Obtaining Exponents in Fermat, Markov and Catalan Diophantine Equations

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Abstract
We propose a procedure which allows to determine the only admissible exponents in Diophantine equations of three kinds: Fermat, Markov and Catalan. First, we use a set of rewriting rules for each Diophantine equation. Second, we obtain the parametrized versions of the Diophantine equations. Next, we differentiate each of parametrized equations obtaining the identities for Fermat, Markov and Catalan equations.

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1. Introduction

We start with a set of rewriting rules for the Fermat, Markov and Catalan equations transforming those into equations which allow their partial differentiation. This leads us to certain identities. From the obtained identities we receive a sequence the limits of which determine the equations admissible exponents.

2. Fermat Diophantine equation

We can write the Fermat's Diophantine equation as

$$a^n + b^n = c^n, \tag{1}$$

where a, b, c, n are natural numbers (i.e. integers greater than zero).

We will replace a with $x + kx$, where $x \in \mathbb{Q}$ and $k \in \mathbb{N}$, rewriting (1) as

$$(x + kx)^n + b^n = c^n \tag{2}$$

or

$$(k + 1)^n x^n + b^n = c^n. \tag{3}$$

Setting $x = \frac{1}{k+1}, \frac{2}{k+1}, \frac{3}{k+1}, \ldots$, will make $a = 1, 2, 3, \ldots$, in (1). Similarly, we will have

$$a^n + (y + ky)^n = c^n \tag{4}$$

and

$$a^n + b^n = (z + kz)^n, \tag{5}$$

where $y, z = \frac{1}{k+1}, \frac{2}{k+1}, \frac{3}{k+1}, \ldots$.

We introduce a real parameter $p \in \mathbb{R}$ into (2) as follows

$$(x + kpx)^n + b^n p = c^n p. \tag{6}$$

Additional parameters $q, r \in \mathbb{R}$ will also be appropriately introduced into (4) and (5) as

$$a^n q + (y + kqy)^n = c^n q \tag{7}$$

and

$$a^n r + b^n r = (z + krz)^n, \tag{8}$$

respectively. Taking partial derivative of (6) with respect to p, we obtain

$$nk(x + kpx)^{n-1} x + b^n = c^n \tag{9}$$

or

$$nk(kp + 1)^{n-1} x^n + b^n = c^n. \tag{10}$$

Setting $p = 1$, we get

$$nk(k + 1)^{n-1} x^n + b^n = c^n. \tag{11}$$

The coefficients of x^n in (11) and (3) must be equal, hence

$$nk(k + 1)^{n-1} = (k + 1)^n. \tag{12}$$

Therefrom, we obtain

$$nk(k + 1)^n/(k + 1) = (k + 1)^n \tag{13}$$

and in this way we may write the equation for the values of the exponent n as a function of k

$$n = \frac{k + 1}{k}. \tag{14}$$

We can now determine the admissible values for n as a function of $k = 1, 2, 3, \ldots, \infty$, which will be $2/1, 3/2, 4/3, \ldots, 1$, respectively. Equations (7) and (8) are treated in a similar fashion with the parameters q and r and partial differentiation. We reject all n that are not natural numbers, leaving us with $n = 1$ and $n = 2$ as the only acceptable exponent values in (1).

3. Markov Diophantine equation

We write the Markov Diophantine equation as

$$a^n + b^n + c^n = 3abc, \tag{15}$$

where $a, b, c, n \in \mathbb{N}$. Without loss of generality we rewrite (15) as

$$(x + kx)^n + b^n + c^n = 3abc \tag{16}$$

or

$$(k + 1)^n x^n + b^n + c^n = 3abc, \tag{17}$$

where $a = x+kx$, $k \in \mathbb{N}$. We also have $x = \frac{1}{k+1}, \frac{2}{k+1}, \frac{3}{k+1}, \ldots$. In this way $a = 1, 2, 3, \ldots$, as assumed in the Markov Diophantine equation (15).

After introducing the real parameter p, we differentiate both sides of (18) with respect to p

$$(x + kpx)^n + b^n p + c^n p = 3abcp \tag{18}$$

and we establish the admissible exponents, as in the case of Fermat Diophantine equation. We come to the conclusion that the only possible exponents in the Markov Diophantine equation (15) are 1 and 2.

4. Catalan Diophantine equation

We write the Catalan Diophantine equation

$$a^i - b^j = 1, \tag{19}$$

where $a, b, i, j \in \mathbb{Z}$ are integers greater than one.

Without loss of generality we rewrite (19) as

$$(x + kx)^i - b^j = 1 \tag{20}$$

or

$$(k + 1)^i x^i - b^j = 1, \tag{21}$$

where $a = x + kx$ and $k \in \mathbb{N}$. We also have $x = \frac{2}{k+1}, \frac{3}{k+1}, \ldots$, thus allowing $a = 2, 3, \ldots$, as assumed in the Catalan Diophantine equation (19).

After introducing the real parameter p, we differentiate both sides of (22) with respect to p

$$(x + kpx)^i - b^j p = p \tag{22}$$

and we determine the exponents. We come to the conclusion that the only admissible exponent i in the Catalan Diophantine equation (19) is equal to 2.

With $i = 2$, we obtain the following Catalan Diophantine equation

$$a^2 - b^j = 1 \tag{23}$$

or

$$b^j = a^2 - 1. \tag{24}$$

Since $b > 1$, we can set $b = 2$, obtaining

$$2^j = a^2 - 1. \tag{25}$$

One can verify that for $j = 3$ we have $2^3 = 8$ and $a = 3$. Then we have the Catalan equation solution

$$a^i - b^j = 1 \tag{26}$$

$a = 3, i = 2, b = 2, j = 3$, which gives

$$3^2 - 2^3 = 1. \tag{27}$$

5. Conclusion

Using a relatively straightforward set of rewriting rules for the Diophantine equations of Fermat, Markov, and Catalan, we were able to transform each into a form that allowed the introduction of rational parameters x, y, z and real parameters p, q, r. When differentiated with respect to the latter, the resulting equations gave us a way to determine the admissible values for the exponent n in the Fermat and Markov, and exponents i, j in Catalan Diophantine equations.

References

[1] Wikipedia.org *Diophantine equation* (2026) and the References cited therein