An operator calculus and corrector theory for binomial coefficients at negative integer indices

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Abstract

We study a self-contained combinatorial equality, discovered by the author some forty years ago, which extends $\binom{n}{p}$ to negative integer indices without invoking any extension of the factorial: $\binom{n}{p} = \binom{-p-1}{-n-1}$, $n \geq p \geq 0$. Extending binomial coefficients to negative arguments is not new — it goes back to Loeb's 1992 construction, further developed by Sprugnoli and, in the q-analogue setting, by Formichella and Straub, and at the level of the factorial itself by Roman — and we show precisely how the equality above relates to, and differs from, that literature. The equality also opens ground the ordinary binomial coefficient cannot reach, and §2.1 develops it: when the two indices have opposite signs, $\binom{n}{p}$ is nonzero where the classical coefficient vanishes, Pascal's rule fails, and it is restored throughout that regime by a single forced multiplicative factor $\frac{n-2p}{n}$ — a corrector of a new kind, multiplicative rather than additive. On the restored rule we obtain three further results, each proved and each in closed form: the corrector of D_1 at every order m, a single formula that vanishes identically on the anti-diagonal $n+2p=0$; the corresponding form of Vandermonde's identity at every depth k; and, on the latter, the corrector of D_1 at any depth and order simultaneously, which reduces to the first at $k=1$, so that the modified Pascal rule is exactly its depth-one case.

The rest of the paper is a small operator calculus built on top of the equality: six index-level para-pseudo-derivatives $D_1, \ldots, D_6$, whose action on Vandermonde's identity produces a nonzero residual term we call the corrector. We give closed quotient and product laws for the corrector at any depth and order, show the six classes collapse into exactly two families related by an exchange of depth and order, compute the operator algebra generated by D_1, D_2, D_3 (closed commutators, a duality by conjugation), and prove that the corrector's relative value at negative order (the discrete integral) is exactly the reciprocal of its value at the corresponding positive order — a duality specific to one of the two families and provably absent from the other. We give a q-analogue of the operator algebra and resolve, for the class D_1, the open question of extending the founding equality itself to the q-setting. We introduce a companion family of division operators, whose prefactor depends on the column rather than the row, so that their correctors are rational rather than binomial; here the founding equality exposes a genuine asymmetry, since — unlike D_1 — a division operator differs from its own mirror, and its corrector is therefore not mirror-invariant, showing the invariance of our main theorem to be special to D_1. Finally we show that D_1, D_2, D_3 coincide, for a single application, with classical operators of Rota's finite operator calculus, and that this correspondence does not extend to the full operator algebra under composition.

1 Introduction

The binomial coefficient $\binom{n}{p}$ is classically defined for $(n,p) \in \mathbb{N}^2$ with $n \geq p \geq 0$. Extending it to negative integer arguments is not a new problem: it goes back at least to Loeb's 1992 extension via a limiting Gamma-function construction [1], further studied by Sprugnoli [2] and, in the q-analogue setting, by Formichella and Straub [3]; at the level of the factorial itself, a negative-integer factorial was studied by Roman [4]. This paper does not claim to be the first extension of $\binom{n}{p}$ to negative indices. Its contribution is a self-contained combinatorial equality, discovered independently by the author some forty years ago,

$$\binom{n}{p} = \binom{-p-1}{-n-1} \qquad (n \geq p \geq 0), \tag{A}$$

used as the starting point for an operator calculus acting directly on the indices of $\binom{n}{p}$ — the para-pseudo-derivatives $D_1, \ldots, D_6$ — rather than as an endpoint in itself.

Why this is useful. Applying a discrete operator such as D_1 directly to the indices of a translation identity — an identity relating $\binom{n}{p}$ at shifted indices, such as Vandermonde's — does not, in general, preserve the identity: a nonzero residual term appears. We call this residual the *corrector* (§5): the residual is the phenomenon, the corrector is the closed value of it that we study. Its systematic study yields results that do not reduce to classical facts about $\binom{n}{p}$ alone:

1. a full treatment of the opposite-sign regime $np < 0$, where the classical coefficient vanishes but $\binom{n}{p}$ does not (§2.1). Four results, each with a proof: Pascal's rule fails there and is restored throughout the regime by the forced multiplicative factor $\frac{n-2p}{n}$ (§2.1); the corrector of D_1 on the restored rule is a single closed formula at every order m (§2.2), vanishing identically on the anti-diagonal $n + 2p = 0$; iterating the rule yields Vandermonde's identity at every depth k in this regime (§2.3); and on the latter the corrector of D_1 is again a single closed formula in (n, p, k, m) (§2.4), which reduces to (2.2) at $k = 1$ — so the modified Pascal rule is exactly its depth-one case;

2. a closed-form quotient and product law for the corrector, valid at any depth and any order (§5.1);

3. a classification of $D_1, \ldots, D_6$ into exactly two families, related by an exchange of the depth of the identity and the order of the operator (§7), together with a closed operator algebra: the commutators $[D_1, D_2]$, $[D_1, D_3]$, and a duality $D_2 \leftrightarrow D_3$ by conjugation (§8);

4. a duality between D_1 and its inverse, the discrete integral: the corrector at negative order is exactly the reciprocal of the corrector at the corresponding positive order (§9.1) — true for the family containing D_1, and false for the other;

5. a q-analogue of the operator algebra (§10), including a proof (not merely a numerical check) that the founding equality itself extends consistently to the q-setting for class D_1 (§10.1), resolving the open question raised in an earlier version of this note;

6. a companion family of division operators (§5.3), whose column-dependent prefactor makes their correctors rational rather than binomial, and for which the founding equality is no longer an invariance: a division operator differs from its own mirror, so, in contrast to D_1 (Theorem 1), its corrector is not mirror-invariant.

None of these results is a restatement of the consistency of (A) — that consistency, as §3 shows, follows quickly from known results once the right reference is used. Section 11 makes precise the relation of D_1, D_2, D_3 to Rota's finite operator calculus, and clarifies exactly which part of the theory is classical and which is not.

2 The founding equality

Definition 1 (Founding equality)**.** For $(n,p) \in \mathbb{N}^2$ with $n \geq p$, we define the negative-index coefficient by

$$\binom{-p-1}{-n-1} := \binom{n}{p}, \qquad \text{equivalently} \quad \binom{-p}{-n} := \binom{n-1}{p-1}. \tag{B}$$

The resulting identity, read in the classical direction, is the founding equality

$$\binom{n}{p} = \binom{-p-1}{-n-1}. \tag{A}$$

Substituting $p \mapsto p+1$, $n \mapsto n+1$ in (B) recovers (A); they are a single convention stated in two mirror forms. This definition assumes no prior notion of the factorial of a negative number: it gives meaning to $\binom{n}{p}$ when both indices are negative by reducing it to an ordinary, well-defined binomial coefficient.

Remark 1 (Factorial form)**.** Written with factorials, (B) reads $\dfrac{(-p)!}{(-n)!} = \dfrac{(n-1)!}{(p-1)!}$ (the common $(n-p)!$ cancels); relabelling $p \mapsto n$, $n \mapsto n+m$ gives, in particular,

$$\frac{(-n)!}{(-n-m)!} = \frac{(n+m-1)!}{(n-1)!}.$$

This is the *unsigned* form; its signed counterpart $(-1)^m \dfrac{(n+m-1)!}{(n-1)!}$ is Loeb's (Proposition 1). Without a convention for the isolated factorial of a negative integer, only such a ratio is meaningful, since an isolated $(-n)!$ is a pole of Γ; the next remark fixes a convention that does assign it a value.

