

Quadratic form binary optimization and optimal solution obtained from a new integral

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Abstract

In this article, I introduce a new integration method based on algebraic tools. This new integral enables an optimal solution with polynomial complexity P to be obtained for an NP-hard problem. The optimization problem involves minimizing a quadratic form with binary variables. The optimization algorithm is presented at the end of the report. The project concludes with numerical simulations to demonstrate this new concept of an integral, and the R programs are provided in the appendix so that the reader can test the new concept. Applications of this optimization problem span diverse fields such as economics, finance, machine learning and statistical physics (the Ising Model).

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1 Introduction

In this report, I present a new integral computed over binary variables. This integral enables us to obtain an optimal solution to a combinatorial optimization problem.

The problem statement employs a new type of projection operator $\mathcal{P}_{\Omega_L}()$; which we describe in the first section.

The optimization problem involves minimizing a quadratic form with binary variables; the problem itself is outlined in the first section.

This combinatorial optimization problems is NP-Hard, and we present a polynomial-time (P) optimization algorithm based on this new integration concept.

Applications of this optimization problem are found in economics, finance, machine learning, and stastical physics (The Ising model).

The report begins by introducing new types of vectors ξ , which are eigenvectors weighted by the square root of the eigenvalues.

Using the vectors ξ and the Gram-Schmidt matrix, I propose a sequential series expansion based on bit values of 0 and 1.

By employing new operators $\mathcal{A}_{[\theta_1, \theta_2]}()$, $(\cdot|\cdot)$, we provide a condensed sequential expansion for inclusion of bits values at 0 and 1.

This series expansion utilize new notations and takes the form of conditional series expansion.

Drawing on primal and dual theorems, we standardize the operators used for the primal and dual series expansions.

We introduce a symetric operator $(\cdot, \cdot)$ derived form operators $(\cdot|\cdot)$.

Based on the preceding discussion, we introduce the concept of an integral over binary variables. To do so, we must introduce the concept of a τ -topology.

We then express the primal and dual combinatorial optimization problem-involving quadratic forms with binary variables-as a primal integral to be maximized and a dual integral to be minimized.

At dual integral, I will propose an optimization algorithm for the new integral.

The report will conclude with a computer simulation using R software.

The R programs will be detailed in the appendix.

The numerical simulation will demonstrate the validation of this new integral fo studying the minimization of the quadratic form with binary variables.

2 Operator $\mathcal{P}_{\Omega_L}()$

In this section, we define the operator $\mathcal{P}_{\Omega_L}()$ that will be used throughout the report. To do so, let us consider an initial quadratic form:

$$Q(V, H) = V^t . H . V$$

where $H \in \mathbb{R}_{+,..}^{n \times n}$ is a symmetric matrix, $V \in \mathbb{R}_{+,..}^n$.

The set Ω_L will be the indices of the vector V where we have canceled the components.

This set therefore corresponds to the indices where bits=0 in the vector V .

L is the number of the bits=0.

We have for example for $L = 2$: $\mathcal{P}_{\Omega_L=\{1,4\}}(V \equiv [v_1, v_2, v_3, v_4, v_5]) = [0, v_2, v_3, 0, v_5]$

The complementary set Ω_L^C corresponds to the bits equal to 1 where we have not zeroed out the vector components V .

In this case, L corresponds to the number of bits equal to 1.

For the previous example, Ω_L^C corresponds to $\{2, 3, 5\}$ and the size L is equal to 3.

In the following, we will therefore use the following projected quadratic form:

$$Q(V, H) = \mathcal{P}_{\Omega_L}^t(V)^t . H . \mathcal{P}_{\Omega_L}(V)$$

and the goal of this project will be to find the subset of indices Ω_L of the size L fixed which minimize the following optimization problem:

$$\min_{\Omega_L} \{ \mathcal{P}_{\Omega_L}^t(V) . H . \mathcal{P}_{\Omega_L}(V) \}$$

Subject to $L = \text{Constant}$.

This problem is NP-hard; however, we introduce a new concept of integration over binary variables that enables an optimal solution to be obtained in polynomial time P .

In the following, we will develop this new integral using algebraic tools.

3 Introduction to vectors $\{\xi_{v_j}\}$ and theorem

Theorem:

Given a vector $V \in \mathbb{R}_{+,\dots}^n$, a symmetric matrix $H \in \mathbb{R}_{+,\dots}^{n \times n}$, a basis of orthonormal eigenvectors $\{X_1 = [x_{i,1}], X_2 = [x_{i,2}], X_3 = [x_{i,3}], \dots, X_n = [x_{i,n}]\}$, for $i = 1, 2, 3, \dots, n$, the spectrum of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ associated with the matrix H and the vectors ξ_{v_j} :

$$\xi_{v_j} = [\sqrt{\lambda_1}x_{1,j}, \sqrt{\lambda_2}x_{2,j}, \sqrt{\lambda_3}x_{3,j}, \dots, \sqrt{\lambda_n}x_{n,j}]$$

In this case, we can write the quadratic form $Q(V, H) = V^t.H.V$ as follows:

$$Q(V, H) = V^t.H.V = \left\| \sum_{i=1}^n v_j \xi_{v_j} \right\|_2^2$$

Proof:

$$\begin{aligned} Q(V, H) &= V^t.H.V \\ &= V^t.H.V \\ &= V^t.H. \sum_{i=1}^n \langle V | X_i \rangle . X_i \\ &= V^t \sum_{i=1}^n \lambda_i \langle V | X_i \rangle X_i \\ &= \sum_{j=1}^n \{ \langle V | X_j \rangle X_j \} . \sum_{i=1}^n \{ \langle V | X_j \rangle X_j \} \end{aligned}$$

The basis $\{X_i\}$ is orthonormal, therefore we have the following relationship:

$\{ \langle V | X_j \rangle X_j \} . \{ \lambda_i \langle V | X_i \rangle X_i \}$ where δ_{ij} is the Kronecker symbol.

$$\begin{aligned} Q(V, H) &= \sum_{i=1}^n \{ \langle V | X_i \rangle . X_i^t \} . \{ \lambda_i \langle V | X_i \rangle X_i \} \\ &= \sum_{i=1}^n \lambda_i \langle V | X_i \rangle^2 . \|X_i\|^2 \text{ however } \{X_i\} \text{ is a orthonormal's basis } \implies \|X_i\|^2 = 1 \\ &= \sum_{i=1}^n \lambda_i \langle V | X_i \rangle^2 \end{aligned}$$

We will consider the following terms: $\sqrt{\lambda_i} < V|X_i >$ using the following eigenvector table:

X_1	X_2	X_3	$\cdots$	X_j	$\cdots$	X_{n-1}	X_n
$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$\cdots$	$x_{1,j}$	$\cdots$	$x_{1,n-1}$	$x_{1,n}$
$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$\cdots$	$x_{2,j}$	$\cdots$	$x_{2,n-1}$	$x_{2,n}$
$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$\cdots$	$x_{3,j}$	$\cdots$	$x_{3,n-1}$	$x_{3,n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{j,1}$	$x_{j,2}$	$x_{j,3}$	$\cdots$	$x_{j,j}$	$\cdots$	$x_{j,n-1}$	$x_{j,n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{n,1}$	$x_{n,2}$	$x_{n,3}$	$\cdots$	$x_{n,j}$	$\cdots$	$x_{n,n-1}$	$x_{n,n}$

We can write the expression $\sqrt{\lambda_i} < V|X_i >$ in a structured table as follows:

$\sqrt{\lambda_1}v_1x_{1,1}$	$\sqrt{\lambda_1}v_2x_{1,2}$	$\sqrt{\lambda_1}v_3x_{1,3}$	$\cdots$	$\sqrt{\lambda_1}v_jx_{1,j}$	$\cdots$	$\sqrt{\lambda_1}v_{n-1}x_{1,n-1}$	$\sqrt{\lambda_1}v_nx_{1,n}$
$\sqrt{\lambda_2}v_1x_{2,1}$	$\sqrt{\lambda_2}v_2x_{2,2}$	$\sqrt{\lambda_2}v_3x_{2,3}$	$\cdots$	$\sqrt{\lambda_2}v_jx_{2,j}$	$\cdots$	$\sqrt{\lambda_2}v_{n-1}x_{2,n-1}$	$\sqrt{\lambda_2}v_nx_{2,n}$
$\sqrt{\lambda_3}v_1x_{3,1}$	$\sqrt{\lambda_3}v_2x_{3,2}$	$\sqrt{\lambda_3}v_3x_{3,3}$	$\cdots$	$\sqrt{\lambda_3}v_jx_{3,j}$	$\cdots$	$\sqrt{\lambda_3}v_{n-1}x_{3,n-1}$	$\sqrt{\lambda_3}v_nx_{3,n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\sqrt{\lambda_j}v_1x_{j,1}$	$\sqrt{\lambda_j}v_2x_{j,2}$	$\sqrt{\lambda_j}v_3x_{j,3}$	$\cdots$	$\sqrt{\lambda_j}v_jx_{j,j}$	$\cdots$	$\sqrt{\lambda_j}v_{n-1}x_{j,n-1}$	$\sqrt{\lambda_j}v_nx_{j,n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\sqrt{\lambda_n}v_1x_{n,1}$	$\sqrt{\lambda_n}v_2x_{n,2}$	$\sqrt{\lambda_n}v_3x_{n,3}$	$\cdots$	$\sqrt{\lambda_n}v_jx_{n,j}$	$\cdots$	$\sqrt{\lambda_n}v_{n-1}x_{n,n-1}$	$\sqrt{\lambda_n}v_nx_{n,n}$

We can write the table as follows:

$$v_1 \cdot \begin{pmatrix} \sqrt{\lambda_1}x_{11} \\ \sqrt{\lambda_2}x_{21} \\ \sqrt{\lambda_3}x_{31} \\ \vdots \\ \sqrt{\lambda_n}x_{n1} \end{pmatrix}, v_2 \cdot \begin{pmatrix} \sqrt{\lambda_1}x_{12} \\ \sqrt{\lambda_2}x_{22} \\ \sqrt{\lambda_3}x_{32} \\ \vdots \\ \sqrt{\lambda_n}x_{n2} \end{pmatrix}, v_3 \cdot \begin{pmatrix} \sqrt{\lambda_1}x_{13} \\ \sqrt{\lambda_2}x_{23} \\ \sqrt{\lambda_3}x_{33} \\ \vdots \\ \sqrt{\lambda_n}x_{n3} \end{pmatrix}, \dots, v_n \cdot \begin{pmatrix} \sqrt{\lambda_1}x_{1n} \\ \sqrt{\lambda_2}x_{2n} \\ \sqrt{\lambda_3}x_{3n} \\ \vdots \\ \sqrt{\lambda_n}x_{nn} \end{pmatrix}$$

