426}{480} \right\rfloor + \left\lfloor \frac{n+412}{480} \right\rfloor + \left\lfloor \frac{n+409}{480} \right\rfloor + \left\lfloor \frac{n+394}{480} \right\rfloor + \left\lfloor \frac{n+385}{480} \right\rfloor + \left\lfloor \frac{n+381}{480} \right\rfloor + \\
& \left\lfloor \frac{n+373}{480} \right\rfloor + \left\lfloor \frac{n+359}{480} \right\rfloor + \left\lfloor \frac{n+344}{480} \right\rfloor + \left\lfloor \frac{n+340}{480} \right\rfloor + \left\lfloor \frac{n+326}{480} \right\rfloor + \left\lfloor \frac{n+317}{480} \right\rfloor + \left\lfloor \frac{n+313}{480} \right\rfloor + \\
& \left. \left\lfloor \frac{n+299}{480} \right\rfloor + \left\lfloor \frac{n+289}{480} \right\rfloor + \left\lfloor \frac{n+276}{480} \right\rfloor \right)
\end{aligned}$$

The exact same economy can be made with the reverse function of $LB_{2,3,5,7,11}(n)$:

$$\begin{aligned}
KM_{2,3,5,7,11}(n) = & KM_{2,3,5,7}(n) - \left\lfloor \frac{KM_{2,3,5,7}(n)+526}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+500}{528} \right\rfloor - \\
& \left\lfloor \frac{KM_{2,3,5,7}(n)+495}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+485}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+480}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+469}{528} \right\rfloor - \\
& \left\lfloor \frac{KM_{2,3,5,7}(n)+454}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+450}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+434}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+424}{528} \right\rfloor - \\
& \left\lfloor \frac{KM_{2,3,5,7}(n)+419}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+410}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+395}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+379}{528} \right\rfloor - \\
& \left\lfloor \frac{KM_{2,3,5,7}(n)+374}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+359}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+349}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+344}{528} \right\rfloor - \\
& \left\lfloor \frac{KM_{2,3,5,7}(n)+329}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+318}{528} \right\rfloor - \left\lfloor \frac{KM_{2,3,5,7}(n)+304}{528} \right\rfloor
\end{aligned}$$

We have now reached $LB_{2,3,5,7,11,13}(n) =$

$$LB_{2,3,5,7,11}\left(n + \sum_{m=1}^{480} \left\lfloor \frac{n+5760+m-1-KM_{2,3,5,7,11}(13*LB_{2,3,5,7,11}(m))}{5760} \right\rfloor \right)$$

The huge value 5760, which is quasi-primorial since it is the product of prime numbers with the difference that each prime has been reduced by one, could signal a danger zone of heavy calculations for the practicality of the formula like it often happens with well-known prime number formulas using factorials. However, since we decided in the second example to find only the prime numbers smaller than $N = 1000$, it is obvious the results within the brackets of the floor functions will either be 0 or 1, and will serve at this point only as an indicator whether or not to add 1 until we reach the potential value of 2, which will only happen much farther away for a value of n around 5000, that is until we need to calculate for approximately the 5000th prime number. So for $LB_{2,3,5,7,11,13}(n)$, the value of 5760 can practically be replaced with the value of 1000, and the same goes for all the following values of 92 160 for $LB_{2,3,5,7,11,13,17}(n)$, 1 658 880 for $LB_{2,3,5,7,11,13,17,19}(n)$, 36 495 360 for $LB_{2,3,5,7,11,13,17,19,23}(n)$, 1 021 870 080 for $LB_{2,3,5,7,11,13,17,19,23,29}(n)$ and 30 656 102 400 for $LB_{2,3,5,7,11,13,17,19,23,29,31}(n)$. As a matter of fact, we don't even need to calculate those values and as soon as the quasi-primorial product goes over the value N we have initially chosen (1000 in the second example) for finding all prime numbers that are below or equal to N , we

simply replace that quasi-primorial and all the following and increasing ones with N.