Remark 2 (A factorial convention, and indices of opposite signs)**.** Since Γ has a pole at each non-positive integer, no value of $(-n)!$ is canonical, and one may instead fix one. Setting

$$(-1)! = (-2)! = -1, \qquad (-n)! = \frac{-1}{(n-1)!} \quad (n \in \mathbb{Z}),$$

a single relation valid for every integer n, with no restriction of sign: for $n \leq 0$ the argument $n-1$ is itself non-positive, so $(n-1)! = -1/(-n)!$ by the same rule, and the displayed relation reduces to the identity $(-n)! = (-n)!$. The convention is thus an involution, readable in either direction, and it returns the ordinary factorial whenever the argument is non-negative (e.g. $n = -3$ gives $(-n)! = 6$ and $-1/(-4)! = 6$).

Equivalence with the founding equality. Since $(-p-1)! = -1/p!$ and $(-n-1)! = -1/n!$,

$$\binom{-p-1}{-n-1} = \frac{(-p-1)!}{(-n-1)!\,\big((-p-1)-(-n-1)\big)!} = \frac{-1/p!}{(-1/n!)\,(n-p)!} = \frac{n!}{p!\,(n-p)!} = \binom{n}{p},$$

which is (A). The two minus signs, one in the numerator and one in the denominator, cancel — and that cancellation is exactly what makes the convention unsigned, where the Γ-residue regularization would leave a factor $(-1)^{n-p}$.

The founding equality and the factorial convention are therefore two equivalent foundations for the same object, and one may start from either. We retain (A)/(B) as the definition, since it presupposes no extension of the factorial at all; the convention is an alternative route to it, not a prerequisite. With this convention the mirror rule $\binom{N}{P} = \binom{-P-1}{-N-1}$ holds for every pair of integers (N, P), including pairs of opposite signs and the case $N < P$ where the classical coefficient vanishes: e.g.

$$\binom{3}{-2} = \binom{1}{-4} = -\frac{1}{20}, \qquad \binom{1}{4} = \binom{-5}{-2} = -\frac{1}{12},$$

while $\binom{n}{p}$ is recovered unchanged for $n \geq p \geq 0$. Note that, unlike the classical coefficient, these values need not be integers, as the two examples show.

The convention has a price, which should be stated plainly: it breaks the factorial recurrence, since $(-n)! = -1/(n-1)!$ while $(-n) \cdot (-n-1)! = +1/(n-1)!$. It is one regularization among others — the Γ-residue regularization carries $(-1)^{n-1}$ instead of a constant -1, and yields Loeb's signed coefficients. Neither is more correct than the other; the constant sign is what makes ours unsigned.

Comparison with Roman's factorial. At the level of the factorial itself, the closest prior object is Roman's negative-integer factorial [4], $[n]! = (-1)^{n-1}/(-n-1)!$ for $n < 0$. Our $(-n)!$ and Roman's $[-n]!$ differ by exactly $(-1)^n$. The distinction is structural. Roman's choice preserves the multiplicative recurrence $[n]! = n\,[n-1]!$ (all $n \neq 0$); ours instead makes the reflection

$$z!\,(-z-1)! = -1$$

a constant — for Roman the same reflection equals $(-1)^z$ — at the cost of that recurrence. These are the two natural sign normalisations of the negative factorial, distinguished by which structure they keep: Roman keeps the recurrence, we keep the constant reflection.

This choice is not aesthetic: it is *forced* by the operator calculus that follows. As an extension of the factorial taken in isolation, neither normalisation is more correct than the other; but for the object studied here the choice ceases to be free. Adopting Roman's normalisation would make the mirror equality carry a sign $(-1)^{n-p}$ (this is exactly the discrepancy with Loeb's extension, Proposition 1), and that sign would propagate into Theorem 1: the term-by-term translation of Vandermonde's identity (V_k) would become alternating in the summation index, so that the Vandermonde closure would fail and the invariance of the corrector — the self-mirror property of D_1 — would be destroyed (numerically, at $(n, p, k) = (6, 3, 2)$, the signed mirror sum equals 4 instead of $\binom{6}{3} = 20$). Proposition 2 already records the order-one form of this: of the four possible sign/magnitude readings, only the unsigned reading achieves $M = X$ identically. The reflection must therefore be constant for the theory to exist; it is in this precise sense, and not by mere convention, that $(-n)! = -1/(n-1)!$ is the required choice. It is on this constant reflection that the operator calculus below is built.

2.1 Pascal's rule for indices of opposite signs

The convention just fixed opens a regime the ordinary binomial coefficient cannot see: n and p of opposite signs ($np < 0$), where $\binom{n}{p}$ is nonzero while the classical coefficient vanishes. Two results describe it. Pascal's rule fails there and is restored by a forced multiplicative factor; and the corrector of D_1 on the restored rule has a closed form at every order.

The ordinary rule fails. Wherever the classical coefficient vanishes, the convention assigns a nonzero value instead, so $\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$ cannot hold: it fails at every opposite-sign pair.

A multiplicative corrector restores it. Throughout the regime $np < 0$ — both $n > 0$, $p < 0$ and $n < 0$, $p > 0$ — the rule becomes

$$\frac{n-2p}{n}\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}. \tag{2.1}$$

Proof. Two factorial ratios suffice, and the convention makes exactly one of them flip sign. For $z \geq 1$ the recurrence $z! = z(z-1)!$ holds, so $z!/(z-1)! = z$; for $z \leq 0$ the convention gives $z! = -1/(-z-1)!$ and $(z-1)! = -1/(-z)!$, whence

$$\frac{z!}{(z-1)!} = \frac{(-z)!}{(-z-1)!} = -z \quad (z \leq 0),$$

the sign opposite to the recurrence value. Take first $n > 0$, $p < 0$: here n and $n-p$ are positive and p is negative, so $(n-1)!/n! = 1/n$, $(n-p)!/(n-p-1)! = n-p$ and $p!/(p-1)! = -p$. Cancelling factorials,

$$\frac{\binom{n-1}{p}}{\binom{n}{p}} = \frac{(n-1)!}{n!} \cdot \frac{(n-p)!}{(n-p-1)!} = \frac{n-p}{n}, \qquad \frac{\binom{n-1}{p-1}}{\binom{n}{p}} = \frac{(n-1)!}{n!} \cdot \frac{p!}{(p-1)!} = \frac{-p}{n},$$

and adding gives $(n-2p)/n$. In the other half, $n < 0$ and $p > 0$, the roles are exchanged: $p!/(p-1)! = p$, while $(n-1)!/n! = -1/n$ and $(n-p)!/(n-p-1)! = -(n-p)$. Each of the two products again undergoes exactly one sign change, so both ratios are unchanged and the sum is the same. $\square$

On the classical domain the second ratio is $+p/n$ instead of $-p/n$, and the two sum to 1: ordinary Pascal. That single sign, and nothing else, is the whole discrepancy. The factor is forced rather than fitted: the quotient of the right-hand side by $\binom{n}{p}$ equals $\frac{n-2p}{n}$ identically on the whole domain. It is therefore a corrector of a new kind — multiplicative rather than additive — and it marks precisely where the unsigned convention pays for its mirror property, since Loeb's signed convention is characterised instead by preserving Pascal's rule everywhere.

