MATRIX THEORETIC AND COMBINATORIAL STRUCTURES IN ADDITION CHAINS

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ABSTRACT. An addition chain of length h that leads to a number n is a sequence of positive integers $s_0 = 1, s_1 = 2, \ldots, s_h = n$ such that $s_i = s_j + s_k$ $(i > j \geq k)$ for each $1 \leq i \leq h$. We introduce a matrix-theoretic framework for studying the properties of arbitrary addition chains that lead to a target integer $n \geq 2$ by associating each chain and the sequence of dominant and lower weight summands that calculate each term in the chain an adjacency matrix that encodes the internal geometry. In this linear-algebraic and matrix theoretic framework, we investigate how rank, rank profiles, singular values, eigenvalues, matrix norms, predecessor depths, and track-interaction terms encode the combinatorial and arithmetic structure of addition chains.

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1. Introduction

Addition chains occupy a classical position at the interface of number theory, combinatorics, and computational arithmetic. An addition chain for a positive integer n is a finite sequence

$$1 = s_0 < s_1 < \cdots < s_h = n$$

such that, for every $i \geq 1$, the entry s_i is the sum of two earlier entries, say

$$s_i = s_{\sigma(i)} + s_{\tau(i)}, \quad i > \sigma(i) \geq \tau(i) \geq 0.$$

The least possible value of h is the classical length of the addition chain, usually denoted by $\ell(n)$. The subject originated in the study of efficient exponentiation and has developed into a substantial area in its own right, with connections to algorithmic number theory, computational complexity, and the optimization of arithmetic programs; see, for example, [2, 13, 9]. The basic tension is already visible in the elementary inequality $\ell(n) \geq \lceil \log_2 n \rceil$: repeated doubling gives a universal logarithmic scale, whereas the freedom to reuse nonconsecutive previously computed values produces a much richer combinatorial optimization problem. The Brauer classical analysis established asymptotically sharp logarithmic upper bounds, while Erdős and later authors developed the fine arithmetic behavior of $\ell(n)$ and the associated families of chains; see [2, 3, 5, 10].

The customary formulation of an addition chain emphasizes the scalar sequence (s_i) and, in optimization questions, its length. However, for structural purposes, the sequence alone suppresses a substantial amount of information. At the i^{th} stage (step) there is not merely a new integer s_i; there is a specific ordered choice of predecessors $(\sigma(i), \tau(i))$ whose sum produces that integer. Distinct decompositions can therefore lead to the same numerical chain, and two chains of the same length can exhibit markedly different patterns of predecessor reuse. This paper takes those predecessor relations as first-class and prioritized mathematical data. Accordingly, throughout the paper, a chain is considered together with a fixed decomposition

$$s_i = s_{\sigma(i)} + s_{\tau(i)}, \qquad i = 1, \ldots, h,$$

and the resulting σ and τ-tracks

$$\left(s_{\sigma(1)}, \ldots, s_{\sigma(h)}\right), \quad \left(s_{\tau(1)}, \ldots, s_{\tau(h)}\right).$$

This refinement is essential: the matrix objects introduced in the sequel encode the chosen decomposition and are consequently invariants of the decorated pair consisting of the chain and its predecessor data, rather than of the underlying numerical sequence considered in isolation.

The central idea is to regard an addition chain as a finite acyclic directed dependency structure. The vertices are the indices $0, 1, \ldots, h$, and the i^{th} vertex points backward to the indices of the summands used to create s_i. In graph-theoretic language, the σ and τ-relations define two directed predecessor systems. Their adjacency matrices are therefore naturally represented by matrices A_σ and A_τ having one nonzero entry in each noninitial row. Since every predecessor index is strictly smaller than the row index, these matrices are strictly lower triangular. The aggregated matrix

$$A = A_\sigma + A_\tau$$

retains the multiplicity of a repeated predecessor: an addition $s_i = 2s_j$ contributes the entry 2 in position (i, j), whereas two distinct predecessors contribute two unit entries in their respective columns. Thus, the matrix A is not merely a drawing device; it records the local branching and repeated-use structure of the chain in a form amenable to the full apparatus of linear algebra. This viewpoint is consonant with the general role of adjacency matrices in algebraic graph theory, where matrix multiplication translates local incidence information into higher-order connectivity information [14].

The lower-triangular form has an immediate conceptual consequence. The dependency graph of an addition chain is acyclic and the nilpotency of the matrices A_σ, A_τ, and A is the linear-algebraic reflection of this acyclicity. More specifically, powers of the track matrices record iterated predecessor maps. For $v \geq 1$, the entry of A_σ^v at (i, j) is nonzero precisely when j is reached from i after v successive applications of the σ-predecessor map; the analogous statement holds for τ. In the present setting, each row contains at most one predecessor of a given track, so the entries remain particularly transparent: the v^{th} power encodes the v-step ancestral relation. This provides a useful bridge between the combinatorial depth of a chain and the algebra of matrix powers. In particular, the vanishing of sufficiently high powers is not an accidental matrix identity but records the finite depth of the underlying dependency structure.

The second layer of the framework retains the arithmetic magnitude by weighting each predecessor edge by the value of the summand that it represents. Let

$$D = \operatorname{diag}(s_0, s_1, \ldots, s_h).$$

The weighted track matrices are then

$$W_\sigma = A_\sigma D, \quad W_\tau = A_\tau D, \quad W = W_\sigma + W_\tau = AD.$$

This simple factorization is one of the organizing principles of the paper. The unweighted matrices describe the incidence geometry of the dependency system, while the diagonal factor D reintroduces the arithmetic scale of the summands. Consequently, matrix products in the weighted model propagate products of predecessor values along ancestral paths. For example, a nonzero entry of W_σ^v is the product of successive σ-track values encountered along a v-step ancestral trajectory. The weighted formalism thus separates two kinds of information that are intertwined in the original scalar recurrence: combinatorial connectivity and arithmetic mass.

This separation makes several standard matrix invariants unexpectedly informative for addition chains. For the unweighted track matrices, rank measures the number of distinct predecessor indices that are actually used by a track, because the nonzero columns correspond exactly to the image of the predecessor map. The same phenomenon persists for higher powers: the rank of A_σ^v (and, analogously, A_τ^v) records the cardinality of the set of vertices reached after v iterations of the corresponding predecessor map. The resulting sequences of ranks, introduced here as the *ancestral-rank profiles*, quantify how rapidly distinct ancestral branches coalesce as one moves backward through the chain. In this sense, rank is not being used merely as a generic linear-algebraic invariant; it becomes a compact statistic for the combinatorial reuse of previously computed quantities.