We can therefore define the vectors ξ_{v_j}

$$\xi_{v_j} = [\sqrt{\lambda_1}x_{1,j}, \sqrt{\lambda_2}x_{2,j}, \sqrt{\lambda_3}x_{3,j}, \dots, \sqrt{\lambda_n}x_{n,j}]$$

By using the relation:

$$Q(H, V) = \sum_{i=1}^n \lambda_i < V|X_i >^2$$

We obtain:

$$Q(V, H) = V^t \cdot H \cdot V = \left\| \sum_{i=1}^n v_i \xi_{v_i} \right\|_2^2$$

Corollary:

Given a vector $V \in \mathbb{R}_{+,..}^n$, a symmetric matrix $H \in \mathbb{R}_{+,..}^{n \times n}$, a basis of orthonormal eigenvectors $\{X_1 = [x_{i,1}], X_2 = [x_{i,2}], X_3 = [x_{i,3}], \dots, X_n = [x_{i,n}]\}$, for $i = 1, 2, 3, \dots, n$, the spectrum of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ associated with the matrix H and the vectors ξ_{v_j} :

$$\xi_{v_j} = [\sqrt{\lambda_1}x_{1,j}, \sqrt{\lambda_2}x_{2,j}, \sqrt{\lambda_3}x_{3,j}, \dots, \sqrt{\lambda_n}x_{n,j}]$$

Now consider a set $\Omega_L \subseteq \{1, 2, 3, \dots, n\}$ and define $\mathcal{P}_{\Omega_L}$ as the operator for adding zeros to the components Ω_L of the vector V :

$$\mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) = \left\| \sum_{j \in \Omega_L^C} v_j \xi_{v_j} \right\|_2^2$$

where Ω_L^C is the complement of the set Ω_L , that is to say the indices where we have not put zeros. (That is, the variables where we put the 1 in the binary vector.)

Proof:

The result is trivial; indeed, by setting the components $j \in \Omega_L$ to zero, we cancel the contributions of these vectors $v_j \xi_{v_j}$ in the evaluation of the initial quadratic form $Q(V, H) = V^t.H.V$, thus obtaining a new quadratic form $\mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V)$ which is evaluated by the expression:

$$\mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) = \left\| \sum_{j \in \Omega_L^C} v_j \xi_{v_j} \right\|_2^2$$

4 Gram-Schmidt matrix and series obtained by adding zeros and variables 1 in the binary vector

To develop the conditional addition series in sparsity, we will start from the Gram-Schmidt matrix on the vectors:

$$\begin{pmatrix} \langle v_1 \xi_{v_1} | v_1 \xi_{v_1} \rangle & \langle v_1 \xi_{v_1} | v_2 \xi_{v_2} \rangle & \cdots & \cdots & \langle v_1 \xi_{v_1} | v_n \xi_{v_n} \rangle \\ \langle v_2 \xi_{v_2} | v_1 \xi_{v_1} \rangle & \langle v_2 \xi_{v_2} | v_2 \xi_{v_2} \rangle & \cdots & \cdots & \langle v_2 \xi_{v_2} | v_n \xi_{v_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle v_j \xi_{v_j} | v_1 \xi_{v_1} \rangle & \cdots & \langle v_j \xi_{v_j} | v_j \xi_{v_j} \rangle & \cdots & \langle v_j \xi_{v_j} | v_n \xi_{v_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle v_n \xi_{v_n} | v_1 \xi_{v_1} \rangle & \cdots & \cdots & \cdots & \langle v_n \xi_{v_n} | v_n \xi_{v_n} \rangle \end{pmatrix}$$

We will analyze some properties that we can extract from this matrix.

The first states that the sum of the elements of the matrix $G(\xi_{v_j})$ gives us the evaluation of the quadratic form $Q(V, H)$, indeed:

$$\text{The sum of the elements in the first row} = \langle v_1 \xi_{v_1} | \sum_{i=1}^n v_i \xi_{v_i} \rangle$$

$$\text{The sum of the elements in the second row} = \langle v_2 \xi_{v_2} | \sum_{i=1}^n v_i \xi_{v_i} \rangle$$

$\vdots$

$$\text{The sum of the elements of the nth line} = \langle v_n \xi_{v_n} | \sum_{i=1}^n v_i \xi_{v_i} \rangle$$

Adding all the lines together gives us:

$$\begin{aligned}
& \langle v_1 \xi_{v_1} \mid \sum_{i=1}^n v_i \xi_{v_i} \rangle + \langle v_2 \xi_{v_2} \mid \sum_{i=1}^n v_i \xi_{v_i} \rangle + \dots + \langle v_n \xi_{v_n} \mid \sum_{i=1}^n v_i \xi_{v_i} \rangle \\
&= \langle \sum_{i=1}^n v_i \xi_{v_i} \mid \sum_{i=1}^n v_i \xi_{v_i} \rangle \\
&= V^t . H . V \\
&= Q(V, H) \\
&= \|G(\xi_j)\|_1
\end{aligned}$$

where $\|\cdot\|_2^2$ is the usual Euclidean norm in $\mathbb{R}_{+,..}^n$ space and $\|\cdot\|_1$ is the matrix 1 norm in space $\mathbb{R}_{+,..}^{n \times n}$.

4.1 Series obtained by adding zeros in the binary vector

Everything that follows will be based on the Gram-Schmidt matrix.

We will now show how to reconstruct the evaluation of a quadratic form $Q(V, H) = V^t \cdot H \cdot V$ by adding zeros to the components of the vector V .

In what follows, we will consider an initial quadratic form $Q(V, H)$ to which we will consecutively add zeros to its components.

If we add a zero in front of the variable v_1 , we see that the first row and first column of the Gram-Schmidt matrix become zero. The goal here is to determine how much was lost during this operation.

Adding a zero to the variable v_1 is equivalent to replacing zeros in the first row and first column of the Gram-Schmidt matrix $G(\xi_{v_i})$.

To obtain this quantity lost due to the addition of zeros, simply sum the elements of the first row: $\langle v_1 \xi_{v_1} | \sum_{i=1}^n v_i \xi_{v_i} \rangle$

and the first column: $\langle v_1 \xi_{v_1} | \sum_{i=1}^n v_i \xi_{v_i} \rangle$ and subtract the quantity that was counted one too many times: $\langle v_1 \xi_{v_1} | v_1 \xi_{v_1} \rangle$. In summary, we therefore obtain:

$$2v_1 \langle \xi_1 | \sum_{i=1}^n v_i \xi_{v_i} \rangle - v_1^2 \|\xi_{v_1}\|_2^2$$

Now we will assume that the first variable v_1 has been replaced by a zero and we will calculate the quantity that will be lost by $Q(V, H)$ by adding a zero to the variable v_2 , given that we have already added a zero to the variable v_1 .

To do this, we simply sum the elements of the second row and the second column, taking into account (as in the previous case) the term that was counted twice. We then obtain the following provisional result:

$$2v_2 \langle \xi_2 | \sum_{i=1}^n v_i \xi_{v_i} \rangle - v_2^2 \|\xi_{v_2}\|_2^2$$

This summary is provisional because we must not forget that the two products $\langle v_1 \xi_{v_1} | v_2 \xi_{v_2} \rangle$ and $\langle v_2 \xi_{v_2} | v_1 \xi_{v_1} \rangle$ have already been taken into consideration, so we still need to subtract these two quantities from the previous result.

The final result of the gradient (or the variation on the quadratic form) obtained by adding a zero to the variable v_2 given the addition of a zero to the variable v_2 becomes:

$$2v_2 \langle \xi_2 | \sum_{i=1}^n v_i \xi_{v_i} \rangle - v_2^2 \|\xi_{v_2}\|_2^2 - 2 \langle v_1 \xi_{v_1} | v_2 \xi_{v_2} \rangle$$

The calculation of the other gradients or variations on the quadratic form is done in the same way as the first two iterations.

We will therefore state the general rule:

Theorem:

The gradient (or the variation on the quadratic form) produced by adding a zero to the variable v_i given the zeros applied to a set of variables Ω_L is evaluated by the following formula:

$$2v_i < \xi_{v_i} | \sum_{h=1}^n v_h \xi_{v_h} > -v_i^2 \|\xi_{v_i}\|_2^2 - 2v_i \sum_{k=1}^{i-1} v_k < \xi_{v_i} | \xi_{v_k} >$$

Adding consecutive zeros to the variables $v_1, v_2, v_3, \dots, v_n$ will cancel the quadratic form, and it is clear that the sum of all the gradients obtained by adding zeros to the variables $v_1, v_2, v_3, \dots, v_n$ will give us the evaluation of the quadratic form.