This allows us to write

$$\text{LB}_{2,3,5,7,11,13}(n) = \text{LB}_{2,3,5,7,11}\left(n + \left\lfloor \frac{n+998}{1000} \right\rfloor + \left\lfloor \frac{n+965}{1000} \right\rfloor + \left\lfloor \frac{n+957}{1000} \right\rfloor + \left\lfloor \frac{n+951}{1000} \right\rfloor + \left\lfloor \frac{n+941}{1000} \right\rfloor + \left\lfloor \frac{n+927}{1000} \right\rfloor + \left\lfloor \frac{n+921}{1000} \right\rfloor + \left\lfloor \frac{n+907}{1000} \right\rfloor + \left\lfloor \frac{n+897}{1000} \right\rfloor + \left\lfloor \frac{n+893}{1000} \right\rfloor + \left\lfloor \frac{n+883}{1000} \right\rfloor + \left\lfloor \frac{n+868}{1000} \right\rfloor + \left\lfloor \frac{n+853}{1000} \right\rfloor + \left\lfloor \frac{n+849}{1000} \right\rfloor + \left\lfloor \frac{n+833}{1000} \right\rfloor + \left\lfloor \frac{n+823}{1000} \right\rfloor + \left\lfloor \frac{n+818}{1000} \right\rfloor\right)$$

By replacing the single “n” outside the brackets of the floor functions with (n-1) and the “n”s inside all the brackets of the floor functions with (n-5), we obtain

$$p_n = p_{2,3,5,7,11,13}(n) = p_{2,3,5,7,11}\left(n - 1 + \left\lfloor \frac{n+993}{1000} \right\rfloor + \left\lfloor \frac{n+960}{1000} \right\rfloor + \left\lfloor \frac{n+952}{1000} \right\rfloor + \left\lfloor \frac{n+946}{1000} \right\rfloor + \left\lfloor \frac{n+936}{1000} \right\rfloor + \left\lfloor \frac{n+922}{1000} \right\rfloor + \left\lfloor \frac{n+916}{1000} \right\rfloor + \left\lfloor \frac{n+902}{1000} \right\rfloor + \left\lfloor \frac{n+892}{1000} \right\rfloor + \left\lfloor \frac{n+888}{1000} \right\rfloor + \left\lfloor \frac{n+878}{1000} \right\rfloor + \left\lfloor \frac{n+863}{1000} \right\rfloor + \left\lfloor \frac{n+848}{1000} \right\rfloor + \left\lfloor \frac{n+844}{1000} \right\rfloor + \left\lfloor \frac{n+828}{1000} \right\rfloor + \left\lfloor \frac{n+818}{1000} \right\rfloor + \left\lfloor \frac{n+813}{1000} \right\rfloor\right)$$

and in those modified equations with the initially chosen number $N = 1000$ as the common denominator of all the fractions inside the floor functions, the first floor function (in this case here $\left\lfloor \frac{n+993}{1000} \right\rfloor$) will always be equal to 1, so it will annul the term - 1 before it and allow us to simplify the expression by erasing both - 1 and $\left\lfloor \frac{n+993}{1000} \right\rfloor$ to get equation 7 shown in the second example.

$$\begin{aligned} \text{KM}_{2,3,5,7,11,13}(n) = & \text{KM}_{2,3,5,7,11}(n) - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+998}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+964}{1000} \right\rfloor - \\ & \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+955}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+948}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+937}{1000} \right\rfloor - \\ & \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+922}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+915}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+900}{1000} \right\rfloor - \\ & \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+889}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+884}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+873}{1000} \right\rfloor - \\ & \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+857}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+841}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+836}{1000} \right\rfloor - \\ & \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+819}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+808}{1000} \right\rfloor - \left\lfloor \frac{\text{KM}_{2,3,5,7,11}(n)+802}{1000} \right\rfloor \end{aligned}$$