The corrector of D_1 on the restored rule. Applying $D_1(m)\binom{n}{p} = (n+1)\cdots(n+m)\binom{n}{p-m}$ to (2.1) leaves a residual in closed form:

$$\Delta_m = -m\,\frac{n+2p}{n}\,(n+1)(n+2)\cdots(n+m-1)\binom{n}{p-m}, \tag{2.2}$$

valid whenever (n,p) and $(n,p-m)$ both lie in the regime.

Proof. Write $R(a,m) := (a+1)\cdots(a+m)$ and apply $D_1(m)$ to (2.1). The left side becomes $\frac{n-2p}{n}R(n,m)\binom{n}{p-m}$. On the right both terms lie in the row $n-1$, so $D_1(m)$ factors $R(n-1,m)$ out of them and leaves $\binom{n-1}{p-m} + \binom{n-1}{p-m-1}$, which by (2.1) applied at $(n, p-m)$ equals $\frac{n-2p+2m}{n}\binom{n}{p-m}$ — this is where the hypothesis on $(n, p-m)$ is used. Subtracting,

$$\Delta_m = \left[\tfrac{n-2p}{n}R(n,m) - \tfrac{n-2p+2m}{n}R(n-1,m)\right]\binom{n}{p-m}.$$

Put $Q := (n+1)\cdots(n+m-1)$, so $R(n,m) = Q(n+m)$ and $R(n-1,m) = nQ$. The bracket is then $\frac{Q}{n}\big[(n-2p)(n+m) - (n-2p+2m)n\big]$, and expanding both products gives

$$(n-2p)(n+m) - (n-2p+2m)n = -m(n+2p),$$

which is (2.2). The vanishing on $n+2p=0$ is immediate from this factorisation, and holds at every order. $\square$

Two features stand out. The entire dependence on the order is carried by the single factor m times a rising product — the same shape as the corrector laws of §5. And Δ_m vanishes identically on the anti-diagonal $n+2p=0$: along that line D_1 crosses the restored rule with no residual at all, at every order.

Vandermonde's identity at depth k. Iterating (2.1) extends the repair to any depth, in one formula and without further computation. For $k \geq 1$,

$$\sum_{i=0}^{k}\binom{k}{i}\binom{n-k}{p-i} = \sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(n-k+1)-2(p-j)}{n-k+1}\binom{n-k+1}{p-j}, \tag{2.3}$$

valid whenever each pair $(n-k+1, p-j)$, $0 \leq j \leq k-1$, lies in the regime. At $k=1$ this is (2.1) itself; at $k=2$ it reads

$$\frac{n-2p+1}{n-1}\binom{n-1}{p-1} + \frac{n-2p-1}{n-1}\binom{n-1}{p} = \binom{n-2}{p-2} + 2\binom{n-2}{p-1} + \binom{n-2}{p}.$$

Proof. Split $\binom{k}{i} = \binom{k-1}{i} + \binom{k-1}{i-1}$ in the left-hand sum and reindex the second part:

$$\sum_i \binom{k}{i}\binom{n-k}{p-i} = \sum_j \binom{k-1}{j}\Big[\binom{n-k}{p-j} + \binom{n-k}{p-j-1}\Big],$$

and each bracket equals $\frac{(n-k+1)-2(p-j)}{n-k+1}\binom{n-k+1}{p-j}$ by (2.1) applied at $(n-k+1, p-j)$ — which is where the hypothesis on those pairs is used. $\square$

The closed-form corrector of D_1 at depth k and order m. Applying $D_1(m)$ to the two sides of (2.3) leaves a residual which again has a single closed form. Write $r := n-k+1$ for the row of the weighted side. Then

$$\Delta(V_k, m) = -\,m\,\frac{(r+1)(r+2)\cdots(r+m-1)}{r}\sum_{j=0}^{k-1}\binom{k-1}{j}(r+2p-2j)\binom{r}{p-m-j}, \tag{2.4}$$

valid whenever (2.3) applies at (n,p) and at $(n, p-m)$. At $k=1$ the sum has one term, $r=n$, and (2.4) reduces to (2.2): the modified Pascal rule is the depth-one case, exactly as V_1 is in the classical theory.

Proof. Write $R(a,m) := (a+1)\cdots(a+m)$, and let $\Delta(V_k,m)$ denote $D_1(m)$ applied term by term to the weighted side, minus $D_1(m)$ applied to the Vandermonde side. The Vandermonde side lies in row $n-k = r-1$ and the weighted side in row r, so

$$\Delta(V_k,m) = R(r,m)\sum_j \binom{k-1}{j} w(r,p-j)\binom{r}{p-j-m} - R(r-1,m)\sum_i \binom{k}{i}\binom{n-k}{p-m-i},$$

with $w(r,q) := \frac{r-2q}{r}$. By (2.3) at $(n, p-m)$ the second sum equals $\sum_j \binom{k-1}{j} w(r, p-m-j)\binom{r}{p-m-j}$, so the two collect over a common $\binom{r}{p-m-j}$:

$$\Delta(V_k, m) = \sum_j \binom{k-1}{j}\binom{r}{p-m-j}\big[R(r,m)\, w(r,p-j) - R(r-1,m)\, w(r,p-m-j)\big].$$

Put $u := r - 2p + 2j$, so $w(r,p-j) = u/r$ and $w(r,p-m-j) = (u+2m)/r$, and set $Q := (r+1)\cdots(r+m-1)$, so that $R(r,m) = Q(r+m)$ and $R(r-1,m) = rQ$. The bracket becomes

$$\frac{Q}{r}\big[(r+m)u - r(u+2m)\big] = \frac{Q}{r}\, m(u-2r) = -\, m\,\frac{Q}{r}\,(r+2p-2j),$$

since $u - 2r = -(r + 2p - 2j)$. This is (2.4). □

One limitation should be stated. Equation (2.3) reduces depth k to depth $k-1$ one row higher, but the reduction cannot be iterated to the end: after the first step the weights depend on the summation index, so the terms no longer pair with equal coefficients. In particular the depth-k sum is not a single multiple of $\binom{n}{p}$ for $k \ge 2$: the quotient $\sum_i \binom{k}{i}\binom{n-k}{p-i}/\binom{n}{p}$ is no product of linear factors of the shape of (2.1). Order condenses completely, as (2.2) shows; depth condenses by exactly one step.

3 Related work, and the precise relation of (A)/(B) to it

Extending $\binom{n}{p}$ to negative arguments. Loeb [1] showed that $\binom{a}{k} := \lim_{\varepsilon\to 0} \Gamma(a+1+\varepsilon)/[\Gamma(k+1+\varepsilon)\Gamma(a-k+1+\varepsilon)]$ satisfies Pascal's rule for all integers $(a,k) \ne (0,0)$ and has a combinatorial interpretation via sets with a negative number of elements. Sprugnoli [2] showed that the classical negation rule must carry a sign function once k can be negative:

$$\binom{a}{k} = (-1)^k \operatorname{sgn}(k)\binom{k-a-1}{k}. \qquad (\star)$$

Formichella and Straub [3] extended all of this to the Gaussian q-binomial coefficients, with a Lucas-type congruence for negative arguments; §10.1 uses their reflection formula.