Based on the above, we can write the following equalities.

$$\begin{aligned} Q(V, H) &= V^t . H . V \\ &= \left\| \sum_{i=1}^n v_i \xi_{v_i} \right\|_2^2 \\ &= \|G(\xi_{v_i})\|_1 \\ &= \sum_{i=1}^n \{ 2v_i < \xi_{v_i} | \sum_{h=1}^n v_h \xi_{v_h} > -v_i^2 \|\xi_{v_i}\|_2^2 - 2v_i \sum_{k=1}^{i-1} v_k < \xi_{v_i} | \xi_{v_k} > \} \\ &= 2v_1 < \xi_{v_1} | \sum_{h=1}^n v_h \xi_{v_h} > -v_1^2 \|\xi_{v_1}\|_2^2 \\ &\quad + 2v_2 < \xi_{v_2} | \sum_{h=1}^n v_h \xi_{v_h} > -\{v_2^2 \|\xi_{v_2}\|_2^2 + 2v_2 v_1 < \xi_{v_2} | \xi_{v_1} >\} \\ &\quad + 2v_3 < \xi_{v_3} | \sum_{h=1}^n v_h \xi_{v_h} > -\{v_3^2 \|\xi_{v_3}\|_2^2 + 2v_3 v_2 < \xi_{v_3} | \xi_{v_2} > + 2v_3 v_1 < \xi_{v_3} | \xi_{v_1} >\} \\ &\quad \vdots \\ &\quad \vdots \\ &\quad + 2v_i < \xi_{v_i} | \sum_{h=1}^n v_h \xi_{v_h} > -v_i^2 \|\xi_{v_i}\|_2^2 - 2v_i \sum_{k=1}^{i-1} v_k < \xi_{v_i} | \xi_{v_k} > \\ &\quad \vdots \\ &\quad \vdots \\ &\quad + 2v_n < \xi_{v_n} | \sum_{h=1}^n v_h \xi_{v_h} > -v_n^2 \|\xi_{v_n}\|_2^2 - 2v_n \sum_{k=1}^{n-1} v_k < \xi_{v_n} | \xi_{v_k} > \end{aligned}$$

Since the order in which we add the zeros is irrelevant, we can define a set Ω_L on which the series will be expanded.

The series on the set Ω_L will be partial and will therefore create a variation on the initial quadratic form $Q(V, H)$ as follows:

$$\Delta Q = Q(V, H) - Q(\mathcal{P}_{\Omega_L}(V), H) = V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V)$$

.

Since the series is a chain rule in graph theory ,we define $Pa(i)$ as the parent indices of index i .

We therefore obtain the general rule:

$$\Delta Q$$

$$= V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V)$$

$$= \sum_{i \in \Omega_L} \{2v_i < \xi_{v_i} | \sum_{h=1}^n v_h \xi_{v_h} > -v_i^2 \|\xi_{v_i}\|_2^2 - 2v_i \sum_{k \in Pa(i)} v_k < \xi_{v_i} | \xi_{v_k} > \}$$

4.2 Series obtained by adding 1 in the binary vector

To give the sequence that occurs on all variables that have not been set to zero (i.e., the areas where bits are set to 1);

We need to define the last variables $v_n, v_{n-1}, v_{n-2}, \dots, v_2, v_1$ and simplify the last gradients: For the last gradient in the position n , we have:

$$2v_n < \xi_{v_n} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_n^2 \|\xi_{v_n}\|_2^2 - 2v_n \sum_{k=1}^{n-1} v_k < \xi_{v_n} | \xi_{v_k} >$$

$$2v_n < \xi_{v_n} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_n^2 \|\xi_{v_n}\|_2^2 - 2 < v_n \xi_{v_n} \left| \sum_{k=1}^{n-1} v_k \xi_{v_k} \right>$$

However we have:

$$\sum_{k=1}^n v_k \xi_{v_k} = \sum_{k=1}^{n-1} v_k \xi_{v_k} + v_n \xi_{v_n} \implies \sum_{k=1}^{n-1} v_k \xi_{v_k} = \sum_{k=1}^n v_k \xi_{v_k} - v_n \xi_{v_n}$$

We obtain for the last gradient:

$$\begin{aligned} & 2v_n < \xi_{v_n} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_n^2 \|\xi_{v_n}\|_2^2 - 2 < v_n \xi_{v_n} \left| \sum_{k=1}^n v_k \xi_{v_k} - v_n \xi_{v_n} \right> \\ & = 2v_n < \xi_{v_n} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_n^2 \|\xi_{v_n}\|_2^2 - 2 < v_n \xi_{v_n} \left| \sum_{k=1}^n v_k \xi_{v_k} \right> + 2 < v_n \xi_{v_n} | v_n \xi_{v_n} > \\ & = v_n^2 \|\xi_{v_n}\|_2^2 \end{aligned}$$

For the gradient in the position $n-1$:

$$\begin{aligned} & 2v_{n-1} < \xi_{v_{n-1}} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_{n-1}^2 \|\xi_{v_{n-1}}\|_2^2 - 2v_{n-1} \sum_{k=1}^{n-2} v_k < \xi_{v_{n-1}} | \xi_{v_k} > \\ & = 2v_{n-1} < \xi_{v_{n-1}} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_{n-1}^2 \|\xi_{v_{n-1}}\|_2^2 - 2 \sum_{k=1}^{n-2} < v_{n-1} \xi_{v_{n-1}} | v_k \xi_{v_k} > \\ & = 2v_{n-1} < \xi_{v_{n-1}} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_{n-1}^2 \|\xi_{v_{n-1}}\|_2^2 - 2 < v_{n-1} \xi_{v_{n-1}} \left| \sum_{k=1}^{n-2} v_k \xi_{v_k} \right> \end{aligned}$$

However we have:

$$\sum_{k=1}^n v_k \xi_{v_k} = \sum_{k=1}^{n-2} v_k \xi_{v_k} + v_{n-1} \xi_{v_{n-1}} + v_n \xi_{v_n} \implies \sum_{k=1}^{n-2} v_k \xi_{v_k} = \sum_{k=1}^n v_k \xi_{v_k} - v_{n-1} \xi_{v_{n-1}} - v_n \xi_{v_n}$$

We obtain for the gradient $n-1$:

$$\begin{aligned} & = 2v_{n-1} < \xi_{v_{n-1}} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_{n-1}^2 \|\xi_{v_{n-1}}\|_2^2 - 2 < v_{n-1} \xi_{v_{n-1}} \left| \sum_{k=1}^n v_k \xi_{v_k} - v_{n-1} \xi_{v_{n-1}} - v_n \xi_{v_n} \right> \\ & = 2v_{n-1} < \xi_{v_{n-1}} \left| \sum_{h=1}^n v_h \xi_{v_h} \right> - v_{n-1}^2 \|\xi_{v_{n-1}}\|_2^2 - 2 < v_{n-1} \xi_{v_{n-1}} \left| \sum_{k=1}^n v_k \xi_{v_k} \right> + 2 < v_{n-1} \xi_{v_{n-1}} | v_{n-1} \xi_{v_{n-1}} > \\ & \quad + 2 < v_{n-1} \xi_{v_{n-1}} | v_n \xi_{v_n} > \\ & = v_{n-1}^2 \|\xi_{v_{n-1}}\|_2^2 + 2v_{n-1} v_n < \xi_{v_{n-1}} | \xi_{v_n} > \end{aligned}$$

As in the previous situation for adding zeros, the order and choice of indices associated with the 1 bits can be arbitrarily chosen.

We can therefore give the gradient rule obtained by adding a 1 bit to a variable v_i given the previous 1-bit variables.

Theorem:

Given a set of zero indices Ω_L and its complement Ω_L^C , a set of 1 bits.

Adding a 1 bit to the variable v_i given the previous 1 bits on the indices of the parent variables $Pa(i)$, the gradient is given by:

$$v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i \sum_{k \in Pa(i)} v_k < \xi_{v_i} | \xi_{v_k} >$$

We can now link the projected quadratic form $\mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V)$ to the series of bit additions 1:

$$\mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) = \sum_{i \in \Omega_L^C} \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i \sum_{k \in Pa(i)} v_k < \xi_{v_i} | \xi_{v_k} >\}$$

5 Operators $\mathcal{A}_{[\theta_1, \theta_2]}()$ and $(\cdot|\cdot)$

To introduce the operators $\mathcal{A}_{[\theta_1, \theta_2]}()$ and $(\cdot|\cdot)$, we need to return to the following two gradients:

$$\left\{ \begin{array}{ll} 2v_i < \xi_{v_i} | \sum_{h=1}^n > v_h \xi_{v_h} - \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i \sum_{k=2}^{i-1} v_k < \xi_{v_i} | \xi_{v_k} >\} & \text{adding of bit 0} \\ v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i \sum_{k=1}^{i-1} v_k < \xi_{v_i} | \xi_{v_k} > & \text{addition of bit 1;} \end{array} \right.$$

Given the i -th gradient for the bit=0:

$$2v_i < \xi_{v_i} | \sum_{h=1}^n > v_h \xi_{v_h} - \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i \sum_{k=2}^{i-1} v_k < \xi_{v_i} | \xi_{v_k} >\}$$

Let us consider only the terms:

$$\begin{aligned} & v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i \sum_{k=1}^{i-1} v_k < \xi_{v_i} | \xi_{v_k} > \\ & = v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_1 < \xi_{v_i} | \xi_{v_1} > + 2v_i v_2 < \xi_{v_i} | \xi_{v_2} > + 2v_i v_3 < \xi_{v_i} | \xi_{v_3} > + \dots + 2v_i v_{i-1} < \xi_{v_i} | \xi_{v_{i-1}} > \end{aligned}$$

We will add the quantity $\|\xi_{v_i}\|_2^2 2$ to each term and remove the quantity $(i-2)v_i^2 \|\xi_{v_i}\|_2^2$:

$$\{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_1 < \xi_{v_i} | \xi_{v_1} >\} + \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_2 < \xi_{v_i} | \xi_{v_2} >\} + \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_1 < \xi_{v_i} | \xi_{v_3} >\} + \dots + \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_{i-1} < \xi_{v_i} | \xi_{v_{i-1}} >\} - (i-2)v_i^2 \|\xi_{v_i}\|_2^2$$

By replacing the bit gradient with bit 0:

$$\begin{aligned} & 2v_i < \xi_{v_i} | \sum_{h=1}^n > v_h \xi_{v_h} + (i-2)v_i^2 \|\xi_{v_i}\|_2^2 - \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_1 < \xi_{v_i} | \xi_{v_1} >\} - \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_2 < \xi_{v_i} | \xi_{v_2} >\} \\ & - \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_1 < \xi_{v_i} | \xi_{v_3} >\} - \dots - \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_{i-1} < \xi_{v_i} | \xi_{v_{i-1}} >\} \end{aligned}$$

by putting:

$$\left\{ \begin{array}{l} \mathcal{A}_{[i-2,1]}(v_i) = (i-2)v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i < \xi_{v_i} | \sum_{h=1}^n v_h \xi_{v_h} > \\ (v_k | v_l) = v_k^2 \|\xi_{v_k}\|_2^2 + 2v_k v_l < \xi_{v_k} | \xi_{v_l} > \end{array} \right.$$

