Exact Global Modes of Two-Term Weighted Sums of i.i.d. Geometric Random Variables

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August 2026

Abstract

Let X and Y be independent geometric random variables on $\mathbb{Z}_{\geq 0}$ with $\mathbb{P}(X = k) = p(1-p)^k$. For positive integers a, b, we determine the global modes of $Z = aX + bY$. The probability mass function may have gaps and local rises and falls. We prove that every global mode is a multiple of $\mathrm{lcm}(a, b)$ and that there are at most two global modes. A logarithmic threshold gives their exact locations and determines whether one or two modes occur. We also give an exact formula for every probability and a small-p asymptotic formula for the mode. The proof parametrizes the solutions of $ax + by = n$ by residue classes and then compares the maxima of consecutive blocks.

Keywords: geometric distribution; weighted sum; global mode; lattice distribution; log-concavity; restricted partition function.
MSC 2020: 60E05; 60C05; 05A17.

1 Introduction

Let X and Y be independent and identically distributed geometric random variables, and let a, b be positive integers. We consider $Z = aX + bY$. The problem is to determine all integers n for which $\mathbb{P}(Z = n)$ is maximal. In particular, we ask whether three or more global modes can occur and whether the mode locations can be written exactly.

This is not an ordinary convolution problem when the weights are different. The event $Z = n$ is described by the nonnegative integer solutions of $ax + by = n$. The support may have gaps, and probabilities in neighboring residue classes need not change monotonically. Thus the full probability mass function need not be unimodal.

We give a complete answer for two i.i.d. geometric variables. Every global mode is a multiple of the least common multiple of the two weights. On this subsequence, the consecutive probability ratios decrease strictly. There is therefore either one global mode or two consecutive candidates with equal probability. An explicit equation determines the second case.

Linear combinations of geometric observations also appear in distributional characterizations. For example, Arnold and Villaseñor [1] study an identity involving $X_1 + 2X_2$. General results on discrete log-concavity and strong unimodality were given by Keilson and Gerber [5]; see also Johnson, Kemp, and Kotz [4]. Bhati and Joshi [2], and Chakraborty, Ong, and Biswas [3], studied related reweighted geometric families. In those papers, "weighted geometric distribution" means a reweighted one-variable distribution, not the deterministic linear combination studied here. To the best of our knowledge, the restriction to multiples of the least common multiple and the exact mode classification below have not been stated before.

2 The problem and an exact probability formula

Throughout, X, Y are independent and identically distributed with

$$\mathbb{P}(X = k) = pq^k, \qquad k \in \mathbb{Z}_{\geq 0}, \quad p \in (0,1), \quad q = 1 - p.$$

Let $a, b \in \mathbb{Z}_{>0}$ and $Z = aX + bY$. Put

$$d = \gcd(a,b), \qquad A = \frac{\min(a,b)}{d}, \qquad B = \frac{\max(a,b)}{d}.$$

Then $A \leq B$ and $\gcd(A,B) = 1$. After interchanging X and Y if necessary, exchangeability gives $Z = dW$ in distribution, where $W = AX + BY$. Here d is the lattice span of Z, while the least-common-multiple block length used below is

$$L = \operatorname{lcm}(a,b) = dAB.$$

We first give an exact formula for each residue class. The case $A = B$ necessarily means $A = B = 1$ and will be treated separately. Suppose $A < B$. For every $r \in \{0, \ldots, AB-1\}$, let x_r be the unique integer in $\{0, \ldots, B-1\}$ satisfying

$$Ax_r \equiv r \pmod{B}, \qquad c_r = \frac{r - Ax_r}{B}.$$

Notice that $-A < c_r < A$.

Proposition 2.1 (Exact mass in a block)**.** *For $m \in \mathbb{Z}_{\geq 0}$, $0 \leq r < AB$, and $Q = q^{B-A}$,*

$$\mathbb{P}(W = ABm + r) = \begin{cases} p^2 q^{Am + x_r + c_r} \dfrac{1 - Q^{m+1}}{1 - Q}, & c_r \geq 0, \\ p^2 q^{Am + x_r + c_r} \dfrac{1 - Q^{m}}{1 - Q}, & c_r < 0. \end{cases}$$

The second expression is zero when $m = 0$.