The weighted matrices support a complementary metric and spectral interpretation. Because every row of a weighted track matrix contains at most one nonzero entry, the Gram matrices $W_\sigma^T W_\sigma$ and $W_\tau^T W_\tau$ are diagonal. Their diagonal entries are determined by two quantities: the arithmetic size s_j^2 of a predecessor and the multiplicity with which that predecessor is selected by the relevant track. It follows that the singular values of a weighted track matrix naturally separate into an arithmetic factor and a combinatorial multiplicity factor. This is precisely the type of interaction for which singular value analysis is useful: unlike eigenvalues of the nilpotent matrices themselves, which are all zero, the singular values of the weighted track matrices retain quantitative information about how heavily different chain elements participate in the predecessor structure. The use of singular values and Gram matrices here is entirely within the standard framework of matrix analysis; see [15].

An important special class of addition chains arises when the larger summand at every step is the immediately preceding term, that is, when $\sigma(i) = i - 1$ for all i. These are the classical Brauer chains. In the matrix language developed in this paper, such a chain is characterized by the fact that the σ-track uses every predecessor index exactly once. Equivalently, the corresponding track matrix has the full possible rank h. This observation motivates the *σ-rank defect*

$$\Delta_\sigma(\mathcal{C}_n) = h - \operatorname{rank}(A_\sigma),$$

which measures the extent to which the distinguished predecessor track fails to traverse the chain without repetition of target columns. Thus, the rank defect provides a linear-algebraic formulation of a classical combinatorial dichotomy: zero defect corresponds to the immediate-predecessor pattern of a Brauer chain, whereas positive defect records predecessor reuse within the σ-track. The point is not to replace the usual definition of a Brauer chain but to exhibit a matrix criterion that can be manipulated together with the other invariants introduced in the paper.

The weighted setting also permits invariants that measure interaction between the two summand tracks. The Frobenius inner product

$$\langle W_\sigma, W_\tau \rangle_F = \sum_{i=1}^{h} s_{\sigma(i)} s_{\tau(i)}$$

quantifies, step by step, the simultaneous arithmetic contribution of the two predecessors. It is a genuine coupling term: when the two tracks are regarded separately, their squared Frobenius norms measure the individual weighted masses, while

$$\|W\|_F^2 = \|W_\sigma\|_F^2 + \|W_\tau\|_F^2 + 2\langle W_\sigma, W_\tau \rangle_F$$

recovers the total weighted mass together with the cross-interaction. In this sense, the decoupling invariant introduced later in the paper is a quantitative measure of how strongly the two predecessor systems overlap on the arithmetic scale. At the same time, the noncommutativity of the two track matrices is captured by the commutators

$$[A_\sigma, A_\tau] = A_\sigma A_\tau - A_\tau A_\sigma, \quad [W_\sigma, W_\tau] = W_\sigma W_\tau - W_\tau W_\sigma.$$

A vanishing commutator expresses a strong compatibility between the two ancestral transition mechanisms, whereas a nonzero commutator records an order-sensitive interaction between them. This opens a natural algebraic direction for distinguishing and classifying addition chains beyond their lengths.

The framework proposed here should be viewed as complementary to the extensive literature on optimal and near-optimal addition chains. Classical work has concentrated predominantly on the minimum length problem, asymptotic lower and upper bounds, structural properties of optimal chains, and computational methods for determining $\ell(n)$ or enumerating optimal chains; see [2, 3, 5, 9, 10]. The present approach asks a different question. Instead of collapsing a chain to the single scalar h, it retains the full predecessor architecture and subjects that architecture to matrix-theoretic analysis. The resulting invariants are therefore intended to describe structural features that length alone cannot distinguish: concentration or dispersion of predecessor reuse, ancestral coalescence, arithmetic weighting of dependencies, interaction of the two summand tracks, and algebraic compatibility between their transition operators.

1.1. **Organization of the paper.** The paper is organized accordingly. Section 2 introduces the unweighted matrices A_σ, A_τ, and A, proves their decomposition and triangularity properties, and establishes the basic entry-counting identities. Section 3 passes to the weighted matrices W_σ, W_τ, and W, emphasizing the factorization through the diagonal matrix D and deriving the corresponding triangular and nilpotent structure. Section 4 gives concrete examples, including a non-Brauer chain and a Brauer chain, in order to display explicitly how different predecessor architectures appear in matrix form. Section 5 develops the exponential or iterated-ancestry viewpoint through powers of the track matrices and introduces the associated structural interpretation of higher powers. Section 6 studies matrix invariants, including rank, ancestral-rank profiles, eigenvalues, and singular values, thus linking predecessor multiplicities to standard linear-algebraic quantities. Section 7 introduces the σ and τ-rank defects and proves the matrix criterion characterizing Brauer chains through the vanishing σ-rank defect. Finally, Section 8 studies commutators and the Frobenius decoupling invariant, and formulates local and global problems concerning commutativity of the two weighted track operators. The overall progression is from incidence structure to weighted structure, to iterated ancestry, to linear-algebraic invariants, and finally to interaction phenomena between the two tracks.

Taken together, these constructions place addition chains in a matrix-theoretic setting without discarding their arithmetic origin. The resulting matrices are sparse, nilpotent, and highly structured, yet they retain enough information to distinguish predecessor patterns that are invisible at the level of the chain length. This suggests a broader program in which combinatorial optimization objects arising from arithmetic computation can be studied through structured linear-algebraic representations, with rank, singular values, matrix norms, commutators, and higher powers serving as structural probes. The present paper develops the first layer of such a framework for ordinary addition chains and records a collection of exact identities and classification criteria that may provide a basis for further investigation of the relationship between arithmetic generation processes and finite-dimensional matrix geometry

2. Unweighted matrices associated with an addition chain

We let $n \geq 2$ be an integer and

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. In this section, we introduce three matrices associated with a fixed addition chain and associated σ and τ track.

Definition 2.1. (The sigma track adjacency matrix) We denote the σ track adjacency matrix $A_\sigma := (a^\sigma_{ij})_{0 \leq i,j \leq h}$ associated with the fixed σ track of the fixed chain by

$$(A_\sigma)_{ij} := a^\sigma_{ij} = \begin{cases} 1 & \text{if} \quad \sigma(i) = j, \quad 1 \leq i \leq h \\ 0 & \text{otherwise} \end{cases}$$

with $a^\sigma_{00} := 0$.