For the gradient with bit 0, we finally obtain:

$$\boxed{\mathcal{A}_{[i-2,1]}(v_i) - \sum_{k=1}^{i-1} (v_i | v_k)}$$

Given the i-th gradient for the bit=1, we have:

$$v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i \sum_{k=1}^{i-1} v_k \langle \xi_{v_i} | \xi_{v_k} \rangle$$

$$= v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_1 \langle \xi_{v_i} | \xi_{v_1} \rangle + 2v_i v_2 \langle \xi_{v_i} | \xi_{v_2} \rangle + 2v_i v_3 \langle \xi_{v_i} | \xi_{v_3} \rangle + \dots + 2v_i v_{i-1} \langle \xi_{v_i} | \xi_{v_{i-1}} \rangle$$

We will add the quantity $\|\xi_{v_i}\|^2 2$ to each term and remove the quantity $(i-2)v_i^2 \|\xi_{v_i}\|_2^2$:

$$\{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_1 \langle \xi_{v_i} | \xi_{v_1} \rangle\} + \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_2 \langle \xi_{v_i} | \xi_{v_2} \rangle\} + \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_3 \langle \xi_{v_i} | \xi_{v_3} \rangle\} + \dots + \{v_i^2 \|\xi_{v_i}\|_2^2 + 2v_i v_{i-1} \langle \xi_{v_i} | \xi_{v_{i-1}} \rangle\} - (i-2)v_i^2 \|\xi_{v_i}\|_2^2$$

by putting:

$$\begin{cases} \mathcal{A}_{[2-i,0]}(v_i) = (2-i)v_i^2 \|\xi_{v_i}\|_2^2 \\ (v_k | v_l) = v_k^2 \|\xi_{v_k}\|_2^2 + 2v_k v_l \langle \xi_{v_k} | \xi_{v_l} \rangle \end{cases}$$

For the gradient with bit 1, we finally obtain:

$$\mathcal{A}_{[2-i,0]}(v_i) + \sum_{k=1}^{i-1} (v_i | v_k)$$

We can now define the operators $\mathcal{A}_{[\theta_1, \theta_2]}(v)$ and $(v_i | v_k)$ in general as follows:

$$\begin{cases} \mathcal{A}_{[\theta_1, \theta_2]}(v) = \theta_1 \cdot v^2 \|\xi_v\|_2^2 + 2\theta_2 \cdot v \langle \xi_v | \sum_{h=1}^n v_h \xi_h \rangle \\ (v_k | v_l) = v_k^2 \|\xi_{v_k}\|_2^2 + 2v_k v_l \langle \xi_{v_k} | \xi_{v_l} \rangle \end{cases}$$

The choice of variables and their order are not important; the series of adding bits 0 is defined and is related to the variation of the quadratic form ΔQ as follows:

$$\Delta Q = V^t \cdot H \cdot V - \mathcal{P}_{\Omega_L}^t(V) \cdot H \cdot \mathcal{P}_{\Omega_L}(V) = \sum_{i \in \Omega_L} \{\mathcal{A}_{[\theta-2,1]}(v_i) - \sum_{k \in Pa(i)} (v_i | v_k)\}$$

where θ takes the successive values $(1, 2, 3, 4, \dots, \#\Omega_L)$ at each new gradient due to the addition of a bit=0

the series of adding bits 1 is defined and is related to the projection of the quadratic form $\mathcal{P}_{\Omega_L}^t(V) \cdot H \cdot \mathcal{P}_{\Omega_L}(V)$ as follows:

$$\mathcal{P}_{\Omega_L}^t(V) \cdot H \cdot \mathcal{P}_{\Omega_L}(V) = \sum_{i \in \Omega_L^C} \{\mathcal{A}_{[2-\theta,0]}(v_i) + \sum_{k \in Pa(i)} (v_i | v_k)\}$$

where θ takes the successive values $(1, 2, 3, 4, \dots, \#\Omega_L^C)$ at each new gradient due to the addition of a bit=1.

To present the results, we will present the series for binary vectors=0 for the variables $v_2, v_4, v_6, v_7, v_{11}$ ($\Omega_L = (2, 4, 6, 7, 11)$):

$$\begin{aligned}
& \Delta Q \\
&= V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) \\
&= \mathcal{A}_{[-1,1]}(v_2) \\
&+ \mathcal{A}_{[0,1]}(v_4) - (v_4|v_2) \\
&+ \mathcal{A}_{[1,1]}(v_6) - (v_6|v_2) - (v_6|v_4) \\
&+ \mathcal{A}_{[2,1]}(v_7) - (v_7|v_2) - (v_7|v_4) - (v_7|v_6) \\
&+ \mathcal{A}_{[3,1]}(v_{11}) - (v_{11}|v_2) - (v_{11}|v_4) - (v_{11}|v_6) - (v_{11}|v_7)
\end{aligned}$$

For the binary vectors=1 for the variables $v_2, v_4, v_6, v_7, v_{11}$ ($\Omega_L^C = (2, 4, 6, 7, 11)$)

$$\begin{aligned}
& \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) \\
&= \mathcal{A}_{[-1,0]}(v_2) \\
&+ \mathcal{A}_{[0,0]}(v_4) + (v_4|v_2) \\
&+ \mathcal{A}_{[-1,0]}(v_6) + (v_6|v_2) + (v_6|v_4) \\
&+ \mathcal{A}_{[-2,0]}(v_7) + (v_7|v_2) + (v_7|v_4) + (v_7|v_6) \\
&+ \mathcal{A}_{[-3,0]}(v_{11}) + (v_{11}|v_2) + (v_{11}|v_4) + (v_{11}|v_6) + (v_{11}|v_7)
\end{aligned}$$

6 Series for the operators $\mathcal{A}_{[\frac{L-3}{2}, 1]}$, (v_k, v_l) and addition of bit=0

We will state a theorem allowing us to homogenize the operator $\mathcal{A}_{[\theta_1, \theta_2]}$ at each addition of bit=0.

Theorem:

Consider a set of indices Ω_L for which we add bits = 0.

In this case, we obtain the following relation:

$$\Delta Q = V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) = \sum_{i \in \Omega_L} \{ \mathcal{A}_{[\frac{L-3}{2}, 1]}(v_i) - \sum_{k \in Pa(i)} (v_i, v_k) \}$$

where we have:

$$(v_i, v_k) = \frac{1}{2} \{ (v_i|v_k) + (v_k|v_i) \}$$

and L is the number of zeros added to the binary vector.

Proof:

The proof is done by induction on a set of variables $(v_1, v_2, v_3, \dots, v_L)$:

For $L=1$:

We know that the first addition of zeros to a variable v_1 is calculated by:

$$\mathcal{A}_{[-2, 1]}(v_1) = \mathcal{A}_{[-1, 1]}(v_1)$$

By using the theorem for $L = 1$ we have:

$$\mathcal{A}_{[\frac{L-3}{2}, 1]}(v_1) = \mathcal{A}_{[-1, 1]}(v_1)$$

We have demonstrated the relationship for $L = 1$.

For $L=2$:

We know that the series on the variables v_1, v_2 can be written as:

$$\mathcal{A}_{[-1, 1]}(v_1) + \mathcal{A}_{[0, 1]}(v_2) - (v_1|v_2)$$

or

$$\mathcal{A}_{[-1, 1]}(v_2) + \mathcal{A}_{[0, 1]}(v_1) - (v_2|v_1)$$

Adding the two previous terms and dividing by two, we obtain the following series on the variables v_1, v_2 :

$$\frac{1}{2} \mathcal{A}_{[-1, 1]}(v_1) + \frac{1}{2} \mathcal{A}_{[-1, 1]}(v_2) + \frac{1}{2} \mathcal{A}_{[0, 1]}(v_2) + \frac{1}{2} \mathcal{A}_{[0, 1]}(v_1) - \frac{1}{2} (v_2|v_1) - \frac{1}{2} (v_1|v_2)$$

But we know that:

$$\begin{aligned}
& \frac{1}{2}\mathcal{A}_{[-1,1]}(v_1) + \frac{1}{2}\mathcal{A}_{[0,1]}(v_1) \\
&= \frac{1}{2}\{-v_1^2\|\xi_{v_1}\|_2^2 + 2v_1 < \xi_{v_1} | \sum_{h=1}^n v_h \xi_{v_h} > + 2v_1 < \xi_{v_1} | \sum_{h=1}^n v_h \xi_{v_h} >\} \\
&= \frac{1}{2}\{-v_1^2\|\xi_{v_1}\|_2^2 + 4v_1 < \xi_{v_1} | \sum_{h=1}^n v_h \xi_{v_h} >\} \\
&= \frac{1}{2}\mathcal{A}_{[-1,2]}(v_1)
\end{aligned}$$

Similarly, we can show that:

$$\frac{1}{2}\mathcal{A}_{[-1,1]}(v_2) + \frac{1}{2}\mathcal{A}_{[0,1]}(v_2) = \frac{1}{2}\mathcal{A}_{[-1,2]}(v_2)$$

So we finally obtain:

$$\begin{aligned}
& \frac{1}{2}\mathcal{A}_{[-1,1]}(v_1) + \frac{1}{2}\mathcal{A}_{[-1,1]}(v_2) + \frac{1}{2}\mathcal{A}_{[0,1]}(v_2) + \frac{1}{2}\mathcal{A}_{[0,1]}(v_1) - \frac{1}{2}(v_2|v_1) - \frac{1}{2}(v_1|v_2) \\
&= \frac{1}{2}\mathcal{A}_{[-1,2]}(v_1) + \frac{1}{2}\mathcal{A}_{[-1,2]}(v_2) - \frac{1}{2}(v_2|v_1) - \frac{1}{2}(v_1|v_2) \\
&= \mathcal{A}_{[-\frac{1}{2},1]}(v_1) + \mathcal{A}_{[-\frac{1}{2},1]}(v_2) - \frac{1}{2}\{(v_2|v_1) + (v_1|v_2)\}
\end{aligned}$$

By putting:

$$(v_1, v_2) = \frac{1}{2}\{(v_2|v_1) + (v_1|v_2)\}$$

We obtain the series:

$$\mathcal{A}_{[-\frac{1}{2},1]}(v_1) + \mathcal{A}_{[-\frac{1}{2},1]}(v_2) - (v_2, v_1)$$

We have demonstrated the relationship for $L = 2$.