Proposition 1 (Precise relation to Loeb's extension)**.** *For all $n \ge p \ge 0$,*

$$\binom{-p-1}{-n-1}_{\text{Loeb}} = (-1)^{n-p}\binom{-p-1}{-n-1} \overset{(A)}{=} (-1)^{n-p}\binom{n}{p}.$$

The two conventions agree exactly when $n-p$ is even, and are of opposite sign when $n-p$ is odd.

Proof. By definition (A), $\binom{-p-1}{-n-1} \overset{(A)}{=} \binom{n}{p}$. For two negative integer arguments Loeb's extension satisfies Kronenburg's reflection rule $\binom{-N}{-K} = (-1)^{N-K}\binom{K-1}{N-1}$, valid for $K \ge N \ge 1$ (equivalently, apply $(\star)$ and the ordinary Pascal symmetry). With $N = p+1$, $K = n+1$ this gives $\binom{-p-1}{-n-1}_{\text{Loeb}} = (-1)^{n-p}\binom{n}{p}$. For instance $(n,p) = (7,4)$: $\binom{-5}{-8} \overset{(A)}{=} \binom{7}{4} = 35$, while Loeb's value is -35 (odd difference); for $(n,p) = (5,3)$ (even difference) both give $\binom{5}{3} = 10$. □

Neither convention is more correct than the other: Loeb's is characterized by Pascal's rule holding for all integer pairs $\ne (0,0)$; (A)/(B) is characterized instead by consistency under the para-pseudo-derivative classes $D_1, \ldots, D_6$ defined below. What matters is stating the relation precisely rather than treating (A) as unrelated to the existing literature.

Finite operator calculus. Rota's finite operator calculus [5] studies shift-invariant delta operators on polynomial sequences. Section 11 makes precise how D_1, D_2, D_3 relate to it.

4 The para-pseudo-derivative

We now make precise what kind of object a para-pseudo-derivative is, what it acts on, and in what sense the six-class verification below is a nontrivial, falsifiable claim rather than a tautology.

4.1 Domain and linearity

Let V be the $\mathbb{Q}$-vector space freely spanned by symbols $\binom{n+j}{p+i}$, for (n,p) a fixed base point and i, j ranging over $\mathbb{Z}$. A para-pseudo-derivative is a $\mathbb{Q}$-linear map $D : V \to V$, determined by its action on a single generator $\binom{n+j}{p+i}$ via an explicit formula, extended to all of V by linearity.

Remark 3 (On the terminology)**.** The name is descriptive: *pseudo-* because D shares the formal syntax of differentiation (product, quotient, and chain rules, order-m iteration) without being a limit of a rate of change; *para-* because it acts alongside the classical derivative, on the discrete indices (n,p) directly rather than on a continuous variable. It is the definition, not the name, that determines the object.

4.2 The magnitude convention, and why it is not arbitrary

Class D_1 is generated by

$$D_1\binom{a}{b} := |a+1| \cdot \binom{a}{b-1}. \tag{†}$$

The absolute value matters only when $a < 0$: for $a \geq 0$ it is vacuous. We verify that it is not an ad hoc patch, but the unique choice, among the natural alternatives, for which the class is self-consistent at every (n,p).

Proposition 2 (Necessity of the magnitude convention)**.** *For $n \geq p \geq 0$, define the direct value $X := (n+1)\binom{n}{p-1}$ and the mirror value M obtained by applying a rule of the form $c \cdot \binom{a}{b-1}$ to $\binom{a}{b} = \binom{-p-1}{-n-1}$, where $c \in \{a+1, |a+1|\}$ and $\binom{a}{b-1}$ is read either via (A)/(B) (unsigned) or via the signed convention of Proposition 1. Of the four resulting combinations, exactly one satisfies $M = X$ for all $n \geq p \geq 0$: $c = |a+1|$ together with the unsigned reading via (A)/(B) — i.e. (†) itself.*

Proof. Write $a = -p-1$, so $\binom{a}{b-1} = \binom{n+1}{p}$ by (A). For $p \geq 0$, $|a+1| = |-p| = p$ identically, so the unsigned+magnitude mirror value is $p\binom{n+1}{p} = (n+1)\binom{n}{p-1} = X$ by the classical Pascal ratio (2). For unsigned without magnitude, $(a+1)\binom{a}{b-1} = -p\binom{n+1}{p} = -X$ for every $n \geq p \geq 1$. For the signed reading, $\binom{a}{b-1}_{\text{signed}} = (-1)^{(n+1)-p}\binom{n+1}{p}$, and neither the signed nor the signed+magnitude value is identically X. □

The reason this is not surprising a posteriori: the prefactor $(n+1)$ in (2) is a Pascal-ratio multiplicity, always positive on the domain where (2) is stated. When (†) is applied to the mirror pair $(a,b) = (-p-1, -n-1)$, the raw value $a+1 = -p$ is negative but is asked to play the role of the same multiplicity — hence the magnitude is what is structurally meant. The same convention is used, without further tuning, for $D_2, \ldots, D_6$; a full case-by-case proof of necessity in the style of Proposition 2 is left for future work.

4.3 The six classes, condensed

Each class D_i is defined by an order-1 identity, established by one of three routes: D_1 by the mirror-consistency mechanism of §4.2; D_2, D_3 by direct factorial verification; and D_4, D_5, D_6 from Pascal's relation and (composed or iterated) application of D_1.

Remark 4 (This mechanism is specific to D_1)**.** The check of Proposition 2, attempted the same way for D_2 and D_3, succeeds under none of the four sign/magnitude combinations, because D_2, D_3 are diagonal (they multiply without shifting any index): applying such a rule to the mirror pair produces a scalar multiple of $\binom{n}{p}$ itself, never the independently shifted object that the identities of §5 and §8 relate it to.

Proposition 3 (D_5 shares D_1's mirror mechanism; D_4 does not)**.** *Let $D_5(m)\binom{a}{b} := a(a-1)\cdots(a-m+1)\binom{a-m}{b}$ and $D_4(m)\binom{a}{b} := (a+1)\cdots(a+m)\binom{a+m}{b}$. Applying $D_5(m)$'s rule to the mirror pair $\binom{-p-1}{-n-1}$, with the unsigned+magnitude convention of (†), reproduces $D_5(m)\binom{n}{p}$ exactly, for all $n \geq p+m \geq 0$. No sign/magnitude convention makes the analogous statement true for D_4.*

Proof. For D_5: the direct value is $X = F_m(n)\binom{n-m}{p}$, $F_m(n) := n(n-1)\cdots(n-m+1)$. Applying D_5's rule to $(a,b) = (-p-1, -n-1)$ shifts a down by m to $-p-1-m$, with magnitude $(p+1)(p+2)\cdots(p+m)$ (each factor negative, so the magnitude flips every sign). By (A), $\binom{-p-1-m}{-n-1} = \binom{n}{p+m}$. So the mirror value is $(p+1)\cdots(p+m)\binom{n}{p+m}$; both this and X reduce, by cancellation of factorials, to $n!/[p!\,(n-p-m)!]$ — identical. For D_4: applying D_4's rule to $(-p-1, -n-1)$ shifts a up by m to $-p-1+m$; by (A) this reduces to $\binom{n-m}{p}$, not $\binom{n+m}{p}$ as $X = (n+1)\cdots(n+m)\binom{n+m}{p}$ requires — the mirror shift and the direct shift move the row index in opposite senses, so no scalar correction can reconcile them. Concretely, $n=6$, $p=3$, $m=1$: $X = 7\binom{7}{3} = 245$, while every mirror convention gives 45 or its sign variants. □

Remark 5. The relevant distinction is not shift-versus-diagonal but the direction of the shift relative to (A)/(B). D_1 (shifts p down) and D_5 (shifts n down) both work; D_4 (shifts n up) does not. The mirror mechanism is shared by exactly $\{D_1, D_5\}$ among the six classes.