Definition 2.2. (The tau track adjacency matrix) We denote the τ track adjacency matrix $A_\tau := (a^\tau_{ij})_{0 \leq i,j \leq h}$ associated with the fixed τ track of the fixed addition chain by

$$(A_\tau)_{ij} := a^\tau_{ij} = \begin{cases} 1 & \text{if} \quad \tau(i) = j, \quad 1 \leq i \leq h \\ 0 & \text{otherwise} \end{cases}$$

with $a^\tau_{00} := 0$.

Definition 2.3. (The chain unweighted adjacency matrix) We denote the unweighted chain adjacency matrix $A := (a_{ij})_{0 \leq i,j \leq h}$ of the fixed addition chain by

$$(A)_{ij} := a_{ij} = \begin{cases} 2 & \text{if} \quad \sigma(i) = \tau(i) = j, \quad 1 \leq i \leq h \\ 1 & \text{if} \quad \sigma(i) = j, \ \tau(i) \neq j, \quad 1 \leq i \leq h \\ 1 & \text{if} \quad \tau(i) = j, \quad \sigma(i) \neq j, \quad 1 \leq i \leq h \\ 0 & \text{if} \quad \sigma(i), \tau(i) \neq j \end{cases}$$

with $a_{00} := 0$.

Proposition 2.4. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. Denote by A, A_σ, A_τ an adjacency matrix, σ and τ-track unweighted adjacency matrix associated with the chain, respectively. We have

$$A = A_\sigma + A_\tau.$$

Proposition 2.5 (Triangularity). *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. Denote by A, A_σ, A_τ an adjacency matrix, σ and τ-track unweighted adjacency matrix associated with the chain, respectively. The matrices A, A_σ, A_τ are strictly lower triangular. Furthermore, they are nilpotent of order $h+1$, which means

$$A^{h+1} = A_\sigma^{h+1} = A_\tau^{h+1} = 0.$$

Proof. By definition $(A_\sigma)_{ij} = 1$ if and only if $\sigma(i) = j$. Similarly, $(A_\tau)_{ij} = 1$ if and only if $\tau(i) = j$. The assertion that A_σ, A_τ are both strictly lower triangular follows from the fact that $\sigma(i), \tau(i) < i$, which implies that $(A_\sigma)_{ij} = 0 = (A_\tau)_{ij}$ for all $j \geq i$. Consequently, the matrix $A := A_\sigma + A_\tau$ is also strictly lower triangular. The nilpotency of the matrices A, A_σ, A_τ is a standard matrix-theoretic fact for all strictly lower triangular matrices. □

Proposition 2.6 (The total entry-length relation). *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. Denote by A, A_σ, A_τ an adjacency matrix, σ and τ-track unweighted adjacency matrix associated with the chain, respectively. We have

$$\sum_{i,j} a_{ij}^{\sigma} = \sum_{i,j} a_{ij}^{\tau} = h = \frac{1}{2} \sum_{i,j} a_{ij}.$$

Proof. For a fixed row (except for the first row), each row of matrices A_σ, A_τ has exactly one 1. This implies $\sum_j a_{ij}^{\sigma} = 1 = \sum_j a_{ij}^{\tau}$. Because the first row of the matrices A_σ, A_τ have all zero entries, we deduce $\sum_{i,j} a_{ij}^{\sigma} = \sum_{i,j} a_{ij}^{\tau} = h$. Similarly, it can be observed that each row of the matrix A (except for the first row) has a row sum of exactly 2. This implies $\sum_{i,j} a_{ij} = 2h$. □

3. Weighted matrices associated with an addition chain

We let $n \geq 2$ be an integer and

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. In this section, we introduce three matrices associated with a fixed addition chain and associated σ and τ track. These matrices are the corresponding weighted versions of those introduced in the previous section.

Definition 3.1 (The sigma track weighted adjacency matrix)**.** We denote the σ track weighted adjacency matrix $W_\sigma := (w_{ij}^\sigma)_{0\leq i,j\leq h}$ associated with the fixed σ track of the fixed chain by

$$(W_\sigma)_{ij} := w_{ij}^\sigma = \begin{cases} s_{\sigma(i)} & \text{if} \quad \sigma(i) = j, \quad 1 \leq i \leq h \\ 0 & \text{otherwise} \end{cases}$$

with $w_{00}^\sigma := 0$.

Definition 3.2. (The tau track weighted adjacency matrix) We denote the τ track weighted adjacency matrix $W_\tau := (w_{ij}^\tau)_{0\leq i,j\leq h}$ associated with the fixed τ track of the fixed addition chain by

$$(W_\tau)_{ij} := w_{ij}^\tau = \begin{cases} s_{\tau(i)} & \text{if} \quad \tau(i) = j, \quad 1 \leq i \leq h \\ 0 & \text{otherwise} \end{cases}$$

with $w_{00}^\tau := 0$.

Definition 3.3. (The weighted adjacency matrix of a chain) We denote the unweighted chain adjacency matrix $W := (a_{ij})_{0\leq i,j\leq h}$ of the fixed addition chain by

$$(W)_{ij} := w_{ij} = \begin{cases} 2s_j & \text{if} \quad \sigma(i) = \tau(i) = j, \quad 1 \leq i \leq h \\ s_j & \text{if} \quad \sigma(i) = j, \ \tau(i) \neq j, \quad 1 \leq i \leq h \\ s_j & \text{if} \quad \tau(i) = j, \quad \sigma(i) \neq j, \quad 1 \leq i \leq h \\ 0 & \text{if} \quad \sigma(i), \tau(i) \neq j \end{cases}$$

with $w_{00} := 0$.

Proposition 3.4. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^h$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^h$ a τ track of the chain. Denote by W, W_σ, W_τ a weighted adjacency matrix, σ and τ-track weighted adjacency matrix associated with the chain, respectively. We have

$$W = W_\sigma + W_\tau.$$

Proposition 3.5 (Weighted triangularity)**.** *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^h$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^h$ a τ track of the chain. Denote by W, W_σ, W_τ a weighted adjacency matrix, σ and τ-track weighted adjacency matrix associated with the chain, respectively. The matrices W, W_σ, W_τ are strictly lower triangular. Furthermore, they are nilpotent of order $h + 1$, which means

$$W^{h+1} = W_\sigma^{h+1} = W_\tau^{h+1} = 0.$$

Proof. By definition $(W_\sigma)_{ij} = s_{\sigma(i)}$ if and only if $\sigma(i) = j$. Similarly, $(W_\tau)_{ij} = s_{\tau(i)}$ if and only if $\tau(i) = j$. The assertion that W_σ, W_τ are both strictly lower triangular follows from the fact that $\sigma(i), \tau(i) < i$, which implies that $(W_\sigma)_{ij} = 0 = (W_\tau)_{ij}$ for all $j \geq i$. Consequently, the matrix $W := W_\sigma + W_\tau$ is also strictly lower triangular.