We will assume that the theorem is true for the iterate $L = P$ and demonstrate that this theorem remains true for the iterate $L = P + 1$. To do this, let's start again from the P order expansion

and add the final gradient in the last row:

$$\begin{aligned}
& \mathcal{A}_{[\frac{P-3}{2}, 1]}(v_1) \\
& + \mathcal{A}_{[\frac{P-3}{2}, 1]} - (v_2) - (v_2, v_1) \\
& + \mathcal{A}_{[\frac{P-3}{2}, 1]} - (v_3) - (v_3, v_1) - (v_3, v_2) \\
& + \mathcal{A}_{[\frac{P-3}{2}, 1]} - (v_4) - (v_4, v_1) - (v_4, v_2) - (v_4, v_3) \\
& \vdots \\
& \vdots \\
& + \mathcal{A}_{[\frac{P-3}{2}, 1]} - (v_P) - (v_P, v_1) - (v_P, v_2) - (v_P, v_3) - \dots - (v_P, v_{P-1}) \\
& + \mathcal{A}_{[P-1, 1]}(v_{P+1}) - (v_{P+1}|v_1 - (v_{P+1}|v_2)) - (v_{P+1}|v_3)) - (v_{P+1}, v_P) \\
& = \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_1) - \frac{1}{2}v_1^2 \cdot \|\xi_{v_1}\|_2^2 \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_2) - \frac{1}{2}v_2^2 \cdot \|\xi_{v_2}\|_2^2 - (v_2, v_1) \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_3) - \frac{1}{2}v_3^2 \cdot \|\xi_{v_3}\|_2^2 - (v_3, v_1) - (v_3, v_2) \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_4) - \frac{1}{2}v_4^2 \cdot \|\xi_{v_4}\|_2^2 - (v_4, v_1) - (v_4, v_2) - (v_4, v_3) \\
& \vdots \\
& \vdots \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_P) - \frac{1}{2}v_P^2 \cdot \|\xi_{v_P}\|_2^2 - (v_P, v_1) - (v_P, v_2) - (v_P, v_3) - \dots - (v_P, v_{P-1}) \\
& + \mathcal{A}_{[P-1, 1]} - (v_{P+1}|v_1) - (v_{P+1}|v_2) - (v_{P+1}|v_3) - \dots - (v_{P+1}|v_P) \\
& = \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_1) - \frac{1}{2}v_1^2 \cdot \|\xi_{v_1}\|_2^2 \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_2) - \frac{1}{2}v_2^2 \cdot \|\xi_{v_2}\|_2^2 - (v_2, v_1) \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_3) - \frac{1}{2}v_3^2 \cdot \|\xi_{v_3}\|_2^2 - (v_3, v_1) - (v_3, v_2) \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_4) - \frac{1}{2}v_4^2 \cdot \|\xi_{v_4}\|_2^2 - (v_4, v_1) - (v_4, v_2) - (v_4, v_3) \\
& \vdots \\
& \vdots \\
& + \mathcal{A}_{[\frac{P-2}{2}, 1]}(v_P) - v_P^2 \cdot \|\xi_{v_P}\|_2^2 - (v_P, v_1) - (v_P, v_2) - (v_P, v_3) - \dots - (v_P, v_{P-1}) \\
& + \mathcal{A}_{[P-1, 1]}(v_{P+1}) - \sum_{k=1}^P v_k \|\xi_{v_k}\|_2^2 - \sum_{k=1}^P (v_{P+1}|v_k)
\end{aligned}$$

The last row of the series becomes:

$$\begin{aligned}
& \mathcal{A}_{[P-1,1]}(v_{P+1}) - \frac{1}{2} \sum_{k=1}^P v_k \|\xi_{v_k}\|_2^2 - \sum_{k=1}^P (v_{P+1} | v_k) \\
&= (P-1)v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} < \xi_{v_{P+1}} | \sum_{h=1}^n v_h \xi_{v_h} > - \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 - P.v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 \\
&- 2.v_{P+1} < \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} > \\
&= -v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} < \xi_{v_{P+1}} | \sum_{h=1}^n v_h \xi_{v_h} > - \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 - 2.v_{P+1} < \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} > \\
&= -v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} < \xi_{v_{P+1}} | \sum_{h=1}^n v_h \xi_{v_h} > - \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 - 2.v_{P+1} < \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} >
\end{aligned}$$

The theorem is true if the last row above is equal to the last gradient of the following theorem:

$$\begin{aligned}
& \mathcal{A}_{[\frac{P-2}{2},1]}(v_{P+1}) - \sum_{k=1}^P (v_{P+1}, v_k) \\
&= \frac{1}{2} (P-2) v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} < \xi_{v_{P+1}} | \sum_{k=1}^n v_k \xi_{v_k} > \\
&- \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} v_1 < \xi_{v_{P+1}} | \xi_{v_1} > \} - \frac{1}{2} \{ v_1^2 \|\xi_{v_1}\|_2^2 - 2v_1 v_{P+1} < \xi_{v_1} | \xi_{v_{P+1}} > \} \\
&- \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} v_2 < \xi_{v_{P+1}} | \xi_{v_2} > \} - \frac{1}{2} \{ v_2^2 \|\xi_{v_2}\|_2^2 - 2v_2 v_{P+1} < \xi_{v_2} | \xi_{v_{P+1}} > \} \\
&- \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} v_3 < \xi_{v_{P+1}} | \xi_{v_3} > \} - \frac{1}{2} \{ v_3^2 \|\xi_{v_3}\|_2^2 - 2v_3 v_{P+1} < \xi_{v_3} | \xi_{v_{P+1}} > \} \\
&\vdots \\
&- \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} v_P < \xi_{v_{P+1}} | \xi_{v_P} > \} - \frac{1}{2} \{ v_P^2 \|\xi_{v_P}\|_2^2 - 2v_P v_{P+1} < \xi_{v_P} | \xi_{v_{P+1}} > \} \\
&= \frac{1}{2} (P-2) v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} < \xi_{v_{P+1}} | \sum_{k=1}^n v_k \xi_{v_k} > - \frac{P}{2} v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 \\
&- v_{P+1} < \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} > - \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 - v_{P+1} < \sum_{k=1}^P v_k \xi_{v_k} | \xi_{v_{P+1}} >
\end{aligned}$$

Finally we obtain:

$$= -v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2v_{P+1} < \xi_{v_{P+1}} | \sum_{h=1}^n v_h \xi_{v_h} > - \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 - 2.v_{P+1} < \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} >$$

We proved the theorem by induction.

To present the results, we will present the series for binary vectors=0 for the variables $(v_2, v_4, v_6, v_7, v_{11}, v_{14})$ ($\Omega_L = (2, 4, 6, 7, 11, 14)$) and $L = 6$:

$$\begin{aligned}
& \Delta Q \\
&= V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) \\
&= \mathcal{A}_{[\frac{3}{2}, 1]}(v_2) \\
&+ \mathcal{A}_{[\frac{3}{2}, 1]}(v_4) - (v_4, v_2) \\
&+ \mathcal{A}_{[\frac{3}{2}, 1]}(v_6) - (v_6, v_2) - (v_6, v_4) \\
&+ \mathcal{A}_{[\frac{3}{2}, 1]}(v_7) - (v_7, v_2) - (v_7, v_4) - (v_7, v_6) \\
&+ \mathcal{A}_{[\frac{3}{2}, 1]}(v_{11}) - (v_{11}, v_2) - (v_{11}, v_4) - (v_{11}, v_6) - (v_{11}, v_7) \\
&+ \mathcal{A}_{[\frac{3}{2}, 1]}(v_{14}) - (v_{14}, v_2) - (v_{14}, v_4) - (v_{14}, v_6) - (v_{14}, v_7) - (v_{14}, v_{11})
\end{aligned}$$

7 Series for the operators $\mathcal{A}_{[\frac{3-L}{2}, 0]}$, (v_k, v_l) and addition of bit=1

We will state a theorem allowing us to homogenize the operator $\mathcal{A}_{[\theta_1, \theta_2]}$ at each addition of bit=1.

Theorem:

Consider a set of indices Ω_L for which we add bits = 0 and Ω_L^C for which we add bits =1.

In this case, we obtain the following relation:

$$\mathcal{P}_{\Omega_L}^t(V) \cdot H \cdot \mathcal{P}_{\Omega_L}(V) = \sum_{i \in \Omega_L^C} \{ \mathcal{A}_{[\frac{3-L}{2}, 0]}(v_i) + \sum_{k \in Pa(i)} (v_i, v_k) \}$$

where we have:

$$(v_i, v_k) = \frac{1}{2} \{ (v_i|v_k) + (v_k|v_i) \}$$

and L is the number of bits=1 added to the binary vector.

Proof:

The theorem can also be proven by induction, as with bits=0.

Let us consider a set of variables $(v_1, v_2, v_3, ..v_L)$.