D_1 **(worked example).** Define $\binom{n}{p}' := (n+1)\binom{n}{p-1}$. Applying the same rule to $\binom{-p-1}{-n-1}$ and reducing via (B), with the magnitude convention:

$$\binom{-p-1}{-n-1}' = |-p-1+1|\binom{-p-1}{-n-2} = p\binom{n+1}{p} \quad \text{(via (B))}.$$

Direct factorial expansion of both $(n+1)\binom{n}{p-1}$ and $p\binom{n+1}{p}$ gives the same rational function, so

$$(n+1)\binom{n}{p-1} = p\binom{n+1}{p}, \tag{2}$$

and at general order m,

$$(n+1)(n+2)\cdots(n+m)\binom{n}{p-m} = p(p-1)\cdots(p-m+1)\binom{n+m}{p}. \tag{4}$$

For D_2, D_3 the order-m identities follow from the lemma

$$F_m(b)\binom{a}{b} = F_m(a)\binom{a-m}{b-m}, \qquad F_m(x) := x(x-1)\cdots(x-m+1), \tag{††}$$

valid for integers $a \geq b \geq 0$ by direct factorial expansion. Applying (††) with $a = n$, $b = p$ gives D_2's order-m identity $F_m(p)\binom{n}{p} = F_m(n)\binom{n-m}{p-m}$; with $a = n$, $b = n - p$ and the symmetry $\binom{n}{p} = \binom{n}{n-p}$ it gives D_3's, $F_m(n-p)\binom{n}{p} = F_m(n)\binom{n-m}{p}$.

Class	Order-1 identity	Description
D_1	$(n+1)\binom{n}{p-1} = p\binom{n+1}{p}$	shift $p \to p-1$
D_2	$p\binom{n}{p} = n\binom{n-1}{p-1}$	diagonal, $p \to p$
D_3	$(n-p)\binom{n}{p} = (p+1)\binom{n}{p+1}$	diagonal, decreasing
D_4/D_5	iterated Pascal correction	row shift via Pascal's rule
D_6	$\Delta_k = \big[(n+1)\cdots(n+k) - (n-1)\cdots(n+k-2)\big]\binom{n}{p-k}$	ternary Vandermonde ($k=2$)

5 The corrector, for class D_1

This is where the paper's actual content begins. Let $X := D_1(m)\binom{n}{p} = (n+1)\cdots(n+m)\binom{n}{p-m}$.

Proposition 4 (Vandermonde's identity)**.**

$$\binom{n}{p} = \sum_{i=0}^{k} \binom{k}{i}\binom{n-k}{p-i}. \tag{V_k}$$

Proof. Classical; e.g. compare coefficients of x^p on both sides of $(1+x)^n = (1+x)^k(1+x)^{n-k}$. □

Proposition 5 (Corrector)**.** *For the identity* (V_k)*,*

$$\Delta(V_k, m) := \big[(n+1)\cdots(n+m) - (n-k+1)\cdots(n-k+m)\big]\binom{n}{p-m}. \tag{6.1}$$

Proof. All terms of (V_k) share the lower argument $n-k$; $D_1(m)$ applies the same factor $(n-k+1)\cdots(n-k+m)$ to each, and the sum closes, by (V_k) applied to $(n, p-m)$, into $\binom{n}{p-m}$. The gap with X is the gap between the two multiplicative factors. □

Remark 6 (D_6's correction terms are a special case)**.** D_6's three correction terms are exactly $\Delta(V_2, m)$ for $m = 1, 2, 3$: expanding (6.1) at $k = 2$ gives $\Delta(V_2, 1) = 2\binom{n}{p-1}$, $\Delta(V_2, 2) = 2(2n+1)\binom{n}{p-2}$, $\Delta(V_2, 3) = 6(n+1)^2\binom{n}{p-3}$. D_6 is the $k = 2$ instance of the single general formula.

5.1 Quotient and product laws

Let $R_{V_k}(n, m) := 1 - \Delta(V_k, m)/X$.

Proposition 6.

$$R_{V_k}(n, m) = \frac{F_k(n)}{F_k(n+m)} = \frac{\binom{n}{k}}{\binom{n+m}{k}}, \qquad F_k(a) := a(a-1)\cdots(a-k+1). \tag{6.2}$$

Proof. By (6.1), $R_{V_k}(n, m) = \frac{(n-k+1)\cdots(n-k+m)}{(n+1)\cdots(n+m)}$. Writing numerator and denominator as factorial ratios gives $\frac{n!\,(n+m-k)!}{(n-k)!\,(n+m)!}$; on the other hand $F_k(n)/F_k(n+m) = \frac{n!/(n-k)!}{(n+m)!/(n+m-k)!}$, the same expression. The same ratio is $\binom{n}{k}/\binom{n+m}{k}$ — the probability that a uniform random k-subset of $[n+m]$ lies inside a fixed n-subset, a reading we use for the two laws below. □

Proposition 7 (Quotient law)**.**

$$Q(V_k/V_{k+1}, m) := \frac{R_{V_k}}{R_{V_{k+1}}} - 1 = \frac{m}{n-k}. \tag{6.3}$$

Proof. By the previous proposition $R_{V_k}/R_{V_{k+1}} = \frac{F_k(n)/F_{k+1}(n)}{F_k(n+m)/F_{k+1}(n+m)}$. Since $F_{k+1}(a) = F_k(a)(a-k)$, this is $\frac{n+m-k}{n-k} = 1 + \frac{m}{n-k}$; subtract 1. $\square$

Proposition 8 (Product law)**.** *For two translation identities I, J with relative correctors $r_I := 1 - R_I$, $r_J := 1 - R_J$,*

$$S(I \cdot J, m) := R_I R_J - 1 = -(r_I + r_J - r_I r_J). \tag{6.4}$$

Proof. $R_I R_J - 1 = (1 - r_I)(1 - r_J) - 1 = -r_I - r_J + r_I r_J$, direct expansion. $\square$

Remark 7 (Inclusion-exclusion reading)**.** Since each R is a probability (Proposition 6), r_I, r_J are complementary failure probabilities and the law is inclusion-exclusion: $r_I + r_J - r_I r_J = 1 - R_I R_J$ is the probability that at least one of two independent events fails. The joint corrector is thus never the naive sum $r_I + r_J$ but that sum minus their product; the second-order term $r_I r_J$ is exactly the overlap a Leibniz-type product rule must subtract.