The nilpotency of the matrices W, W_σ, W_τ is a standard matrix-theoretic fact for all strictly lower triangular matrices. $\square$

Proposition 3.6. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^h$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^h$ a τ track of the chain. We have

$$W_\sigma = A_\sigma D \quad \textit{and} \quad W_\tau = A_\tau D$$

where $D = \operatorname{diag}(s_0, s_1, \ldots, s_h)$ is a diagonal matrix defined by

$$D_{ij} = \begin{cases} s_j & \textit{if} \quad i = j \\ 0 & \textit{if} \quad i \neq j. \end{cases}$$

Furthermore, we have

$$W = (A_\sigma + A_\tau)D = AD.$$

Proof. We let i, j be fixed indices with $i, j \in \{0, \ldots, h\}$. We can write

$$(A_\sigma D)_{ij} = \sum_{k=0}^{h} a_{ik}^{\sigma} d_{kj}$$

where $(D)_{kj} = d_{kj}$ is the entry on the row k column j of the diagonal matrix D. Because the addition chain $\mathcal{C}_n$ has been fixed, there exists a unique $k' \in \{0, \ldots, h\}$ such that $\sigma(i) = k'$. The matrix D is such that $d_{kj} = s_j$ whenever $k = j$ and zero otherwise. This implies that

$$(A_\sigma D)_{ij} = \sum_{k=0}^{h} a_{ij}^{\sigma} d_{kj} = s_{\sigma(i)}$$

whenever $\sigma(i) = k' = j$ and zero otherwise. Thus, $(A_\sigma D)_{ij} = (W_\sigma)_{ij}$. A similar argument also yields

$$(A_\tau D)_{ij} := \begin{cases} s_{\tau(i)} & \text{if} \quad \tau(i) = j \\ 0 & \text{otherwise} \end{cases} = (W_\tau)_{ij}.$$

The relation $W = AD$ is easily obtained using $W = W_\sigma + W_\tau$ and $A = A_\sigma + A_\tau$. $\square$ [1]

4. Some examples

4.1. A non-Brauer optimal chain. Consider the addition chain $C_{13} : 1, 2, 4, 5, 8, 13$ with

$$2 = 1 + 1, \quad 4 = 2 + 2, \quad 5 = 4 + 1, \quad 8 = 4 + 4, \quad 13 = 8 + 5.$$

Thus

$$(\sigma(1), \ldots, \sigma(5)) = (0, 1, 2, 2, 4),$$

and

$$(\tau(1), \ldots, \tau(5)) = (0, 1, 0, 2, 3).$$

[1] The introduced and developed matrix-theoretic framework is amenable to linear-algebraic and graph theoretic approaches.

The unweighted adjacency matrix for the σ and τ tracks is

$$A_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$A_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Consequently

$$A = A_\sigma + A_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The weighted matrices are

$$W_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \end{pmatrix}, \quad W_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \end{pmatrix}.$$

Consequently,

$$W = W_\sigma + W_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 8 & 0 \end{pmatrix}.$$

4.2. **A Brauer chain as a benchmark.** For the Brauer chain

$$1, 2, 3, 5, 8, 13,$$

we have $\sigma(i) = i - 1$ for $1 \leq i \leq 5$, so

$$A_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We obtain the τ track adjacency matrix

$$A_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

so we obtain

$$A = A_\sigma + A_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

We construct the corresponding weighted matrices as follows:

$$W_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \end{pmatrix}$$

and

$$W_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \end{pmatrix}$$

so we have

$$W = W_\sigma + W_\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 8 & 0 \end{pmatrix}.$$

5. The exponential profile of matrices associated with addition chains

We let $n \geq 2$ be an integer and

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. In this section, we study the structure of powers of matrices associated with a fixed addition chain and associated σ and τ tracks. These matrices are the corresponding exponential versions of those introduced in the previous section.

Proposition 5.1 (The exponential unweighted adjacency matrices)**.** *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. Let $v \geq 2$ be a fixed integer. The matrices A_σ^v, A_τ^v have entries that satisfy

$$(A_\sigma^v)_{ij} := \begin{cases} 1 & \textit{if} \quad \sigma^v(i) = j \\ 0 & \textit{otherwise} \end{cases}$$

and

$$(A_\tau^v)_{ij} = \begin{cases} 1 & \textit{if} \quad \tau^v(i) = j \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, the matrices A_σ^v, A_τ^v are strictly lower triangular.

Proof. Here, we argue by induction. We let $i, j \in \{0, \ldots, h\}$ be fixed indices and write

$$(A_\sigma^2)_{ij} = \sum_{k=0}^{h} a_{ik}^{\sigma} a_{kj}^{\sigma}.$$

Because the addition chain $\mathcal{C}_n$ has been fixed, there exists a unique $k' \in \{0, \ldots, h\}$ such that $\sigma(i) = k'$. If $\sigma(k') = j$ (i.e, $\sigma^2(i) = j$) then $(A_\sigma^2)_{ij} = 1$ and zero otherwise (i.e, whenever $\sigma(k') \neq j$). The observation is therefore true when the exponent $r = 2$. We assume that for some $r \geq 3$, we have

$$(A_\sigma^{r-1})_{ij} := a_{ij}^{\sigma^{r-1}} = \begin{cases} 1 & \text{if} \quad \sigma^{r-1}(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

Now, we write

$$(A_\sigma^r)_{ij} = (A_\sigma^{r-1} A_\sigma)_{ij} = \sum_{k=0}^{h} a_{ik}^{\sigma^{r-1}} a_{kj}^{\sigma}.$$

Because the addition chain $\mathcal{C}_n$ has been fixed, there exists a unique $k' \in \{0, \ldots, h\}$ such that $\sigma^{r-1}(i) = k'$. Thus, $(A_\sigma^r)_{ij} = 1$ whenever $\sigma(k') = j$ (i.e, $\sigma^r(i) = j$) and zero otherwise. This completes the inductive argument. We can certainly adapt a similar argument to show that

$$(A_\tau^v)_{ij} = \begin{cases} 1 & \text{if} \quad \tau^v(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