Proof. All integral solutions of $Ax + By = ABm + r$ are

$$x = x_r + Bt, \qquad y = Am + c_r - At,$$

with $t \in \mathbb{Z}$. Since $0 \leq x_r < B$, the constraint $x \geq 0$ is equivalent to $t \geq 0$. The condition $y \geq 0$ gives $t \leq m$ when $c_r \geq 0$ and $t \leq m - 1$ when $c_r < 0$. Moreover,

$$x + y = Am + x_r + c_r + (B - A)t.$$

Summing $p^2 q^{x+y}$ over the permitted values of t gives the result. □

The same formula gives the coefficients of $p^2/((1 - qs^A)(1 - qs^B))$. It will be used to find the modes.

3 Solution of the mode problem

For a discrete random variable V, a *global mode* is an integer n maximizing $\mathbb{P}(V = n)$. We do not require the full mass sequence to be unimodal.

Lemma 3.1 (Maximum in each block)**.** *Suppose $A < B$. For every $m \in \mathbb{Z}_{\geq 0}$ and $1 \leq r < AB$,*

$$\mathbb{P}(W = ABm + r) < \mathbb{P}(W = ABm).$$

Proof. At $r = 0$ one has $x_0 = c_0 = 0$, so Proposition 2.1 gives

$$H_m := \mathbb{P}(W = ABm) = p^2 q^{Am} \frac{1 - Q^{m+1}}{1 - Q}. \tag{1}$$

If $r > 0$ and $c_r \geq 0$, the geometric sum has the same number of terms as in (1), but $x_r + c_r > 0$ gives an additional factor smaller than one. If $c_r < 0$, the geometric sum has one fewer term. Also,

$$x_r + c_r = \frac{r + (B - A)x_r}{B} > 0.$$

Thus both cases give a value strictly smaller than H_m. □

Therefore, in each half-open block $[ABm, AB(m + 1))$, the unique maximum is at the left endpoint. We now compare these block maxima.

Theorem 3.2 (Exact mode classification)**.** *Let X, Y be i.i.d. geometric variables on $\mathbb{Z}_{\geq 0}$ with parameter p, and let $Z = aX + bY$, where a, b are positive integers.*

If $a \neq b$, define d, A, B, L as above and

$$T = T_{A,B}(q) := \frac{\log\big((1 - q^A)/(1 - q^B)\big)}{(B - A)\log q}. \tag{2}$$

Then $T > 0$, and:

(i) *if $T \notin \mathbb{Z}$, Z has the unique global mode*

$$L\lfloor T \rfloor;$$

(ii) *if $T = N \in \mathbb{Z}_{>0}$, Z has exactly two global modes*

$$L(N - 1) \quad \text{and} \quad LN.$$

Equivalently, the double-mode case occurs exactly when, for some $N \in \mathbb{Z}_{>0}$,

$$q^{(B-A)N}(1 - q^B) = 1 - q^A. \tag{3}$$

If $a = b = d$, write $R = q/p$. When $R \notin \mathbb{Z}$, the unique global mode is $d\lfloor R \rfloor$; when $R = N \in \mathbb{Z}_{>0}$, the two global modes are $d(N - 1)$ and dN.

Proof. Assume first that $a \neq b$, hence $A < B$. Since $0 < (1 - q^A)/(1 - q^B) < 1$ and $0 < q < 1$, both the numerator and denominator in (2) are negative; hence $T > 0$. By Lemma 3.1, every global mode of W is of the form ABm. From (1),

$$\frac{H_{m+1}}{H_m} = q^A \frac{1 - q^{(B-A)(m+2)}}{1 - q^{(B-A)(m+1)}}. \tag{4}$$

These ratios decrease strictly with m. Indeed, with $Q = q^{B-A}$, the function $(1 - Qt)/(1 - t)$ increases strictly in $t \in (0, 1)$, whereas $t = Q^{m+1}$ decreases strictly in m. Rearranging (4) gives

$$\frac{H_{m+1}}{H_m} \geq 1 \iff q^{(B-A)(m+1)} \geq \frac{1 - q^A}{1 - q^B} \iff m + 1 \leq T.$$

It follows that (H_m) increases to a unique maximum unless one ratio equals one. In that case, exactly two consecutive terms are equal and maximal. This proves the result for W. Multiplication by d gives the stated locations for Z. Equation (3) is the condition $T = N$.