5.2 Matrix formalism

Representing $D_1(m)$ on $V^{(n)} := (\binom{n}{0}, \ldots, \binom{n}{N})^\top$ by the sub-diagonal matrix $M_{D_1(m)}$ with entry $\lambda_m := (n+1)\cdots(n+m)$ at position $(p, p-m)$:

Proposition 9. *$M_{D_1(m)}$ is nilpotent: $M^k_{D_1(m)} = 0$ once $km > N$.*

Proof. $M_{D_1(m)}$ has nonzero entries only at $(p, p-m)$; its k-th power has nonzero entries only at $(p, p-km)$, requiring $p \geq km$. Since p ranges over $0, \ldots, N$, no such p exists once $km > N$. $\square$

5.3 A division operator, and the failure of mirror invariance

The corrector construction is not limited to D_1, and its extensions show that the mirror invariance of Theorem 1 is a special property of D_1, not a general law. Consider the division operator $E\binom{n}{p} := \frac{1}{p}\binom{n+1}{p-1}$, whose prefactor $1/p$ depends on the column. It is diagonal, $E\binom{n}{p} = \frac{n+1}{(n-p+1)(n-p+2)}\binom{n}{p}$, and iterates cleanly to $E^m\binom{n}{p} = \frac{(p-m)!}{p!}\binom{n+m}{p-m}$. Applied term by term to Pascal's rule $\binom{n+1}{p+1} = \binom{n}{p+1} + \binom{n}{p}$ it does not close, since the divisor differs between the columns $p+1$ and p; at order 1,

$$\frac{\binom{n+2}{p}}{p+1} = \frac{\binom{n+1}{p}}{p+1} + \frac{\binom{n+1}{p-1}}{p} - \frac{\binom{n+1}{p}}{(p+1)(n-p+2)}.$$

More generally, E^m multiplies the first two terms by $(p+1-m)!/(p+1)!$ and the third by $(p-m)!/p!$; the first two combine, by Pascal's rule at $(n+m, p+1-m)$, into $\frac{(p+1-m)!}{(p+1)!}\binom{n+m}{p-m}$, so the order-$m$ corrector has the closed form

$$\Delta^E_m = \binom{n+m}{p-m}\left[\tfrac{(p+1-m)!}{(p+1)!} - \tfrac{(p-m)!}{p!}\right] = -\frac{m\,(p-m)!}{(p+1)!}\binom{n+m}{p-m}.$$

Unlike the correctors of D_1, which are binomial coefficients, Δ^E_m is a rational function of the indices — the signature of a column-dependent prefactor.

Now apply the founding equality. Because D_1 multiplies by the row, the mirror of its rule through (A)/(B) is D_1 again; this self-mirror property is exactly what makes Theorem 1 hold. The division operator has no such symmetry: since E divides by the column, its mirror divides by the row,

$$G\binom{n}{p} := \frac{1}{n+1}\binom{n+1}{p-1}, \qquad \Delta_m^G = -\frac{m\,n!}{(n+m+1)!}\binom{n+m+1}{p+1-m},$$

a different operator with a different corrector. The two are genuine duals under the column/row exchange $p \leftrightarrow n$ — Δ_m^E carries $(p-m)!/(p+1)!$ and Δ_m^G carries $n!/(n+m+1)!$ — but they are not equal: e.g. at $(n,p,m) = (5,3,2)$, $\Delta^E = -\frac{7}{12}$ while $\Delta^G = -\frac{1}{6}$. Hence E is not mirror-invariant, and Theorem 1 is seen to be specific to self-mirror operators such as D_1, not a feature of the corrector construction in general. (A further member of the family, $\binom{n}{p} \mapsto \frac{1}{n+1}\binom{n}{p+1}$, raises the column and has the analogous rational corrector $-\binom{n-1}{p+1}/[(n+1)(p+2)]$ at order 1.)

The same asymmetry appears on Vandermonde's identity. Because its right-hand side lies in a single row $n-k$, the row-divider G has a constant divisor there and yields a clean corrector, $\Delta_{V_k}^G = -k\binom{n+1}{p-1}/[(n+1)(n-k+1)]$, whereas E, dividing by the (varying) column, does not factor and its corrector is a sum. An operator's corrector is thus simplest precisely on the identities whose translated terms share the index that operator divides by; on powers of the identity the corrector obeys the general law $C_m = m\,A^{m-1}C_1$ of §5, with C_1 the operator's order-one corrector.

6 Invariance of the corrector under the mirror equality

Theorem 1. *For all $k \geq 1$, $m \geq 1$, and (n,p) in the usual domain, translating (V_k) term by term through (A)/(B) and applying $D_1(m)$ directly to the resulting negative-index symbols gives a corrector $\Delta_{\text{mirror}}(V_k, m)$ equal to the ordinary corrector: $\Delta_{\text{mirror}}(V_k, m) = \Delta(V_k, m)$.*

Proof. By (4) applied to (n,p), $D_1(m)$ on the mirror left side gives exactly X. Applied to each mirror term (base $-(n-k)-1$), it gives, via (4) applied to $(n-k, p-i)$, a common factor $(n-k+1)\cdots(n-k+m)$ times $\binom{n-k}{p-i-m}$; summing over i closes via (V_k) into $(n-k+1)\cdots(n-k+m)\binom{n}{p-m}$ — the same subtracted term as in (6.1). □

7 Two families

Define $D_4(m)\binom{a}{b} := (a+1)\cdots(a+m)\binom{a+m}{b}$, $D_5(m)\binom{a}{b} := a(a-1)\cdots(a-m+1)\binom{a-m}{b}$.

Theorem 2. *D_4 and D_6 share D_1's corrector law, $r = 1 - F_k(n)/F_k(n+m)$; D_2, D_3, D_5 share the law $r = 1 - F_m(n-k)/F_m(n)$ — the same law with $k \leftrightarrow m$ exchanged and the base translated by k.*

Proof. For D_4: applied to each term of (V_k) (common base $n-k$), $D_4(m)$ factors out $(n-k+1)\cdots(n-k+m)$, which closes via (V_k) at $(n+m,p)$ into $\binom{n+m}{p}$; this common factor cancels in R_{D_4}, giving the same law as R_{D_1}. For D_5: same argument with common factors $F_m(n)$ and $F_m(n-k)$, giving $r_{D_5} = 1 - F_m(n-k)/F_m(n)$. For D_2, D_3: use (††); applying it term by term factors out $F_m(n-k)$ and closes via (V_k), giving $r = 1 - F_m(n-k)/F_m(n)$. For D_6: Theorem 1 already identifies its mirror corrector with D_1's. □

This gives two families: $\mathcal{A} = \{D_1, D_4, D_6\}$, $\mathcal{B} = \{D_2, D_3, D_5\}$. The quotient law of Family $\mathcal{B}$ lives in the complementary direction from Family $\mathcal{A}$'s: at fixed depth k, varying order m, $Q_{\mathcal{B}}(m, m+1, k) =$

$k/(n-k-m)$, exactly the depth-quotient law of Family $\mathcal{A}$ evaluated at base $n-k$, order k, depths $m, m+1$.

8 Operator algebra

Extend D_1, D_2, D_3 to linear operators on $f(n,p)$: $D_1(m)[f](n,p) := (n+1)\cdots(n+m)f(n,p-m)$, $D_2(m)[f] := F_m(p)f$, $D_3(m)[f] := F_m(n-p)f$.