We noticed previously that the $LB_{2,3\dots p(x-1), p_x}(n)$ function gives prime numbers until it reaches the value n equalled to

$$KM_{2, 3\dots p(x-1), p_x}((p_{x+1})^2)$$

where it produces the composed number $(p_{x+1})^2$. Once we have transformed $LB_{2, 3\dots p(x-1), p_x}(n)$ into $p_{2, 3\dots p(x-1), p_x}(n)$ by replacing the first “ n ” with $(n-1)$ and all the “ n ”s inside the brackets of the floor functions with $(n-x+1)$, because of the shifting phenomenon,

$p_{2, 3\dots p(x-1), p_x}(n)$ will be valid and give prime numbers up to $n =$

$$KM_{2, 3\dots p(x-1), p_x}((p_{x+1})^2) + x - 2$$

SIMILARITIES BETWEEN PRIME NUMBERS AND LUCKY NUMBERS

Lucky numbers are numbers produced by a sieve just like prime numbers, but their sieve eliminates numbers based on their position rather than on their non-divisibility. The first ten lucky numbers are 1, 3, 7, 9, 13, 15, 21, 25, 31 and 33. [2]

Lucky numbers and prime numbers share many common properties. It seems that every even number can be expressed as a sum of two lucky numbers just like for primes in Goldbach's conjecture. The number of lucky numbers is also infinite. The quantity of twin lucky numbers and twin primes appears to be asymptotically the same.

Lucky numbers can also be generated with a set of nested equations similar to the one just shown for prime numbers. If L_n denotes the n th lucky number,

$$\text{for } n = 1 \text{ to } 2, L_n = L_2(n) = 2n - 1$$

$$\text{for } n = 3 \text{ to } 6, L_n = L_{2,3}(n) = L_2\left(n + \left\lfloor \frac{n-1}{2} \right\rfloor\right)$$

$$\text{for } n = 7 \text{ to } 8, L_n = L_{2,3,7}(n) = L_{2,3}\left(n + \left\lfloor \frac{n-1}{6} \right\rfloor\right)$$

$$\text{for } n = 9 \text{ to } 12, L_n = L_{2,3,7,9}(n) = L_{2,3,7}\left(n + \left\lfloor \frac{n-1}{8} \right\rfloor\right)$$

$$\text{for } n = 13 \text{ to } 14, L_n = L_{2,3,7,9,13}(n) = L_{2,3,7,9}\left(n + \left\lfloor \frac{n-1}{12} \right\rfloor\right)$$

$$\text{for } n = 15 \text{ to } 20, L_n = L_{2,3,7,9,13,15}(n) = L_{2,3,7,9,13}\left(n + \left\lfloor \frac{n-1}{14} \right\rfloor\right)$$

$$\text{for } n = 21 \text{ to } 24, L_n = L_{2,3,7,9,13,15,21}(n) = L_{2,3,7,9,13,15}\left(n + \left\lfloor \frac{n-1}{20} \right\rfloor\right)$$

$$\text{for } n = 25 \text{ to } 30, L_n = L_{2,3,7,9,13,15,21,25}(n) = L_{2,3,7,9,13,15,21}\left(n + \left\lfloor \frac{n-1}{24} \right\rfloor\right)$$

More generally, the continuing pattern is

$$\text{for } n = L_x \text{ to } L_{x+1}-1, L_n = L_{2,3,\dots,L_x}(n) = L_{2,3,\dots,L_{x-1}}\left(n + \left\lfloor \frac{n-1}{(L_x)-1} \right\rfloor\right)$$

As we can see, the generating process is way much simpler horizontally for lucky numbers than it is for prime numbers. However, from a vertical standpoint, that's the exact opposite. Under $N = 1000$, there are 153 lucky numbers with $L_{153} = 997$. To generate those, we would need 32 equations with 1 floor function each up to $L_{2,3,\dots,151}(n)$, compared to the set of 12 nested equations with a total of 76 floor functions for producing the 168 prime numbers below 1000.