We have:

$L=1$:

Adding a bit=1 gives us:

$$\mathcal{A}_{[2-i, 0]}(v_1) = \mathcal{A}_{[-1, 0]}(v_1)$$

By using the theorem for $L = 1$, we obtain the same result:

$$\mathcal{A}_{[\frac{3-L}{2}, 0]}(v_1) = \mathcal{A}_{[-1, 0]}(v_1)$$

$L = 2$:

The series on the variables (v_1, v_2) can be calculated as:

$$\mathcal{A}_{[-1, 0]}(v_1) + \mathcal{A}_{[0, 0]}(v_2) + (v_1|v_2)$$

or

$$\mathcal{A}_{[-1, 0]}(v_2) + \mathcal{A}_{[0, 0]}(v_1) + (v_2|v_1)$$

Adding and dividing by two, we obtain:

$$\frac{1}{2} \mathcal{A}_{[-1, 0]}(v_1) + \frac{1}{2} \mathcal{A}_{[0, 0]}(v_1) + \frac{1}{2} \mathcal{A}_{[-1, 0]}(v_2) + \frac{1}{2} \mathcal{A}_{[0, 0]}(v_2) + \frac{1}{2} (v_1|v_2) + \frac{1}{2} (v_2|v_1)$$

However we have:

$$\begin{aligned} & \frac{1}{2}\mathcal{A}_{[-1,0]}(v_1) + \frac{1}{2}\mathcal{A}_{[0,0]}(v_1) \\ &= \mathcal{A}_{[-\frac{1}{2},2]}(v_1) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2}\mathcal{A}_{[-1,0]}(v_2) + \frac{1}{2}\mathcal{A}_{[0,0]}(v_2) \\ &= \mathcal{A}_{[-\frac{1}{2},2]}(v_2) \end{aligned}$$

Finally we obtain the result:

$$\begin{aligned} & \frac{1}{2}\mathcal{A}_{[-1,0]}(v_1) + \frac{1}{2}\mathcal{A}_{[0,0]}(v_1) + \frac{1}{2}\mathcal{A}_{[-1,0]}(v_2) + \frac{1}{2}\mathcal{A}_{[0,0]}(v_2) + \frac{1}{2}(v_1|v_2) + \frac{1}{2}(v_2|v_1) \\ &= \mathcal{A}_{[-\frac{1}{2},2]}(v_1) + \mathcal{A}_{[-\frac{1}{2},2]}(v_2) + \frac{1}{2}\{(v_1|v_2) + (v_2|v_1)\} \\ &= \mathcal{A}_{[-\frac{1}{2},2]}(v_1) + \mathcal{A}_{[-\frac{1}{2},2]}(v_2) + (v_1, v_2) \end{aligned}$$

We will assume that the theorem is true for the iterate $L = P$ and demonstrate that this theorem remains true for the iterate $L = P + 1$. To do this, let's start again from the P order expansion

and add the final gradient in the last row:

$$\begin{aligned}
&= \mathcal{A}_{[\frac{3-P}{2}, 0]}(v_1) \\
&+ \mathcal{A}_{[\frac{3-P}{2}, 0]}(v_2) + (v_2, v_1) \\
&+ \mathcal{A}_{[\frac{3-P}{2}, 0]}(v_3) + (v_3, v_1) + (v_3, v_2) \\
&+ \mathcal{A}_{[\frac{3-P}{2}, 0]}(v_4) + (v_4, v_1) + (v_4, v_2) + (v_4, v_3) \\
&\vdots \\
&\vdots \\
&+ \mathcal{A}_{[\frac{3-P}{2}, 0]}(v_P) + (v_P, v_1) + (v_P, v_2) + (v_P, v_3) + \dots + (v_P, v_{P-1}) \\
&+ \mathcal{A}_{[1-P, 0]}(v_{P+1}) + (v_{P+1}|v_1) + (v_{P+1}|v_2) + (v_{P+1}|v_3) + \dots + (v_{P+1}|v_P) \\
\\
&= \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_1) + \frac{1}{2} v_1^2 \|\xi_{v_1}\|_2^2 \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_2) + \frac{1}{2} v_2^2 \|\xi_{v_2}\|_2^2 + (v_2, v_1) \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_3) + \frac{1}{2} v_3^2 \|\xi_{v_3}\|_2^2 + (v_3, v_1) + (v_3, v_2) \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_4) + \frac{1}{2} v_4^2 \|\xi_{v_4}\|_2^2 + (v_4, v_1) + (v_4, v_2) + (v_4, v_3) \\
&\vdots \\
&\vdots \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_P) + \frac{1}{2} v_P^2 \|\xi_{v_P}\|_2^2 + (v_P, v_1) + (v_P, v_2) + (v_P, v_3) + \dots + (v_P, v_{P-1}) \\
&+ \mathcal{A}_{[1-P, 0]}(v_{P+1}) + (v_{P+1}|v_1) + (v_{P+1}|v_2) + (v_{P+1}|v_3) + \dots + (v_{P+1}|v_P) \\
\\
&= \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_1) \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_2) + (v_2, v_1) \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_3) + (v_3, v_1) + (v_3, v_2) \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_4) + (v_4, v_1) + (v_4, v_2) + (v_4, v_3) \\
&\vdots \\
&\vdots \\
&+ \mathcal{A}_{[\frac{2-P}{2}, 0]}(v_P) + (v_P, v_1) + (v_P, v_2) + (v_P, v_3) + \dots + (v_P, v_{P-1}) \\
&+ \mathcal{A}_{[1-P, 0]}(v_{P+1}) + \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 + \sum_{k=1}^P (v_{P+1}|v_k)
\end{aligned}$$

The last term can be developed as follows:

$$\begin{aligned}
& \mathcal{A}_{[1-P,0]}(v_{P+1}) + \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 + \sum_{k=1}^P (v_{P+1} | v_k) \\
&= (1-P) v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 + P v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2 v_{P+1} \langle \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} \rangle \\
&= v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 + 2 v_{P+1} \langle \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} \rangle
\end{aligned}$$

The theorem is true if the above expression is equal to the following last gradient:

$$\begin{aligned}
& \mathcal{A}_{[\frac{2-P}{2},0]}(v_{P+1}) + \sum_{k=1}^P (v_{P+1}, v_k) \\
&= \frac{2-P}{2} v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 \\
&+ \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2 v_{P+1} v_1 \langle \xi_{v_{P+1}} | \xi_{v_1} \rangle \} + \frac{1}{2} \{ v_1^2 \|\xi_{v_1}\|_2^2 + 2 v_1 v_{P+1} \langle \xi_{v_1} | \xi_{v_{P+1}} \rangle \} \\
&+ \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2 v_{P+1} v_2 \langle \xi_{v_{P+1}} | \xi_{v_2} \rangle \} + \frac{1}{2} \{ v_2^2 \|\xi_{v_2}\|_2^2 + 2 v_2 v_{P+1} \langle \xi_{v_2} | \xi_{v_{P+1}} \rangle \} \\
&+ \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2 v_{P+1} v_3 \langle \xi_{v_{P+1}} | \xi_{v_3} \rangle \} + \frac{1}{2} \{ v_3^2 \|\xi_{v_3}\|_2^2 + 2 v_3 v_{P+1} \langle \xi_{v_3} | \xi_{v_{P+1}} \rangle \} \\
&\vdots \\
&\vdots \\
&+ \frac{1}{2} \{ v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + 2 v_{P+1} v_P \langle \xi_{v_{P+1}} | \xi_{v_P} \rangle \} + \frac{1}{2} \{ v_P^2 \|\xi_{v_P}\|_2^2 + 2 v_P v_{P+1} \langle \xi_{v_P} | \xi_{v_{P+1}} \rangle \} \\
&= \frac{2-P}{2} v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + \frac{P}{2} v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + v_{P+1} \langle \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} \rangle + \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 \\
&+ v_{P+1} \langle \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} \rangle \\
&= v_{P+1}^2 \|\xi_{v_{P+1}}\|_2^2 + \frac{1}{2} \sum_{k=1}^P v_k^2 \|\xi_{v_k}\|_2^2 + 2 v_{P+1} \langle \xi_{v_{P+1}} | \sum_{k=1}^P v_k \xi_{v_k} \rangle
\end{aligned}$$

We proved the theorem by induction.

To present the results, we will present the series for binary vectors=1 for the variables $(v_2, v_4, v_6, v_7, v_{11}, v_{14})$ ($\Omega_L^C = (2, 4, 6, 7, 11, 14)$) and $L = 6$:

$$\begin{aligned}
& \mathcal{P}_{\Omega_L}^t(V) \cdot H \cdot \mathcal{P}_{\Omega_L}(V) \\
&= \mathcal{A}_{[-\frac{3}{2}, 0]}(v_2) \\
&+ \mathcal{A}_{[-\frac{3}{2}, 0]}(v_4) + (v_4, v_2) \\
&+ \mathcal{A}_{[-\frac{3}{2}, 0]}(v_6) + (v_6, v_2) + (v_6, v_4) \\
&+ \mathcal{A}_{[-\frac{3}{2}, 0]}(v_7) + (v_7, v_2) + (v_7, v_4) + (v_7, v_6) \\
&+ \mathcal{A}_{[-\frac{3}{2}, 0]}(v_{11}) + (v_{11}, v_2) + (v_{11}, v_4) + (v_{11}, v_6) + (v_{11}, v_7) \\
&+ \mathcal{A}_{[-\frac{3}{2}, 0]}(v_{14}) + (v_{14}, v_2) + (v_{14}, v_4) + (v_{14}, v_6) + (v_{14}, v_7) + (v_{14}, v_{11})
\end{aligned}$$

8 Introduction to the topology $\tau(v_i, v_j)$ and the integral

In this section, we will define the primal topology $\tau^P(v_i, v_j)$ and the dual topology $\tau^D(v_i, v_j)$.

We call them primal and dual because these topologies will be used to propose an optimal solution to a primal and dual combinatorial optimization problem related to a quadratic form with binary variables.

From these topologies, we will define a new integral and establish the link between this integral, the quadratic form, and the primal and dual topologies $\tau^{(P,D)}(v_i, v_j)$.

Based on everything that has been done previously, we define the primal and dual topologies as follows:

Primal topology $\tau^P(v_i, v_j)$:

$$\tau^P(v_i, v_j) = \{\mathcal{A}_{[\frac{L-3}{2(L-1)}, \frac{1}{L-1}]}(v_i) + \{\mathcal{A}_{[\frac{L-3}{2(L-1)}, \frac{1}{L-1}]}(v_j)\} - (v_i, v_j)$$

Dual topology $\tau^D(v_i, v_j)$:

$$\tau^D(v_i, v_j) = \{\mathcal{A}_{[\frac{3-L}{2(L-1)}, 0]}(v_i) + \mathcal{A}_{[\frac{3-L}{2(L-1)}, 0]}(v_j)\} + (v_i, v_j)$$

We will make the connection between the new integrals, the quadratic form and the primal, dual topologies $\tau^{(P,D)}(v_i, v_j)$:

Quadratic form, primal integral and topology $\tau^P(v_i, v_j)$:

$$\Delta Q = V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) = \int_{i \in \Omega_L} \mu^P(v_i | Pa(v_i)) = \sum_{i \in \Omega_L} \sum_{k \in Pa(i)} \tau^P(v_i, v_k)$$

Where Ω_L is the sequence of indices of binary variables=0.