Proposition 10.

$$[D_1(m_1), D_2(1)] = -m_1 D_1(m_1), \qquad [D_1(m_1), D_3(1)] = +m_1 D_1(m_1),$$

and $[D_2, D_3] = 0$ *identically.*

Proof. Write $\lambda := (n+1)\cdots(n+m_1)$. Then $D_1(m_1) \circ D_2(1)[f](n,p) = \lambda(p-m_1)f(n,p-m_1)$ and $D_2(1) \circ D_1(m_1)[f](n,p) = p\lambda f(n,p-m_1)$; subtracting gives $-m_1 D_1(m_1)[f]$. Similarly $D_1(m_1) \circ D_3(1) = \lambda(n-p+m_1)f(n,p-m_1)$ and $D_3(1) \circ D_1(m_1) = (n-p)\lambda f(n,p-m_1)$; subtracting gives $+m_1 D_1(m_1)$. Finally $D_2(m_2) \circ D_3(m_3) = F_{m_2}(p)F_{m_3}(n-p)f$ is symmetric in the two scalar (diagonal) multiplications, which commute. ☐

Proposition 11 (Duality by conjugation)**.** *Let* $S[f](n,p) := f(n, n-p)$. *Then* $S^2 = \mathrm{Id}$ *and* $S \circ D_2(m) \circ S = D_3(m)$ *for all* m*;* S *does not conjugate* D_1 *to itself.*

Proof. $S^2[f](n,p) = f(n, n-(n-p)) = f(n,p)$, so $S^2 = \mathrm{Id}$. Next $S \circ D_2(m) \circ S[f](n,p) = F_m(n-p)f(n,p) = D_3(m)[f](n,p)$. A direct check ($S \circ D_1(1) \circ S[f](n,p) = (n+1)f(n,p+1)$, generically different from $D_1(1)[f] = (n+1)f(n,p-1)$) shows S does not conjugate D_1 to itself. ☐

Proposition 12 (Periodic compositions)**.** *For a word* W *of* r *order-1 factors from* $\{D_1, D_2, D_3\}$ *with* s *copies of* D_1*,* $W\binom{n}{p} \propto \binom{n}{p-s}$*: the net shift depends only on the count of* D_1 *factors, e.g.*

$$(D_1 \circ D_2)^k \binom{n}{p} = (n+1)^k F_k(p-1)\binom{n}{p-k}, \qquad (D_2 \circ D_1)^k \binom{n}{p} = (n+1)^k F_k(p)\binom{n}{p-k}.$$

Proof. Process the word right to left, tracking the current lower index q (initially p). A factor $D_2(1)$ or $D_3(1)$ multiplies by q or $n-q$ and leaves q unchanged; $D_1(1)$ multiplies by $(n+1)$ and decreases q by 1. After r factors q has decreased by s. Iterating k times telescopes along $p, p-s, \ldots$; for $s=1$ it evaluates to $(n+1)^k F_k(p-1)$ or $(n+1)^k F_k(p)$. ☐

9 Discrete integral, and its corrector duality

Definition 2. For $\kappa \geq 1$, $D_1(-\kappa)\binom{n}{p} := \binom{n}{p+\kappa}/R_\kappa(n)$, where $R_\kappa(a) := (a+1)\cdots(a+\kappa)$.

Proposition 13. $D_1(\kappa)\big[D_1(-\kappa)\binom{n}{p}\big] = \binom{n}{p}$*: the discrete integral inverts* $D_1(\kappa)$*.*

Proof. Let $g(n,p) := D_1(-\kappa)\binom{n}{p} = \binom{n}{p+\kappa}/R_\kappa(n)$. Then $D_1(\kappa)[g](n,p) = R_\kappa(n)g(n,p-\kappa) = R_\kappa(n)\binom{n}{p}/R_\kappa(n) = \binom{n}{p}$. ☐

9.1 A duality between the derivative and the integral

Proposition 14. *$R_{V_k}(n, -\kappa) = R_\kappa(n)/R_\kappa(n-k)$.*

Proof. $X_{-\kappa} = \binom{n}{p+\kappa}/R_\kappa(n)$. Applying $D_1(-\kappa)$ term by term to (V_k) (base $n-k$) and closing via (V_k) at $(n, p+\kappa)$ gives $\binom{n}{p+\kappa}/R_\kappa(n-k)$; the ratio gives the result. $\square$

Theorem 3 (Derivative/integral duality). *For all $k, \kappa \geq 1$ and (n,p) in the usual domain, $R_{V_k}(n, -\kappa) = 1/R_{V_k}(n, \kappa)$.*

Proof. By the previous proposition $R_{V_k}(n, -\kappa) = R_\kappa(n)/R_\kappa(n-k)$; by (6.2) at $m = \kappa$, $R_{V_k}(n, \kappa) = R_\kappa(n-k)/R_\kappa(n)$. The product is 1. $\square$

The naive substitution $m \mapsto -\kappa$ directly in (6.2) does not give this value — it would give $F_k(n)/F_k(n-\kappa)$, which differs from the correct $F_k(n+\kappa)/F_k(n)$.

Proposition 15 (No analogous duality for Family $\mathcal{B}$). *Extending D_5 to negative order the same way, $R^{D_5}_{V_k}(n, -\mu) \cdot R^{D_5}_{V_k}(n, \mu) \neq 1$ in general. For $k = 1$, $n = 20$, $p = 4$, $\mu = 1$: $R_{D_5}(n, \mu) = 19/20$ and $R_{D_5}(n, -\mu) = 21/20$, whose product is $399/400 \neq 1$.*

Proof. One counterexample suffices, as displayed. The mechanism: D_1 acts on p, separate from the depth-shift of (V_k) on n, so the sum closes identically in both directions; D_5 (and D_2) act on the same index n that the depth of (V_k) shifts, breaking the telescoping closure. $\square$

9.2 The failure is exact, but the duality is restored asymptotically

Proposition 16. *For $k = 1$ and any $\mu \geq 1$, $R^{D_5}_{V_1}(n, \mu) \cdot R^{D_5}_{V_1}(n, -\mu) = \frac{(n-\mu)(n+\mu)}{n^2} = 1 - \frac{\mu^2}{n^2}$.*

Proof. $F_\mu(n-1)/F_\mu(n) = (n-\mu)/n$ and $R_\mu(n)/R_\mu(n-1) = (n+\mu)/n$ by telescoping; the product follows. $\square$

Proposition 17 (Asymptotic restoration). *For every $k \geq 1$ and $\mu \geq 1$, $\lim_{n\to\infty} R^{D_5}_{V_k}(n, \mu) \cdot R^{D_5}_{V_k}(n, -\mu) = 1$.*

Proof. The product is a ratio of two polynomials in n of equal degree 2μ with the same leading coefficient; the limit of the ratio of leading terms is 1. $\square$

Proposition 18. *$[D_1(1), D_5(m_2)]\binom{n}{p} = +m_2 \cdot n(n-1)\cdots(n-m_2+1) \cdot \binom{n-m_2}{p-1}$: D_1 and D_5, though acting on distinct indices, do not commute.*

Proof. Write $F_{m_2}(n) := n(n-1)\cdots(n-m_2+1)$. Then $D_1(1) \circ D_5(m_2)[f](n,p) = (n+1)F_{m_2}(n)f(n-m_2, p-1)$, while $D_5(m_2) \circ D_1(1)[f](n,p) = F_{m_2}(n)(n-m_2+1)f(n-m_2, p-1)$: the prefactor from D_1 is evaluated at n in the first case, at $n - m_2$ in the second. Subtracting, $[D_1(1), D_5(m_2)][f] = m_2 F_{m_2}(n) f(n-m_2, p-1)$. $\square$

Remark 8. An earlier version of this note stated this commutator with a minus sign, following the founding document without independent re-derivation. Direct symbolic recomputation gives $+m_2$, not $-m_2$; the qualitative conclusion (non-commutation) is unaffected, but the sign is corrected here.

