Entropy and the Damped Harmonic Oscillator: A Bridge Between Mechanics and Thermodynamics

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Abstract

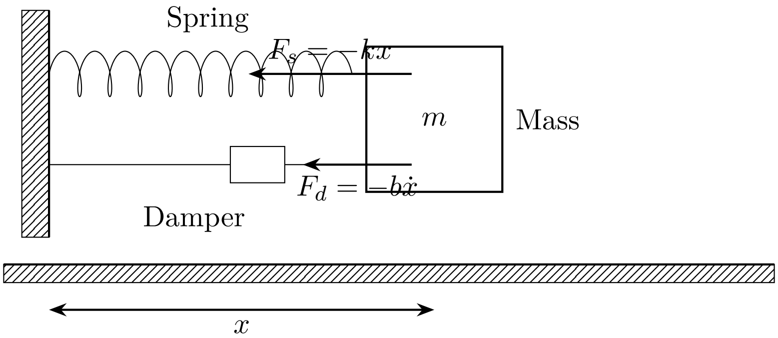
We show that the decay of the energy envelope in a damped harmonic oscillator can be explicitly connected to the increase of entropy in the surroundings. The result is a simple, pedagogical bridge between mechanics and the Second Law of Thermodynamics.

1 The Damped Harmonic Oscillator

A damped harmonic oscillator is an oscillator that is subjected to a restoring force and a velocity-dependent damping force, causing its mechanical energy to dissipate over time.

1.1 Spring–Mass–Damper System

Consider a mass attached to a spring and a damper, as shown below.



The equation of motion for this system is:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}(t) \tag{1}$$

Dividing through by m, we obtain the standard form:

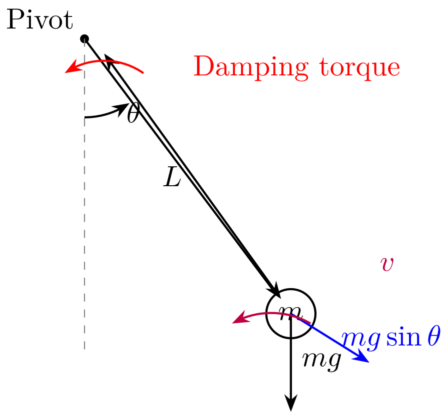
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \frac{F_{\text{ext}}(t)}{m} \tag{2}$$

where:

- $\omega_0 = \sqrt{\dfrac{k}{m}}$ is the undamped natural frequency,
- $\zeta = \dfrac{b}{2\sqrt{km}}$ is the damping ratio,
- $F_{\text{ext}}(t)$ is any external forcing.

In the absence of external forcing ($F_{\text{ext}} = 0$), the system is a freely damped oscillator.

1.2 Damped Pendulum



For small angular displacements, the equation of motion is:

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta} + \omega_0^2\theta = 0 \tag{3}$$

where:

$$\omega_0 = \sqrt{\frac{g}{L}}$$

and ζ is the damping ratio.

2 The Amplitude Envelope

The amplitude envelope is the smooth curve that traces the peak displacement of a damped oscillator over time.

For an underdamped oscillator, the amplitude decays exponentially:

$$A(t) = A_0 e^{-\beta t} \tag{4}$$

where:

- A_0 is the initial amplitude,
- β is the decay constant,
- t is time.

3 The Energy Envelope

The energy envelope is the footprint of the energy decay of a damped oscillator.

Because damping exists, the oscillator's mechanical energy is continuously converted into other forms such as heat due to friction, air drag, or internal material losses.

The total mechanical energy of a damped harmonic oscillator can be written as:

$$E(t) = \frac{1}{2}k_{\text{eff}}[A(t)]^2 \tag{5}$$

where k_{eff} is the effective stiffness of the oscillator.

Since the amplitude decays according to:

$$A(t) = A_0 e^{-\beta t}$$

we substitute:

$$E(t) = \frac{1}{2} k_{\text{eff}} A_0^2 e^{-2\beta t} \tag{6}$$

Letting:

$$E_0 = \frac{1}{2} k_{\text{eff}} A_0^2$$

we obtain the energy envelope:

$$\boxed{E(t) = E_0 e^{-2\beta t}} \tag{7}$$

4 Lost Mechanical Energy as Heat

The energy lost by the oscillator by time t is the difference between its initial energy and its current energy:

$$E_{\text{loss}}(t) = E_0 - E(t) \tag{8}$$

Substituting the energy envelope:

$$E_{\text{loss}}(t) = E_0 - E_0 e^{-2\beta t} \tag{9}$$

Factoring E_0:

$$\boxed{E_{\text{loss}}(t) = E_0 \left(1 - e^{-2\beta t}\right)} \tag{10}$$

This lost mechanical energy must go somewhere. By the First Law of Thermodynamics, it is not destroyed. Instead, it is converted into heat and dissipated into the surroundings.