Quadratic form, dual integral and topology $\tau^D(v_i, v_j)$:

$$\mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) = \int_{i \in \Omega_L^C} \mu^D(v_i | Pa(v_i)) = \sum_{i \in \Omega_L^C} \sum_{k \in Pa(i)} \tau^D(v_i, v_k)$$

Where Ω_L^C is the sequence of indices of binary variables=1 and Ω_L is the complement sequence of indices of binary variables=0.

To present the results, we will present the series and the integral for binary vectors=0 for the variables $(v_2, v_4, v_6, v_7, v_{11}, v_{14})$ ($\Omega_L = (2, 4, 6, 7, 11, 14)$) and $L = 6$:

$$\begin{aligned}
& \Delta Q \\
&= V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) \\
&= \int_{i \in \Omega_L} \mu^P(v_i | Pa(v_i)) \\
&[2] = 0 \\
&[4] + \tau^P(v_4, v_2) \\
&[6] + \tau^P(v_6, v_2) + \tau^P(v_6, v_4) \\
&[7] + \tau^P(v_7, v_2) + \tau^P(v_7, v_4) + \tau^P(v_7, v_6) \\
&[11] + \tau^P(v_{11}, v_2) + \tau^P(v_{11}, v_4) + \tau^P(v_{11}, v_6) + \tau^P(v_{11}, v_7) \\
&[14] + \tau^P(v_{14}, v_2) + \tau^P(v_{14}, v_4) + \tau^P(v_{14}, v_6) + \tau^P(v_{14}, v_7) + \tau^P(v_{14}, v_{11})
\end{aligned}$$

To present the results, we will present the series and the integral for binary vectors=1 for the variables $(v_2, v_4, v_6, v_7, v_{11}, v_{14})$ ($\Omega_L^C = (2, 4, 6, 7, 11, 14)$) and $L = 6$:

$$\begin{aligned}
& \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) \\
&= \int_{i \in \Omega_L^C} \mu^D(v_i | Pa(v_i)) \\
&[2] = 0 \\
&[4] + \tau^D(v_4, v_2) \\
&[6] + \tau^D(v_6, v_2) + \tau^D(v_6, v_4) \\
&[7] + \tau^D(v_7, v_2) + \tau^D(v_7, v_4) + \tau^D(v_7, v_6) \\
&[11] + \tau^D(v_{11}, v_2) + \tau^D(v_{11}, v_4) + \tau^D(v_{11}, v_6) + \tau^D(v_{11}, v_7) \\
&[14] + \tau^D(v_{14}, v_2) + \tau^D(v_{14}, v_4) + \tau^D(v_{14}, v_6) + \tau^D(v_{14}, v_7) + \tau^D(v_{14}, v_{11})
\end{aligned}$$

9 Optimization of a quadratic form with binary variables

We will now state the combinatorial optimization problem for a quadratic form with binary variables.

We will establish the link between the notion of the integral and the topology associated with a quadratic form with binary variables.

From what follows, we will define the primal problem to be maximized and its dual problem to be minimized.

Primal optimization problem:

The combinatorial optimization problem can be stated in three different ways:

$$\max_{\Omega_L} \{V^t.H.V - \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V)\}$$

This problem can also be written as:

$$\max_{\Omega_L} \int_{i \in \Omega_L} \mu^P(v_i | Pa(v_i))$$

or

$$\max_{\Omega_L} \left\{ \sum_{i \in \Omega_L} \sum_{k \in Pa(i)} \tau^P(v_i, v_k) \right\}$$

Dual optimization problem:

The combinatorial optimization problem can be stated in three different ways:

$$\min_{\Omega_L} \{ \mathcal{P}_{\Omega_L}^t(V).H.\mathcal{P}_{\Omega_L}(V) \}$$

This problem can also be written as:

$$\min_{\Omega_L^c} \int_{i \in \Omega_L^c} \mu^D(v_i | Pa(v_i))$$

or

$$\min_{\Omega_L^c} \left\{ \sum_{i \in \Omega_L^c} \sum_{k \in Pa(i)} \tau^D(v_i, v_k) \right\}$$

9.1 Optimization algorithm when the values $\tau(v_i, v_j)$ are positives

To introduce the optimization algorithm, we will study the most interesting case, namely the dual problem.

For this, we will use an example with an initial sequence in ascending order:

$$\Omega_n^C = [1][2][3][4][5][6][7][8].$$

The series with bits equal to 1 gives us:

$$[1] = 0$$

$$[2] + \tau^D(v_2, v_1)$$

$$[3] + \tau^D(v_3, v_1) + \tau^D(v_3, v_2)$$

$$[4] + \tau^D(v_4, v_1) + \tau^D(v_4, v_2) + \tau^D(v_4, v_3)$$

$$[5] + \tau^D(v_5, v_1) + \tau^D(v_5, v_2) + \tau^D(v_5, v_3) + \tau^D(v_5, v_4)$$

$$[6] + \tau^D(v_6, v_1) + \tau^D(v_6, v_2) + \tau^D(v_6, v_3) + \tau^D(v_6, v_4) + \tau^D(v_6, v_5)$$

$$[7] + \tau^D(v_7, v_1) + \tau^D(v_7, v_2) + \tau^D(v_7, v_3) + \tau^D(v_7, v_4) + \tau^D(v_7, v_5) + \tau^D(v_7, v_6)$$

$$[8] + \tau^D(v_8, v_1) + \tau^D(v_8, v_2) + \tau^D(v_8, v_3) + \tau^D(v_8, v_4) + \tau^D(v_8, v_5) + \tau^D(v_8, v_6) + \tau^D(v_8, v_7)$$

The goal is to find the subset of indices $\Omega_L^C \subset \Omega_n^C$ of size L which we will fix to 5 minimizing the series sum. ($L = 5$)

We consider in this section that the values $\tau(v_i, v_j)$ are positives.

We will show in the next section how to adapt the optimization problem when the values of $\tau(v_i, v_j)$ are positives or negatives.

The last index

For positives values of $\tau(v_i, v_j)$, we will consider the index with the largest value (sum of the row values) at the end of the sequence. Let's assume this is index [3] and swap [3] with [8].

The sequence then becomes [1][2][8][4][5][6][7][3].

The row with index [3] has the highest value, and the pyramidal sum is invariant under permutation (constant). This gives us the smallest subsum of size $n - 1 = 7$ for the following sequence:

$$[1][2][8][4][5][6][7]$$

because the constant minus the largest value gives the smallest value for the preceding sequence.

Since the value of the row [3] was the largest and the values of $\tau(v_i, v_j)$ are positives, we know that the values of preceding rows will be less than that of row [3].

The second to last index

We will now consider the second to last strongest row.

Let's assume it's index [5] and therefore swap index [5] with index [7]

The sequence then becomes [1][2][8][4][7][6][5][3].

We know that index [3] is the row with the highest value and that index [5] is the second to last row with the highest value. Since the sum is invariant under permutation, the sequence [1][2][8][4][7][6] is the smallest subsum of size $n - 2 = 6$.

The low value of the sub-subsum arises from the fact that we removed the highest index at the end [3] and the second to last index highest [5] at the invariant constant by permutation.

Since the value of the row [5] was the largest and the values of $\tau(v_i, v_j)$ are positives, we know that the values of preceding rows will be less than that of row [5].

The third to last index

We will consider the third to last row to be the strongest. Let's assume it's index [1]. Let's swap the row [6] and the row [1].

The sequences becomes [6][2][8][4][7][1][5][3].

We now have the last strongest row with index [3], the second to last strongest row with index [5], and the third to last strongest row with index [1].

Since the sum is invariant under permutations on sequences, the sum minus the strongest gives us the next smallest subsum of size $n - 3 = 5$: [6][2][8][4][7].

The low value of the sub-subsum arises from the fact that we removed the highest index at the end [3], the second to last index highest [5] and the third to last index highest [6] at the invariant constant by permutation.

Since the value of the row [1] was the largest and the values of $\tau(v_i, v_j)$ are positives, we know that the values of preceding rows will be less than that of row [1].

The preceding sequence is therefore the optimal solution to the optimization problem for $L = 5$.

In other words, we must place a bit=1 in the indices [6],[2],[8],[4] and [7] of the binary vector to obtain the smallest value of the quadratic form.

Note that the values taken in order by the indices [3][5][1] go in descending order:

[3] the latest strongest index.

[5] the second latest strongest index.

[1] the third latest strongest index.

9.2 Optimization algorithm when the values $\tau(v_i, v_j)$ are positives or negatives

We considere stricly positives topologies $\tau(v_i, v_j)$.

We will show what to do when the values of $\tau(v_i, v_j)$ are postives or negatives.

To do this , let's startagain with the following series

$$[1] = 0$$

$$[2] + \tau^D(v_2, v_1)$$

$$[3] + \tau^D(v_3, v_1) + \tau^D(v_3, v_2)$$

$$[4] + \tau^D(v_4, v_1) + \tau^D(v_4, v_2) + \tau^D(v_4, v_3)$$

$$[5] + \tau^D(v_5, v_1) + \tau^D(v_5, v_2) + \tau^D(v_5, v_3) + \tau^D(v_5, v_4)$$

$$[6] + \tau^D(v_6, v_1) + \tau^D(v_6, v_2) + \tau^D(v_6, v_3) + \tau^D(v_6, v_4) + \tau^D(v_6, v_5)$$

$$[7] + \tau^D(v_7, v_1) + \tau^D(v_7, v_2) + \tau^D(v_7, v_3) + \tau^D(v_7, v_4) + \tau^D(v_7, v_5) + \tau^D(v_7, v_6)$$

$$[8] + \tau^D(v_8, v_1) + \tau^D(v_8, v_2) + \tau^D(v_8, v_3) + \tau^D(v_8, v_4) + \tau^D(v_8, v_5) + \tau^D(v_8, v_6) + \tau^D(v_8, v_7)$$

Let's consider the value $\tau_{min}^D(v_i, v_j)$ as the smallest negative value.

And let's translate it as follows:

$$\tilde{\tau}^D(v_i, v_j) = \tau^D(v_i, v_j) + |\tau_{min}^D(v_i, v_j)|$$

Therefore, in the series there will be one topology equal to 0 and the rest will be strictly positive.