10 q-analogue

Let $[n]_q := (1-q^n)/(1-q)$, $[n]_q! := [1]_q \cdots [n]_q$, $\binom{n}{p}_q := [n]_q!/([p]_q!\,[n-p]_q!)$.

Proposition 19.

$$[n+1]_q \binom{n}{p-1}_q = [p]_q \binom{n+1}{p}_q \ (D_1), \quad [p]_q \binom{n}{p}_q = [n]_q \binom{n-1}{p-1}_q \ (D_2),$$

$$[n-p]_q \binom{n}{p}_q = [p+1]_q \binom{n}{p+1}_q \ (D_3),$$

each verified by direct q-factorial expansion.

Proof. For D_1: $[n+1]_q \binom{n}{p-1}_q = [n+1]_q!/([p-1]_q!\,[n-p+1]_q!)$ and $[p]_q \binom{n+1}{p}_q = [n+1]_q!/([p-1]_q!\,[n+1-p]_q!)$; these agree since $n-p+1 = n+1-p$. The D_2, D_3 identities follow by the same cancellation, using $[a]_q \cdot [a-1]_q! = [a]_q!$. □

Proposition 20 (q-commutators)**.**

$$[D_1^{(q)}(1), D_2^{(q)}(1)] = -q^{p-1} D_1^{(q)}(1), \qquad [D_1^{(q)}(1), D_3^{(q)}(1)] = +q^{n-p} D_1^{(q)}(1),$$

deformed by different powers of q — unlike the classical case where the two commutators differ only in sign.

Proof. $D_1^{(q)}(1) \circ D_2^{(q)}(1) \binom{n}{p}_q = [n+1]_q [p-1]_q \binom{n}{p-1}_q$ and $D_2^{(q)}(1) \circ D_1^{(q)}(1) \binom{n}{p}_q = [p]_q [n+1]_q \binom{n}{p-1}_q$; the difference is $[n+1]_q([p-1]_q - [p]_q)\binom{n}{p-1}_q$, and $[p]_q - [p-1]_q = q^{p-1}$, giving the first. The second follows from $[n-p+1]_q - [n-p]_q = q^{n-p}$. □

10.1 Resolving the open question for class D_1

Theorem 4 (q-analogue of the founding equality, for D_1)**.** *Define, for $n \geq p \geq 0$,*

$$\binom{-p-1}{-n-1}_q := -q^p \binom{n}{p}_q. \qquad (A_q)$$

Then applying the q-analogue of the D_1 rule to both sides of (A_q) gives the same identity on both sides — i.e. (A_q) is consistent with $D_1^{(q)}$ in the same sense that (A)/(B) is consistent with D_1.

Proof. Direct side: applying $D_1^{(q)}$ to $\binom{n}{p}_q$ gives $[n+1]_q \binom{n}{p-1}_q$. Mirror side: applying the same structural rule to $\binom{-p-1}{-n-1}_q$ gives $[-p]_q \cdot \binom{-p-1}{-n-2}_q$. By (A_q) (with $N = n+1$, $P = p$), $\binom{-p-1}{-n-2}_q = -q^p \binom{n+1}{p}_q$. Using $[-p]_q = -q^{-p}[p]_q$: mirror side $= -q^{-p}[p]_q \cdot (-q^p \binom{n+1}{p}_q) = [p]_q \binom{n+1}{p}_q$, which equals the direct side by the D_1 q-identity. □

Remark 9 (Why D_2, D_3 resist the same argument)**.** D_2, D_3 have no mirror-consistency mechanism to close, q-deformed or not: applying a diagonal rule to the mirror pair produces a scalar multiple of $\binom{n}{p}$ itself, a structural fact unaffected by specializing $q \to 1$. Extending (A_q) to D_2, D_3 is left as an open question.

11 Comparison with Rota's finite operator calculus

For fixed n, let $F_n(x) := (1+x)^n = \sum_p \binom{n}{p} x^p$.

Proposition 21. *Reading off the coefficient of x^p:*

$$(n+1)\binom{n}{p-1} = [x^p]\big((n+1)xF_n(x)\big), \quad p\binom{n}{p} = [x^p]\big(xF_n'(x)\big), \quad (n-p)\binom{n}{p} = [x^p]F_n'(x),$$

corresponding to D_1, D_2, D_3 respectively.

Proof. $(n+1)xF_n(x) = \sum_q (n+1)\binom{n}{q-1}x^q$, giving the first identity. $xF_n'(x) = \sum_p p\binom{n}{p}x^p$, the second. $F_n'(x) = n(1+x)^{n-1} = \sum_p n\binom{n-1}{p}x^p$; combined with $(n-p)\binom{n}{p} = n\binom{n-1}{p}$ this gives the third. □

The identity $(n-p)\binom{n}{p} = n\binom{n-1}{p}$ makes the third line exact: D_3, applied to the coefficients of $F_n(x)$, is d/dx — the canonical delta operator of [5]; $D_2 = x\, d/dx$ is the associated Euler operator; D_1 plays the role of the dual multiplication operator.

Proposition 22 (The correspondence does not extend to composition)**.** *Composing $D_1^c := (n+1)x \cdot (-)$ and $D_3^c := d/dx$ as literal operators on $F_n(x)$, with n held fixed, does not reproduce the combinatorial commutator $[D_1(1), D_3(1)] = D_1(1)$ of §8.*

Proof. $D_3^c(D_1^c F_n) - D_1^c(D_3^c F_n) = (n+1)F_n(x)$, i.e. $[D_3^c, D_1^c] = (n+1)\cdot \mathrm{Id}$ as a classical operator identity, for every n. But $D_1(1)$ is a shift operator (changing p), while $(n+1)\cdot\mathrm{Id}$ fixes p; a nonzero scalar multiple of the identity cannot equal a genuine shift operator on a space where $\binom{n}{p}$, $p = 0, \ldots, n$, are linearly independent. Concretely, for $n = 6$: the classical commutator $[D_1^c, D_3^c]$ has coefficients $(-7, -42, -105, \ldots)$ against the expected $(0, 7, 42, \ldots)$ from $D_1(1)$. □

The combinatorial D_1, D_3 carry n as a parameter re-read at each step of a composition, while the classical operators freeze n from the start. Section 4's order-1 identities are classical facts, while the operator algebra of §8, the corrector theory of §§5–7, and the duality of §9.1 are not reducible to it.

12 Conclusion

The founding equality (A)/(B) is one of several valid, self-consistent extensions of $\binom{n}{p}$ to negative indices; §3 states precisely how it relates to Loeb's canonical extension (a sign $(-1)^{n-p}$), to Roman's negative-integer factorial (a sign $(-1)^n$ at the factorial level, §2), and to Rota's finite operator calculus (§11, exact for a single application, not under composition). The content specific to this paper is the operator calculus built on (A)/(B): the corrector and its quotient/product laws (§5), its invariance under the mirror equality (§6), the two-family classification (§7), the closed operator algebra (§8), the derivative/integral duality (§9.1) — proved for Family $\mathcal{A}$ and disproved for Family $\mathcal{B}$ — and a proven q-analogue closure for D_1 (§10.1).

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