If $a = b = d$, then $Z = d(X + Y)$ and

$$\mathbb{P}(X + Y = n) = (n + 1)p^2 q^n.$$

The adjacent ratio $q(n + 2)/(n + 1)$ gives the result directly. □

Corollary 3.3 (At most two global modes)**.** *For every $p \in (0,1)$ and all positive integer weights a, b, the random variable $aX + bY$ has either one or two global modes, never more.*

Remark 3.4 (Convention for the geometric law). If geometric variables are instead supported on $\{1, 2, \ldots\}$, write $X' = X + 1$ and $Y' = Y + 1$. Every mode listed in Theorem 3.2 is then shifted by $a + b$; the number of modes is unchanged.

Remark 3.5 (The full mass sequence need not be unimodal). For $a, b > 1$, some integers are absent from the support. Even after dividing out the greatest common divisor, probabilities in different residue classes may have local rises and falls. The theorem concerns only the global modes. The block-maximum sequence (H_m) is strictly log-concave because its adjacent ratios in (4) decrease strictly.

4 Examples

4.1 Weights 1 and 2

For $Z = X + 2Y$, one has $A = 1$, $B = 2$, and $L = 2$. A tie with $N = 1$ occurs when

$$q(1 - q^2) = 1 - q, \qquad \text{or equivalently} \qquad q(1 + q) = 1.$$

Thus $q = (\sqrt{5} - 1)/2$ gives the two modes 0 and 2. If q is not in the countable critical set given by (3), the mode is unique. The same result holds for $2X + Y$ by exchangeability.

4.2 Weights 2 and 3

For $Z = 2X + 3Y$, $A = 2$, $B = 3$, and $L = 6$. Every global mode is a multiple of 6, although the support contains every sufficiently large integer. The first tie occurs at the unique root in $(0, 1)$ of

$$q^2(1 + q) = 1,$$

namely $q \approx 0.754877666$, and its modes are 0 and 6. Higher ties are given by $q^N(1-q^3) = 1-q^2$.

5 Consequences, verification, and scope

5.1 Small-success-probability asymptotics

As $p \downarrow 0$ (so $q \uparrow 1$), expanding (2) gives

$$T_{A,B}(q) \sim \frac{\log(B/A)}{(B - A)p}.$$

Therefore, if M_p is any global mode (with either mode allowed in the double-mode case), then

$$M_p \sim \frac{dAB \log(B/A)}{(B - A)p}. \tag{5}$$

Since

$$\mathbb{E}[Z] = \frac{d(A + B)q}{p}, \qquad \mathrm{Var}(Z) = \frac{d^2(A^2 + B^2)q}{p^2},$$

we also have

$$\frac{M_p}{\mathbb{E}[Z]} \longrightarrow \frac{AB \log(B/A)}{(B - A)(A + B)}.$$

The continuous extension at $A = B$ is $1/2$, which agrees with the equal-weight case. When $p \uparrow 1$, the unique mode is eventually zero.

5.2 Exact arithmetic check

The proof above does not use computation. We also made an exact arithmetic check. Define the normalized coefficients $c_n = p^{-2}\mathbb{P}(Z = n)$. They satisfy

$$(1 - qs^a)(1 - qs^b) \sum_{n \geq 0} c_n s^n = 1,$$

that is,

$$c_n = qc_{n-a} + qc_{n-b} - q^2 c_{n-a-b},$$

with $c_0 = 1$ and negative-index terms set to zero. We used exact rational arithmetic for all $1 \leq a, b \leq 12$ and $q \in \{1/10, 2/10, \ldots, 9/10\}$. This gives 1296 cases. We generated the coefficients through six complete L-blocks beyond the predicted maximizing block. In every case, the maximizers in the checked range agreed with Theorem 3.2. This is only a consistency check. The proof over the infinite tail is given by Lemma 3.1 and the ratio argument in (4).

5.3 Scope of the proof

The proof uses two variables with the same geometric ratio q. With more variables, the weighted denumerants are higher-dimensional. With unequal geometric parameters, the terms along each solution set no longer form one geometric progression. These mode problems therefore need separate study.

6 Conclusion

We asked where the global modes of $aX + bY$ are and whether more than two can occur. The residue parametrization shows that the left endpoint of each least-common-multiple block has larger probability than every other point in that block. The consecutive ratios of these block maxima decrease strictly. This gives their exact locations and allows at most one tie. Hence $aX + bY$ has one or two global modes, never more. The full probability mass function need not be unimodal, but its block-maximum sequence is strictly log-concave.

References

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