The optimization problem:

$$\min_{\Omega_L} \{ \mathcal{P}_{\Omega_L}^t(V) . H . \mathcal{P}_{\Omega_L}(V) \}$$

becomes

$$\min_{\Omega_L} \{ \mathcal{P}_{\Omega_L}^t(V) . H . \mathcal{P}_{\Omega_L}(V) + |\tau_{min}^D(v_i, v_j)| . (\frac{L.(L-1)}{2} + 1) \}$$

Since the term:

$$|\tau_{min}^D(v_i, v_j)| . (\frac{L.(L-1)}{2} + 1)$$

is a constant when the value of L is fixed, the two previous optimization problems are equivalent. The constant $|\tau_{min}^D(v_i, v_j)| . (\frac{L.(L-1)}{2} + 1)$ comes from the fact that we summed $\frac{L.(L-1)}{2} + 1$ the value $|\tau_{min}^D(v_i, v_j)|$ when calculating the integral of size L .

In what follows, we will arbitrarily choose the value $\tau^D(v_1, v_2)$ as the value $\tau_{min}^D(v_i, v_j)$.

We will place this topology in the second position.

However, the value in question can be located anywhere in the series. It is then sufficient to place this topology in the second position.

By imposing the second topology $\tau^D(v_1, v_2)$ equal to zeros, we fix the first and second columns of the series.

The series then becomes:

$$\begin{aligned}
[1] &= |\tau_{min}^D(v_1, v_2)| \\
[2] &+ 0 \\
[3] &+ \tilde{\tau}^D(v_3, v_1) + \tilde{\tau}^D(v_3, v_2) \\
[4] &+ \tilde{\tau}^D(v_4, v_1) + \tilde{\tau}^D(v_4, v_2) + \tilde{\tau}^D(v_4, v_3) \\
[5] &+ \tilde{\tau}^D(v_5, v_1) + \tilde{\tau}^D(v_5, v_2) + \tilde{\tau}^D(v_5, v_3) + \tilde{\tau}^D(v_5, v_4) \\
[6] &+ \tilde{\tau}^D(v_6, v_1) + \tilde{\tau}^D(v_6, v_2) + \tilde{\tau}^D(v_6, v_3) + \tilde{\tau}^D(v_6, v_4) + \tilde{\tau}^D(v_6, v_5) \\
[7] &+ \tilde{\tau}^D(v_7, v_1) + \tilde{\tau}^D(v_7, v_2) + \tilde{\tau}^D(v_7, v_3) + \tilde{\tau}^D(v_7, v_4) + \tilde{\tau}^D(v_7, v_5) + \tilde{\tau}^D(v_7, v_6) \\
[8] &+ \tilde{\tau}^D(v_8, v_1) + \tilde{\tau}^D(v_8, v_2) + \tilde{\tau}^D(v_8, v_3) + \tilde{\tau}^D(v_8, v_4) + \tilde{\tau}^D(v_8, v_5) + \tilde{\tau}^D(v_8, v_6) + \tilde{\tau}^D(v_8, v_7)
\end{aligned}$$

The optimization problem now becomes the same as for the positive values of $\tau^D(v_i v_j)$ in the previous section.

The only difference is that the first two indices are fixed from the start.

These first two indices will therefore systematically appear in all other rows of the series during the successive selection of the strongest optimal rows.

Note that the choice of $\tau^D(v_1 v_2)$ how the zero value was arbitrary for presenting the algorithm.

The zero value can be something other than $\tau^D(v_1 v_2)$.

The algorithm then positions the zero value in the second position in the series and imposes these first two indices on the other rows of the series.

10 Numerical simulations with the R software

In this section, we are going to use the programs in the appendix. We consider the initial quadratic form: $V^t.H.V$ with

$$tV = (-3, 69, 3, -35, 68, -29)$$

$$H = \begin{pmatrix} 14.88 & -94.90 & -10.16 & 70.41 & -19.17 & 51.93 \\ -94.90 & -43.91 & -85.99 & 64.09 & -17.84 & -1.82 \\ -10.16 & -85.99 & 71.84 & 70.77 & -21.77 & 78.17 \\ 70.41 & 64.09 & 70.77 & -14.99 & 4.75 & -22.98 \\ -19.17 & -17.84 & -21.77 & 4.75 & 8.62 & -94.75 \\ 51.93 & -1.82 & 78.17 & -22.98 & -94.75 & 1.54 \end{pmatrix}$$

We compute the adjacency MatTopo from the function "MatrixTopoDual()" for $L = 4$. For the function "MatrixTopoDual()" to work, you also need to download the functions "CreateXiVector()", "Prod()" and "A()":

```
> MatTopo=Re(MatrixTopoDual(4,H,tV))
```

$$MatTopo = \begin{pmatrix} 0.000 & -30351.93 & 443.040 & 8709.823 & 21152.293 & 9512.173 \\ -30351.930 & 0.00 & -105069.510 & -385360.787 & -223809.437 & -61969.817 \\ 443.040 & -105069.51 & 0.000 & -20767.097 & 4619.653 & -12954.347 \\ 8709.823 & -385360.79 & -20767.097 & 0.000 & -15444.623 & -52338.603 \\ 21152.293 & -223809.44 & 4619.653 & -15444.623 & 0.000 & 387412.007 \\ 9512.173 & -61969.82 & -12954.347 & -52338.603 & 387412.007 & 0.000 \end{pmatrix}$$

We suppose now that we have to put the bits=1 in the positions 1,3,5,6 and we compute the submatrix SubMatTopo:

```
> SubMatTopo=MatTopo[c(1,3,5,6),c(1,3,5,6)]
```

$$SubMatTopo = \begin{pmatrix} 0.000 & 443.040 & 21152.293 & 9512.173 \\ 443.040 & 0.000 & 4619.653 & -12954.347 \\ 21152.293 & 4619.653 & 0.000 & 387412.007 \\ 9512.173 & -12954.347 & 387412.007 & 0.000 \end{pmatrix}$$

```
> Integral=Re(0.5*sum(SubMatTopo))
```

```
> Integral
```

```
[1] 410184.8
```

Now we will verify if the integral is equal to the projection of the quadratic form:

$$\mathcal{P}_{\Omega_4^C=\{1,3,5,6\}}^t(V).H.\mathcal{P}_{\Omega_4^C=\{1,3,5,6\}}(V)$$

Here $\mathcal{P}_{\Omega_4^C=\{1,3,5,6\}}(V)$ adds bits=1 to the indices $\{1, 3, 5, 6\}$.

We are going to check if we have the same value.

> tR=rep(0,6)

> tR[1]=tV[1]

> tR[3]=tV[3]

> tR[5]=tV[5]

> tR[6]=tV[6]

> R=t(t(tR))

> tR%%*%M%%*%R

[1,] 410184.8

We obtain the same values. The integral is correct:

$$\mathcal{P}_{\Omega_4^C=\{1,3,5,6\}}^t(V).H.\mathcal{P}_{\Omega_4^C=\{1,3,5,6\}}(V) = \int_{i \in \Omega_4^C=\{1,3,5,6\}} \mu^D(v_i | Pa(v_i)) = \sum_{i \in \Omega_4^C=\{1,3,5,6\}} \sum_{k \in Pa(i)} \tau^D(v_i, v_k)$$

11 Conclusion

In this report, I introduced a new integration method for binary variables.

Using algebraic tools, I outlined the steps required to arrive at this new integral. This integration concept enable the proposal of a polynomial-time P approach to solving a combinatorial optimization problem.

The optmization problem aimed to minimize a quadratic form involving binary variables.

By utilizing the series expansion of the integral, we presented an algorithm capable of finding an optimal solution in polynomial time for an NP-Hard problem.

A Program to create ξ vectors with R software

Mat is the matrix for the quadratic form and tR is the vector for the quadratic

```
CreateXiVector<-function(Mat,tR)
{

  NbreVect=ncol(Mat)

  R=t(t(tR))

  SqrtLambda<-sqrt(as.complex(eigen(Mat)[[1]]))

  tP=t(eigen(Mat)[[2]])

  MatVect=tP

  for(i in 1:NbreVect)
  {
    MatVect[i,]<-MatVect[i,]*tR*SqrtLambda[i]

  }

  colnames(MatVect)<-paste("Xi",1:length(tR),sep="")

  Sum=0

  for(j in 1:NbreVect)
  {
    Sum=Sum+MatVect[,j]

  }
  Sum=t(t(Sum))

  colnames(Sum)="XiTotal"

  List=list("Vector_Xi"=MatVect,"Vector_Xi_Total"=Sum)

  return(List)
}
```

B Operator $\mathcal{A}_{[\theta_1, \theta_2]}$ and operator $(,)$ with R software

A is the operator $\mathcal{A}_{[\theta_1, \theta_2]}$ and Prod is the operator $(,)$

```
Prod<-function (NumCompo2, NumCompo1, M, tV )
{
  Xi=CreateXiVector (M, tV ) [[ 1 ]]
  XiT=CreateXiVector (M, tV ) [[ 2 ]]
  Prod=0.5*(sum ( Xi [ , NumCompo2]^2)+2*sum ( Xi [ , NumCompo2]*Xi [ , NumCompo1 ])
+sum ( Xi [ , NumCompo1]^2)+2*sum ( Xi [ , NumCompo1]*Xi [ , NumCompo2 ]))

  return ( Prod )

}

A<-function (NumCompo, Theta1 , Theta2 ,M, tV )
{
  Xi=CreateXiVector (M, tV ) [[ 1 ]]
  XiT=CreateXiVector (M, tV ) [[ 2 ]]

  Asol=Theta1*sum ( Xi [ , NumCompo]^2)+2*Theta2*sum ( Xi [ , NumCompo]*XiT)

  return ( Asol )

}
```

C Dual topological adjacency matrix program with R software

L is the number of the bits=1, M is the matrix of the quadratic form and tR is the vector of the quadratic form.

```
MatrixTopoDual<-function(L,M,tR)
{
N=length(tR)

Mat=diag(N)

for(i in 1:N)
{
for(j in 1:N)
{
Mat[i,j]
=A(i,(3-L)/(2*(L-1)),0,M,tR)+A(j,(3-L)/(2*(L-1)),0,M,tR)+Prod(i,j,M,tR)
}
}

diag(Mat)=0

return(Mat)

}
```

Reference

[1]*Cours d'algèbre linéaire. Université libre de Bruxelles. Author: Dominique Buset.*