The assertion that the matrices A_σ^v, A_τ^v are strictly lower triangular follows from the observation that

$$\sigma^r(i) = j \leq i - r < i$$

for $r \geq 2$. $\square$

Proposition 5.2 (The exponential weighted adjacency matrices)**.** *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The *sequence* $\{s_{\sigma(i)}\}_{i=1}^{h}$ *is a* σ *track of the chain and* $\{s_{\tau(i)}\}_{i=1}^{h}$ *a* τ *track of the chain. Let* $v \geq 2$ *be a fixed integer. The matrices* $W_{\sigma}^{v}, W_{\tau}^{v}$ *have entries that satisfy*

$$(W_{\sigma}^{v})_{ij} := \begin{cases} \prod_{r=1}^{v} s_{\sigma^{r}(i)} & \textit{if} \quad \sigma^{v}(i) = j \\ 0 \quad \textit{otherwise} \end{cases}$$

and

$$(W_{\tau}^{v})_{ij} = \begin{cases} \prod_{r=1}^{v} s_{\tau^{r}(i)} & \textit{if} \quad \tau^{v}(i) = j \\ 0 \quad \textit{otherwise.} \end{cases}$$

Furthermore, the matrices $W_{\sigma}^{v}, W_{\tau}^{v}$ *are strictly lower triangular.*

Proof. Here, we argue by induction. We let $i, j \in \{0, \ldots, h\}$ be fixed indices and write

$$(W_{\sigma}^{2})_{ij} = \sum_{k=0}^{h} w_{ik}^{\sigma} w_{kj}^{\sigma}.$$

Because the addition chain $\mathcal{C}_n$ has been fixed, there exists a unique $k' \in \{0, \ldots, h\}$ such that $\sigma(i) = k'$. If $\sigma(k') = j$ (i.e, $\sigma^{2}(i) = j$) then $(W_{\sigma}^{2})_{ij} = s_{\sigma(i)} s_{\sigma^{2}(i)}$ and zero otherwise (i.e, whenever $\sigma(k') \neq j$). The observation is therefore true when the exponent $r = 2$. We assume that for some $r \geq 3$, we have

$$(W_{\sigma}^{r-1})_{ij} := w_{ij}^{\sigma^{r-1}} = \begin{cases} \prod_{k=1}^{r-1} s_{\sigma^{k}(i)} & \text{if} \quad \sigma^{k}(i) = j \\ 0 \quad \text{otherwise.} \end{cases}$$

Now, we write

$$(W_{\sigma}^{r})_{ij} = (W_{\sigma}^{r-1} W_{\sigma})_{ij} = \sum_{k=0}^{h} w_{ik}^{\sigma^{r-1}} w_{kj}^{\sigma}.$$

Because the addition chain $\mathcal{C}_n$ has been fixed, there exists a unique $k' \in \{0, \ldots, h\}$ such that $\sigma^{r-1}(i) = k'$. Thus, $(W_{\sigma}^{r})_{ij} = \prod_{k=1}^{r} s_{\sigma^{k}(i)}$ whenever $\sigma(k') = j$ (i.e, $\sigma^{r}(i) = j$) and zero otherwise. This completes the inductive argument. We can certainly adapt a similar argument to show that

$$(W_{\tau}^{v})_{ij} = \begin{cases} \prod_{l=1}^{v} s_{\sigma^{l}(i)} & \text{if} \quad \tau^{v}(i) = j \\ 0 \quad \text{otherwise.} \end{cases}$$

The assertion that the matrices $A_{\sigma}^{v}, A_{\tau}^{v}$ are strictly lower triangular follows from the observation that

$$\sigma^{r}(i) = j \leq i - r < i$$

for $r \geq 2$. ☐

6. Invariants of associated addition chain matrices

In this section, we study several invariants of the matrices associated with addition chains introduced in previous sections. We study the rank, singular values, determinants, and spectra of matrices associated with a fixed addition chain.

6.1. **The rank and rank profiles of associated matrices.**

Proposition 6.1. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. Let $v \geq 2$ be a fixed integer and let $\mathrm{Im}(\sigma)$ and $\mathrm{Im}(\tau)$ denote the images of σ and τ, respectively, in this fixed decomposition. We define

$$I_v^{\sigma} = \{\sigma^v(i) \ : \ 1 \leq i \leq h\}$$

and

$$I_v^{\tau} := \{\tau^v(i) \ : \ 1 \leq i \leq h\}$$

The following relations hold:

(1) $\mathrm{rank}(W_\sigma) = \mathrm{rank}(A_\sigma) = |\mathrm{Im}(\sigma)|$

(2) $\mathrm{rank}(W_\tau) = \mathrm{rank}(A_\tau) = |\mathrm{Im}(\tau)|$

(3) $\mathrm{rank}(W_\sigma^v) = \mathrm{rank}(A_\sigma^v) = |I_v^\sigma|$

(4) $\mathrm{rank}(W_\tau^v) = \mathrm{rank}(A_\tau^v) = |I_v^\tau|$.

Proof. We call a column of a matrix nonzero if it has at least a nonzero entry. We observe that for a fixed j, there is one-to-one correspondence between the multisets

$$\{(A_\sigma^v)_{ij} \ : \ (A_\sigma^v)_{ij} > 0, \ i \in \{1, \ldots, h\}\}$$

and the nonzero j^{th} column of the matrix A_σ^v, the multiset

$$\{(W_\sigma^v)_{ij} \ : \ (W_\sigma^v)_{ij} > 0, \ i \in \{1, \ldots, h\}\}$$

and the nonzero j^{th} column of the matrix (W_σ^v), the multiset

$$\{(A_\tau^v)_{ij} \ : \ (A_\tau^v)_{ij} > 0, \ i \in \{1, \ldots, h\}\}$$

and the nonzero j^{th} column of the matrix A_τ^v, the multiset

$$\{(W_\tau^v)_{ij} \ : \ (W_\tau^v)_{ij} > 0, \ i \in \{1, \ldots, h\}\}$$

and the nonzero j^{th} column of the matrix W_τ^v. Because each of the entries

$$(A_\sigma^v)_{ij}, (W_\sigma^v)_{ij}, (A_\tau^v)_{ij}, (W_\tau^v)_{ij} > 0$$

if and only if $\sigma^v(i) = j$, respectively, $\tau^v(i) = j$, and the matrices $A_\sigma, A_\tau, W_\sigma, W_\tau$ have exactly one nonzero entry in each row, we deduce

$$\mathrm{rank}(W_\sigma^v) = \mathrm{rank}(A_\sigma^v) = |I_v^\sigma|$$

and

$$\mathrm{rank}(W_\tau^v) = \mathrm{rank}(A_\tau^v) = |I_v^\tau|.$$

We observe that the properties (1) and (2) are only special cases with $v = 1$. $\square$