The explanation of this huge structural difference is that the extremely irregular and chaotic intervals that exist between each floor function in the heavy summations of the nested equations producing the primes happen to be here perfectly regular for the lucky numbers, as if there was an effective and somewhat hidden merging of all the floor functions into a single one. For example, let's look at the equation $LB_{2,3,5,7}(n)$ previously seen :

$$LB_{2,3,5,7}(n) = LB_{2,3,5} \left(n + \left\lfloor \frac{n+46}{48} \right\rfloor + \left\lfloor \frac{n+35}{48} \right\rfloor + \left\lfloor \frac{n+29}{48} \right\rfloor + \left\lfloor \frac{n+26}{48} \right\rfloor + \left\lfloor \frac{n+20}{48} \right\rfloor + \left\lfloor \frac{n+17}{48} \right\rfloor + \left\lfloor \frac{n+11}{48} \right\rfloor + \left\lfloor \frac{n}{48} \right\rfloor \right)$$

Inside the floor functions, the upper numbers 46, 35, 29, 26, 20, 17, 11 and 0 are totally irregular in their distribution. There are 8 of them spread over a 48 period, meaning the average difference between two consecutive numbers is $48/8 = 6$. If those numbers had instead been regularly spaced with an equal difference of 6 between consecutive numbers like 2, 8, 14, 20, 26, 32, 38 and 44, we would then have had

$$LB_{2,3,5,7}(n) = LB_{2,3,5} \left(n + \left\lfloor \frac{n+2}{48} \right\rfloor + \left\lfloor \frac{n+8}{48} \right\rfloor + \left\lfloor \frac{n+14}{48} \right\rfloor + \left\lfloor \frac{n+20}{48} \right\rfloor + \left\lfloor \frac{n+26}{48} \right\rfloor + \left\lfloor \frac{n+32}{48} \right\rfloor + \left\lfloor \frac{n+38}{48} \right\rfloor + \left\lfloor \frac{n+44}{48} \right\rfloor \right)$$

which could have then be reduced to an equation with a single floor function, just like with the above lucky numbers, by merging all 8 floor functions into a single one :

$$LB_{2,3,5,7}(n) = LB_{2,3,5} \left(n + \left\lfloor \frac{n+2}{6} \right\rfloor \right)$$

For $x > 1$, the reverse function of L_n is

$$L_{2, 3 \dots L_x}^{-1}(n) = RP_{2, 3 \dots L_x}(n) = RP_{2, 3 \dots L(x-1)}(n) - \left\lfloor \frac{RP_{2, 3 \dots L(x-1)}(n)}{L_x} \right\rfloor,$$

for $n = L_{(L_x)-1} + 1$ to $L_{(L(x+1))-1}$

$$\text{for } n = 1 \text{ to } 3, RP_2(n) = n - \left\lfloor \frac{n}{2} \right\rfloor$$

$$\text{if } x = 2, \text{ for } n = 4 \text{ to } 15, RP_{2, 3}(n) = RP_2(n) - \left\lfloor \frac{RP_2(n)}{3} \right\rfloor$$

$$\text{if } x = 3, \text{ for } n = 16 \text{ to } 25, RP_{2, 3, 7}(n) = RP_{2, 3}(n) - \left\lfloor \frac{RP_{2, 3}(n)}{7} \right\rfloor$$

$$\text{if } x = 4, \text{ for } n = 26 \text{ to } 43, RP_{2, 3, 7, 9}(n) = RP_{2, 3, 7}(n) - \left\lfloor \frac{RP_{2, 3, 7}(n)}{9} \right\rfloor$$

$$\text{if } x = 5, \text{ for } n = 44 \text{ to } 51, RP_{2, 3, 7, 9, 13}(n) = RP_{2, 3, 7, 9}(n) - \left\lfloor \frac{RP_{2, 3, 7, 9}(n)}{13} \right\rfloor$$

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