Therefore:

$$Q(t) = E_{\text{loss}}(t) \tag{11}$$

where $Q(t)$ is the total heat dissipated into the surroundings by time t.

5 Entropy Increase of the Surroundings

By the Second Law of Thermodynamics, when heat Q is transferred into surroundings acting as an isothermal thermal reservoir at constant temperature T, the entropy of the surroundings increases.

For an infinitesimal heat transfer:

$$dS_{\text{surr}} = \frac{\delta Q}{T} \tag{12}$$

Since T is constant, integrating gives:

$$\Delta S_{\text{surr}}(t) = \frac{Q(t)}{T} \tag{13}$$

Substituting the expression for the lost mechanical energy:

$$\boxed{\Delta S_{\text{surr}}(t) = \frac{E_0}{T} \left(1 - e^{-2\beta t}\right)} \tag{14}$$

As $t \to \infty$, all mechanical energy is dissipated, yielding the asymptotic total entropy increase:

$$\Delta S_{\text{surr}}(\infty) = \frac{E_0}{T} \tag{15}$$

5.1 Environmental Temperature Dependence and the Entropy Footprint

The total asymptotic entropy footprint $\Delta S_{\text{surr}}(\infty) = \dfrac{E_0}{T}$ reveals an intriguing temperature dependence.

Consider an identical oscillator with initial energy E_0 dissipating its energy in two different isothermal surroundings: a hot environment at temperature T_i and a colder environment at temperature T_f, where $T_f < T_i$.

The total entropy generated in each respective environment as $t \to \infty$ is:

$$\Delta S_i = \frac{E_0}{T_i}, \quad \Delta S_f = \frac{E_0}{T_f} \tag{16}$$

Since $T_f < T_i$, it follows directly that:

$$\Delta S_f > \Delta S_i \tag{17}$$

Microscopically, temperature measures the average thermal agitation per degree of freedom in the surroundings. A colder environment resides in a state of higher microscopic order with fewer occupied thermal microstates. Injecting a fixed macroscopic packet of mechanical energy E_0 into a cold bath creates a far larger relative disturbance—and thus a greater expansion of accessible phase space—than injecting the same energy into an already agitated, high-temperature bath.

Thus, while the oscillator loses the exact same quantity of mechanical energy E_0 in both cases, the entropy footprint it leaves behind depends inversely on the temperature of its environment: **the colder the world it dies in, the deeper the footprint.**

6 Rate of Entropy Production

The rate at which entropy is generated in the surroundings is obtained by differentiating the entropy change with respect to time:

$$\frac{dS_{\text{surr}}}{dt} = \frac{1}{T}\frac{dQ}{dt} \tag{18}$$

Since $Q(t) = E_0 - E(t)$,

$$\frac{dQ}{dt} = -\frac{dE}{dt} \tag{19}$$

Therefore:

$$\frac{dS_{\text{surr}}}{dt} = -\frac{1}{T}\frac{dE}{dt} = \frac{2\beta E_0}{T}e^{-2\beta t} \tag{20}$$

This shows that as the energy envelope decays, entropy is produced continuously. The faster the energy decays, the faster entropy increases.

7 Discussion

The result we have derived connects three fundamental concepts:

1. **Mechanics:** The damped harmonic oscillator loses energy exponentially.
2. **Thermodynamics:** That lost energy becomes heat dissipated into the environment.
3. **Second Law:** That heat increases the entropy of the surroundings, with a total footprint determined by $\frac{E_0}{T}$.

The amplitude envelope is the footprint of decaying order.

The energy envelope is the footprint of declining useful mechanical energy.

The entropy increase is the footprint of the Second Law being satisfied.

All three tell the same story: organized mechanical energy dissipates into disordered thermal energy, and entropy increases—leaving a deeper thermodynamic mark in colder surroundings.

8 Conclusion

We have shown that the energy envelope of a damped harmonic oscillator can be explicitly connected to the entropy increase of its surroundings.

The result is simple and universal:

$$\boxed{\Delta S_{\text{surr}}(t) = \frac{E_0}{T}\left(1 - e^{-2\beta t}\right)}$$

This pedagogical bridge shows that a damped oscillator is not merely a mechanics problem. It is a thermodynamic process unfolding in time, demonstrating how mechanical order degrades into thermal disorder.