Definition 6.2 (The sigma and tau ancestral rank profiles)**.** Let $n \geq 2$ be an integer and

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^h$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^h$ a τ track of the chain. We denote the σ ancestral-rank profile of the fixed addition chain $\mathcal{C}_n$ with fixed decomposition as the h-tuple

$$\mathcal{R}_\sigma(\mathcal{C}_n) = \Big(\mathrm{rank}(A_\sigma), \mathrm{rank}(A_\sigma^2), \ldots, \mathrm{rank}(A_\sigma^h)\Big)$$

and the τ ancestral-rank profile

$$\mathcal{R}_\tau(\mathcal{C}_n) = \Big(\mathrm{rank}(A_\tau), \mathrm{rank}(A_\tau^2), \ldots, \mathrm{rank}(A_\tau^h)\Big)\,.$$

6.2. The eigenvalues and singular values of matrices associated with an addition chain.

Proposition 6.3. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)}+s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^h$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^h$ a τ track of the chain. We define

$$m_j^\sigma := \#\{i \in \{0, \ldots, h\} \ : \ \sigma(i) = j\}.$$

We have $W_\sigma^T W_\sigma = \mathrm{diag}(m_0^\sigma s_0^2, m_1^\sigma s_1^2, \ldots, m_h^\sigma s_h^2)$, which means that the matrix $W_\sigma^T W_\sigma$ is a diagonal matrix with the entries $m_0^\sigma s_0^2, m_1^\sigma s_1^2, \ldots, m_h^\sigma s_h^2$ on the diagonal. Furthermore, the eigenvalues are $m_j^\sigma s_j^2$ for each $j \in \{0, \ldots, h\}$ with the corresponding singular values $s_j\sqrt{m_j^\sigma}$ for each $j \in \{0, \ldots, h\}$.

Proof. We let i, j be fixed indices with $0 \leq i, j \leq h$ and write

$$(W_\sigma^T W_\sigma)_{ij} = \sum_{k=0}^{h} w_{ki}^\sigma w_{kj}^\sigma.$$

If $i = j$, then

$$(W_\sigma^T W_\sigma)_{ij} = \sum_{k=0}^{h} (w_{kj}^\sigma)^2 = m_j^\sigma s_j^2.$$

On the other hand, if $i \neq j$, then $\sigma(k) \neq i$ whenever $\sigma(k) = j$ or vice versa, which implies

$$(W_\sigma^T W_\sigma)_{ij} = \sum_{k=0}^{h} w_{ki}^\sigma w_{kj}^\sigma = 0.$$

This shows that the matrix $W_\sigma^T W_\sigma$ is a diagonal matrix with the entries $m_0^\sigma s_0^2, m_1^\sigma s_1^2, \ldots, m_h^\sigma s_h^2$ on the diagonal. In particular

$$(W_\sigma^T W_\sigma)_{ij} = \begin{cases} m_j^\sigma s_j^2 & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

The assertion that the eigenvalues and singular values are of the form $m_j^\sigma s_j^2$ and $s_j\sqrt{m_j^\sigma}$ for each $j \in \{0, \ldots, h\}$ is standard. $\square$

Proposition 6.4. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. We define

$$m_j^{\tau} := \#\{i \in \{0, \ldots, h\} \ : \ \tau(i) = j\}.$$

We have $W_{\tau}^{T} W_{\tau} = \operatorname{diag}(m_0^{\tau} s_0^2, m_1^{\tau} s_1^2, \ldots, m_h^{\tau} s_h^2)$, which means that the matrix $W_{\tau}^{T} W_{\tau}$ is a diagonal matrix with the entries $m_0^{\tau} s_0^2, m_1^{\tau} s_1^2, \ldots, m_h^{\tau} s_h^2$ on the diagonal. Furthermore, the eigenvalues are $m_j^{\tau} s_j^2$ for each $j \in \{0, \ldots, h\}$ with the corresponding singular values $s_j \sqrt{m_j^{\tau}}$ for each $j \in \{0, \ldots, h\}$.

Proof. We let i, j be fixed indices with $0 \leq i, j \leq h$ and write

$$(W_{\tau}^{T} W_{\tau})_{ij} = \sum_{k=0}^{h} w_{ki}^{\tau} w_{kj}^{\tau}.$$

If $i = j$, then

$$(W_{\tau}^{T} W_{\tau})_{ij} = \sum_{k=0}^{h} (w_{kj}^{\tau})^2 = m_j^{\tau} s_j^2.$$

On the other hand, if $i \neq j$, then $\tau(k) \neq i$ whenever $\tau(k) = j$ or vice versa, which implies

$$(W_{\tau}^{T} W_{\tau})_{ij} = \sum_{k=0}^{h} w_{ki}^{\tau} w_{kj}^{\tau} = 0.$$

This shows that the matrix $W_{\tau}^{T} W_{\tau}$ is a diagonal matrix with the entries $m_0^{\tau} s_0^2, m_1^{\tau} s_1^2, \ldots, m_h^{\tau} s_h^2$ on the diagonal. In particular

$$(W_{\tau}^{T} W_{\tau})_{ij} = \begin{cases} m_j^{\tau} s_j^2 & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

The assertion that the eigenvalues and singular values are of the form $m_j^{\tau} s_j^2$ and $s_j \sqrt{m_j^{\tau}}$ for each $j \in \{0, \ldots, h\}$ is standard. □

Proposition 6.5. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. We define

$$m_j^{\sigma} := \#\{i \in \{0, \ldots, h\} \ : \ \sigma(i) = j\}.$$

We have $A_{\sigma}^{T} A_{\sigma} = \operatorname{diag}(m_0^{\sigma}, m_1^{\sigma}, \ldots, m_h^{\sigma})$, which means that the matrix $A_{\sigma}^{T} A_{\sigma}$ is a diagonal matrix with the entries $m_0^{\sigma}, m_1^{\sigma}, \ldots, m_h^{\sigma}$ on the diagonal. Furthermore, the eigenvalues are m_j^{σ} for each $j \in \{0, \ldots, h\}$ with the corresponding singular values $\sqrt{m_j^{\sigma}}$ for each $j \in \{0, \ldots, h\}$.

