The unity of the prime number theorem, Bertrand's postulate and Euler's proof

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Abstract

For any integer n such that $2n + 1$ is prime and $n \geq 2$, we construct interval $(n, 2n)$. By synthesizing the prime number theorem, the Bertrand–Chebyshev theorem and Euler's proof to analyze the gap between $2n + 1$ and primes in the interval $(n, 2n)$, we prove the Polignac's Conjecture.

For any integer $n \geq 2$, we construct intervals $(n, 2n)$ and $(2n, 4n)$. By synthesizing the Bertrand–Chebyshev theorem and the prime number theorem to analyze the gap of primes between the consecutive intervals, we prove the Polignac's Conjecture.

Keywords: Polignac's Conjecture, Twin prime conjecture, Bertrand–Chebyshev theorem, the prime number theorem

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1. Introduction

One of the most famous problems in mathematics is the Polignac's Conjecture. Significant progress has been achieved to date—most notably by Y. Zhang [1], James Maynard, Terence Tao, and the collaborative Polymath8 project [2]—all of which advanced the underlying sieve-theoretic framework. Nevertheless, the standard sieve method faces a fundamental limitation due to inherent parity obstacles. The conjecture remains unsolved, and genuinely novel approaches are required.

2. Proof I

For any integer n such that $2n+1$ is prime and $n \geq 2$, we construct interval

$$(n, 2n)$$

By the Bertrand–Chebyshev theorem, there is always at least one prime in the interval.

Let p_n denote the prime. The number of p_n is given by the Prime Number Theorem

$$\pi(2n) - \pi(n) \sim Li(2n) - Li(n)$$

The probability that $2n-1$ is prime is estimated as below:

(1) Upper bound

$$\frac{Li(2n) - Li(n)}{\frac{(2n-n)}{2}}$$

(2) Median value I

$$\frac{\frac{2n}{\ln 2n} - \frac{n}{\ln n}}{\frac{(2n-n)}{2}}$$

(3) Median value II

$$\frac{1}{\ln(2n-1)}$$

(4) Median value III

$$\frac{1}{\frac{(2n-n)}{2}}$$

(5) Lower bound

$$\frac{1}{(2n-1)}$$

Below, we prove $E_{\text{lower}}[S] = \sum_{n=2}^{\infty} \frac{1}{(2n-1)} = \infty$

$\because$ ① The sum of the reciprocals of all primes diverges (Euler, 1737)

$$\sum_{p \geq 5} \frac{1}{p} = \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n+1} + \cdots = \infty$$

$$② \ \frac{1}{2n-1} > \frac{1}{2n+1}$$

$$\therefore \ \sum_{n=2}^{\infty} \frac{1}{(2n-1)} = \infty$$

Hence,

$$\lim_{n \to \infty} inf(2n + 1 - p_n) = (2n+1) - (2n-1) = 2$$

The Twin Prime Conjecture holds.

3. Proof II

$\forall\, n \in \mathbb{Z}^{+}, n \geq 2$, we construct consecutive intervals

$$(n, 2n),\ \ (2n, 4n)$$

By the Bertrand–Chebyshev theorem, there is always at least one prime in each of the intervals $(2n, 4n)$ and $(n, 2n)$. Let p_n denote the prime in $(2n, 4n)$, q_n denote the prime in $(n, 2n)$.

The p_n and q_n number are given by the Prime Number Theorem

$$p_n = [\pi(4n) - \pi(2n)] \sim Li(4n) - Li(2n)$$

$$q_n = [\pi(2n) - \pi(n)] \sim Li(2n) - Li(n)$$

As n, $2n$ and $4n$ increase, primes enter, traverse and depart the intervals: the positions occupied by primes undergo continuous reassignment in response to the shifting of n, $2n$ and $4n$.

The probability that $2n+1$ is prime and the probability that $2n-1$ is prime are estimated as below

(1) Upper bound

$$\frac{Li(4n) - Li(2n)}{\frac{(4n-2n)}{2}} \quad and \quad \frac{Li(2n) - Li(n)}{\frac{(2n-n)}{2}}$$

(2) Median value I

$$\frac{\frac{4n}{\ln 4n} - \frac{2n}{\ln 2n}}{\frac{(4n-2n)}{2}} \quad and \quad \frac{\frac{2n}{\ln 2n} - \frac{n}{\ln n}}{\frac{(2n-n)}{2}}$$

(3) Median value II

$$\frac{1}{\ln(2n+1)} \quad and \quad \frac{1}{\ln(2n-1)}$$

(4) Lower bound

$$\frac{1}{\frac{(4n-2n)}{2}} \quad and \quad \frac{1}{\frac{(2n-n)}{2}}$$

The joint probability that both $2n+1$ and $2n-1$ are primes equals the product of their marginal probabilities. If the sum of the joint probability diverges, the conjecture holds.

The convergence or divergence of the sum of the joint probability are listed as below

√ divergence; *X* convergence

q_n \ p_n	$\frac{Li(4n)-Li(2n)}{\frac{(4n-2n)}{2}}$	$\frac{\frac{4n}{\ln 4n}-\frac{2n}{\ln 2n}}{\frac{(4n-2n)}{2}}$	$\frac{1}{\ln(2n+1)}$	$\frac{1}{\frac{(4n-2n)}{2}}$
$\frac{Li(2n)-Li(n)}{\frac{(2n-n)}{2}}$	√	√	√	√
$\frac{\frac{2n}{\ln 2n}-\frac{n}{\ln n}}{\frac{(2n-n)}{2}}$	√	√	√	√
$\frac{1}{\ln(2n-1)}$	√	√	√	√
$\frac{1}{\frac{(2n-n)}{2}}$	√	√	√	*X*

The joint probability cannot equal the product of $\frac{1}{\frac{(4n-2n)}{2}}$ and $\frac{1}{\frac{(2n-n)}{2}}$, because it is impossible for both intervals $(n, 2n)$ and $(2n, 4n)$ to contain exactly one prime number.

Hence, $\lim_{n\to\infty} inf(p_n - q_n) = (2n+1) - (2n-1) = 2$

The Twin Prime Conjecture holds.

For $p_n - q_n = 4$, there exist exactly two possible combinations:

(1) $p_n = 2n+1$ and $q_n = 2n-3$

(2) $p_n = 2n+3$ and $q_n = 2n-1$

The joint probability is double the product of their marginal probabilities. Its sum diverges. Hence,

$$\lim_{n\to\infty} inf(p_n - q_n) = 4$$

Similarly, we can prove $\lim_{n\to\infty} inf(p_n - q_n) = 2k \quad (k = 1,2,3,4,\cdots)$

The Polignac's Conjecture holds.

References

[1] ZHANG Y., Bounded gaps between primes}. *Ann. of Math.* 179 (2014), 1121-1174

[2] D. H. J. POLYMATH. *The bounded gaps between primes, polymath projects---a retrospective*. arXiv: 1409.8361, 2014.