Proof. We let i, j be fixed indices with $0 \leq i, j \leq h$ and write

$$(A_\sigma^T A_\sigma)_{ij} = \sum_{k=0}^{h} a_{ki}^\sigma a_{kj}^\sigma.$$

If $i = j$, then

$$(A_\sigma^T A_\sigma)_{ij} = \sum_{k=0}^{h} (a_{kj}^\sigma)^2 = m_j^\sigma.$$

On the other hand, if $i \neq j$, then $\sigma(k) \neq i$ whenever $\sigma(k) = j$ or vice versa, which implies

$$(A_\sigma^T A_\sigma)_{ij} = \sum_{k=0}^{h} a_{ki}^\sigma a_{kj}^\sigma = 0.$$

This shows that the matrix $A_\sigma^T A_\sigma$ is a diagonal matrix with the entries $m_0^\sigma, m_1^\sigma, \ldots, m_h^\sigma$ on the diagonal. In particular

$$(A_\sigma^T A_\sigma)_{ij} = \begin{cases} m_j^\sigma & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

The assertion that the eigenvalues and singular values are of the form m_j^σ and $\sqrt{m_j^\sigma}$ for each $j \in \{0, \ldots, h\}$ is standard. $\square$

Proposition 6.6. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. We define

$$m_j^\tau := \#\{i \in \{0, \ldots, h\} \;:\; \tau(i) = j\}.$$

We have $A_\tau^T A_\tau = \operatorname{diag}(m_0^\tau, m_1^\tau, \ldots, m_h^\tau)$, which means that the matrix $A_\tau^T A_\tau$ is a diagonal matrix with the entries $m_0^\tau, m_1^\tau, \ldots, m_h^\tau$ on the diagonal. Furthermore, the eigenvalues are m_j^τ for each $j \in \{0, \ldots, h\}$ with the corresponding singular values $\sqrt{m_j^\tau}$ for each $j \in \{0, \ldots, h\}$.

Proof. We let i, j be fixed indices with $0 \leq i, j \leq h$ and write

$$(A_\tau^T A_\tau)_{ij} = \sum_{k=0}^{h} a_{ki}^\tau a_{kj}^\tau.$$

If $i = j$, then

$$(A_\tau^T A_\tau)_{ij} = \sum_{k=0}^{h} (a_{kj}^\tau)^2 = m_j^\tau.$$

On the other hand, if $i \neq j$, then $\tau(k) \neq i$ whenever $\tau(k) = j$ or vice versa, which implies

$$(A_\tau^T A_\tau)_{ij} = \sum_{k=0}^{h} a_{ki}^\tau a_{kj}^\tau = 0.$$

This shows that the matrix $A_{\tau}^{T}A_{\tau}$ is a diagonal matrix with the entries $m_0^{\tau}, m_1^{\tau}, \ldots, m_h^{\tau}$ on the diagonal. In particular

$$(A_{\sigma}^{T}A_{\sigma})_{ij} = \begin{cases} m_j^{\tau} & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

The assertion that the eigenvalues and singular values are of the form m_j^{τ} and $\sqrt{m_j^{\tau}}$ for each $j \in \{0, \ldots, h\}$ is standard. $\square$

7. Matrices and local vs global combinatorial structure of a chain

We study ways to extract the local and global structure of an addition chain from related matrices in the following identities

Proposition 7.1. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ ($i > \sigma(i) \geq \tau(i) \geq 0$) for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. We let

$$I_{(h+1)\times 1} := (\underbrace{1, 1, \ldots, 1}_{h+1})^{T}.$$

The following identities hold:

(1)

$$WI_{(h+1)\times 1} = \begin{pmatrix} 0 \\ s_1 \\ \vdots \\ s_h \end{pmatrix}$$

(2) $(I_{(h+1)\times 1})^{T} W I_{(h+1)\times 1} = \sum\limits_{i=1}^{h} s_i$

(3)

$$\sum_{j=0}^{h} w_{ij}^{\sigma} = s_{\sigma(i)} \quad \textit{and} \quad \sum_{j=0}^{h} w_{ij}^{\tau} = s_{\tau(i)}$$

(4)

$$||W_{\sigma}||_F^2 = \sum_{i=1}^{h} s_{\sigma(i)}^2 \quad \textit{and} \quad ||W_{\tau}||_F^2 = \sum_{i=1}^{h} s_{\tau(i)}^2$$

where $||\cdot||_F$ denotes the Frobenius norm of $\cdot$.

Proof. By proposition 3.4, we can write $WI_{(h+1)\times 1} = (W_{\sigma} + W_{\tau})I_{(h+1)\times 1} = W_{\sigma}I_{(h+1)\times 1} + W_{\tau}I_{(h+1)\times 1}$. The matrices W_{σ}, W_{τ} have exactly one nonzero entry on each row, except the first. In particular, for a fixed index $i \in \{1, \ldots, h\}$, we have

$$(W_{\sigma})_{ij} = \begin{cases} s_{\sigma(i)} & \text{if} \quad \sigma(i) = j \\ 0 & \text{otherwise} \end{cases}$$

and

$$(W_{\tau})_{ij} = \begin{cases} s_{\tau(i)} & \text{if} \quad \tau(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

This implies

$$W_\sigma I_{(h+1)\times 1} = \begin{pmatrix} 0 \\ s_{\sigma(1)} \\ s_{\sigma(2)} \\ \vdots \\ s_{\sigma(h)} \end{pmatrix}$$

and

$$W_\tau I_{(h+1)\times 1} = \begin{pmatrix} 0 \\ s_{\tau(1)} \\ s_{\tau(2)} \\ \vdots \\ s_{\tau(h)} \end{pmatrix}$$

and obtain

$$WI_{(h+1)\times 1} = \begin{pmatrix} 0 \\ s_{\sigma(1)} \\ s_{\sigma(2)} \\ \vdots \\ s_{\sigma(h)} \end{pmatrix} + \begin{pmatrix} 0 \\ s_{\tau(1)} \\ s_{\tau(2)} \\ \vdots \\ s_{\tau(h)} \end{pmatrix} = \begin{pmatrix} 0 \\ s_1 \\ s_2 \\ \vdots \\ s_h \end{pmatrix}.$$

By (1), we deduce

$$I^T_{(h+1)\times 1} WI_{(h+1)\times 1} = \sum_{i=1}^{h} s_i.$$

The identity (3) and (4) can easily be deduced using the structure of the matrices W_σ and W_τ ☐

[2]

8. The sigma and tau-rank defects

In this section, we study the σ and τ-rank defect as a measure of the deviation of the rank of the associated adjacency matrix from the length of the addition chain.

Definition 8.1 (The sigma,tau-rank defect)**.** Let $n \geq 2$ be an integer and

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. We define

$$\Delta_\sigma(\mathcal{C}_n) = h - \operatorname{rank}(A_\sigma) = h - \operatorname{rank}(W_\sigma).$$

We call $\Delta_\sigma(\mathcal{C}_n)$ the σ-rank defect of the fixed chain $\mathcal{C}_n$ with fixed decomposition. Similarly, we define the τ-rank defect

$$\Delta_\tau(\mathcal{C}_n) := h - \operatorname{rank}(A_\tau) = h - \operatorname{rank}(W_\tau).$$

We now develop a matrix theoretic criterion that can be used to detect whether an arbitrary addition chain is *Brauer* or non-Brauer.

[2]These identities are extremely useful for recovering the original arithmetic structure of the addition chain. The framework makes it possible to study an addition chain using matrix-theoretic concepts as invariants and later recovering the results as hidden properties of the chain.

Proposition 8.2 (The Brauer/non-Brauer chain criterion). *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. The addition chain $\mathcal{C}_n$ is Brauer if and only if $\Delta_\sigma(\mathcal{C}_n) = 0$.

Proof. We suppose a fixed addition chain

$$\mathcal{C}_n : s_0 = 1 < s_1 < \cdots < s_h = n$$

with fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i))$ for each $i \in \{1, \ldots, h\}$ is *Brauer*. This implies $\sigma(i) = i - 1$ for each $i \in \{1, \ldots, h\}$. Hence, each row of the matrices A_σ, W_σ has exactly one nonzero entry (except for the first row) at $(A_\sigma)_{ii-1}$ and $(W_\sigma)_{ii-1}$ for each $i \in \{1, \ldots, h\}$. In particular, we can write

$$A_\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$W_\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ s_{\sigma(1)} & 0 & \cdots & 0 & 0 \\ 0 & s_{\sigma(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{\sigma(h)} & 0. \end{pmatrix}$$

Therefore, we deduce $\operatorname{rank}(A_\sigma) = \operatorname{rank}(W_\sigma) = h$ and get $\Delta_\sigma(\mathcal{C}_n) = 0$. Conversely, we suppose $\Delta_\sigma(\mathcal{C}_n) := h - \operatorname{rank}(A_\sigma) = h - \operatorname{rank}(W_\sigma) = 0$. By proposition 6.1, we deduce $\operatorname{rank}(A_\sigma) = \operatorname{rank}(W_\sigma) = |\operatorname{Im}(\sigma)| = \#\{\sigma(1), \sigma(2), \ldots, \sigma(h)\} = h$. Because $\sigma(1) = 0, \sigma(2) \in \{0, 1\}, \sigma(3) \in \{0, 1, 2\}, \sigma(4) \in \{0, 1, 2, 3\}, \ldots, \sigma(h) \in \{0, 1, \ldots, h-1\}$, we deduce

$$\sigma(1) = 0, \sigma(2) = 1, \sigma(3) = 2, \ldots, \sigma(h) = h - 1.$$

In particular, $\sigma(i) = i - 1$ for each $i \in \{1, 2, \ldots, h\}$. This implies that the addition chain $\mathcal{C}_n$ is *Brauer*. □

9. Commutators, decoupling invariants and some problems

In this section, we study the commutators and decoupling invariants of associated matrices of a fixed addition chain.

Definition 9.1 (Unweighted chain matrix commutators). Let $n \geq 2$ be an integer and

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain.

We define the unweighted commutator of the fixed chain corresponding to the fixed track by

$$[A_\sigma, A_\tau] := A_\sigma A_\tau - A_\tau A_\sigma$$

and the weighted commutator corresponding to the fixed track

$$[W_\sigma, W_\tau] := W_\sigma W_\tau - W_\tau W_\sigma.$$

Problem 9.2 (Local existence problem). Let $\mathcal{F}_n$ denote the family of all addition chains that lead n. For which addition chains in the family does there exist a fixed τ and σ track so that the commutator $[W_\sigma, W_\tau] = 0$. In particular, classify all chains in $\mathcal{F}_n$ satisfying for a fixed track $W_\sigma W_\tau = W_\tau W_\sigma$.

Problem 9.3 (Global search problem). Find all integers $n \geq 2$ for which there exists a chain with a fixed σ and τ track so that $W_\sigma W_\tau = W_\tau W_\sigma$.

Definition 9.4 (Decoupling invariants)**.** Let $n \geq 2$ be an integer and

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. We define the decoupling invariants of the fixed chain corresponding to the fixed tracks by

$$\langle W_\sigma, W_\tau \rangle_F := \sum_{i=1}^{h} s_{\sigma(i)} s_{\tau(i)}.$$

Using the relation $W = W_\sigma + W_\tau$, we can write

$$\langle W, W \rangle_F = \langle W_\sigma + W_\tau, W_\sigma + W_\tau \rangle_F = \langle W_\sigma, W_\sigma \rangle_F + \langle W_\tau, W_\tau \rangle_F + 2\langle W_\sigma, W_\tau \rangle_F.$$

Observing that $||W_\sigma||_F^2 = \langle W_\sigma, W_\sigma \rangle_F$ and $||W_\tau||_F^2 = \langle W_\tau, W_\tau \rangle_F$, we deduce the following identity.

Proposition 9.5. *Let $n \geq 2$ be an integer and*

$$\mathcal{C}_n : s_0 < s_1 < \cdots < s_h = n$$

be a fixed addition chain that leads to n with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $i \in \{1, \ldots, h\}$, with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. The sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ is a σ track of the chain and $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the chain. We let $|| \cdot ||_F$ denote the Frobenius norm of $\cdot$. We have

$$\langle W_\sigma, W_\tau \rangle_F = \frac{1}{2}\left(||W||_F^2 - ||W_\sigma||_F^2 - ||W_\tau||_F^2\right).$$

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