ON QUASI CLOSED CHAINS AND QUASI COMPLETE NUMBERS

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Abstract. An addition chain of length h that leads to an integer $n \geq 2$ is a strictly increasing sequence of positive integers $s_0 = 1, s_1 = 2, \ldots, s_h = n$ such that $s_i = s_j + s_k$ with $i > j \geq k \geq 0$ for each $i \in \{1, \ldots, h\}$. We denote the minimal length of an addition chain that leads to an integer target $\cdot$ by $\ell(\cdot)$. In this paper, we introduce the concept of *quasi closed* addition chains and show that numbers $n \geq 2$ that admit a minimal-length quasi closed chains that are also a minimal-length addition (quasi complete numbers) satisfy the Scholz conjecture, precisely the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n).$$

1. Introduction

The addition chains are combinatorial arithmetic structures (sequences) that play an important role in fast exponentiation [5, 6, 7, 8]. Formally speaking, an addition chain of length h that leads to a positive integer $n \geq 2$, introduced by the German mathematician (see, e.g., [1]) Arnold Scholz in 1937, is a strictly increasing sequence of positive integers

$$s_0 = 1, s_1 = 2, \ldots, s_h = n$$

such that $s_i = s_j + s_k$ with $i > j \geq k \geq 0$ for each $i \in \{1, \ldots, h\}$. The smallest positive integer s for which there exists an addition chain leading to $n \geq 2$ such that $\ell(n) = s$ is the minimal length of the chain. The central problem in studies of addition chains is the speculative inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n)$$

conjectured by Arnold Scholz in his foundational studies [1]. This is now widely known as the Scholz conjecture on addition chains.

Various approaches to addressing the Scholz conjecture have been discovered, ranging from structural to digital-arithmetic approaches [9, 10, 11, 4]. The trendiest and most tractable approach is to find a sufficient condition(s) fulfilled by an addition chain that naturally satisfies the inequality $\ell(2^n - 1) \leq n - 1 + \ell(n)$. In this way, positive integers that admit the family of addition chain(s) fulfilling this sufficient condition satisfy the Scholz conjecture. The German mathematician Alfred Brauer (see, e.g., [2]) was one of the pioneers of this line of thought in addressing the Scholz conjecture.

The Brauer addition chain of length h that leads to a positive integer $n \geq 2$ is an addition chain

$$s_0 = 1, s_1 = 2, \ldots, s_h = n$$

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such that $s_i = s_{i-1} + s_j$ for each $i \in \{1, \ldots, h\}$. We denote the minimal length of a Brauer addition chain that leads to $\cdot$ by $\ell^*(\cdot)$. In [2], it is shown that the inequality

$$\ell^*(2^n - 1) \leq n - 1 + \ell^*(n)$$

is satisfied by the family of all Brauer addition chains leading to $n \geq 2$. This implies that the Scholz conjecture is true for all positive integers that admit a minimal length chain, which is also a *Brauer* addition chain (i.e. $\ell^*(n) = \ell(n)$). These numbers are now known in the literature as *Brauer* numbers

Walter Hansen further pursued this attack strategy in his later work on addition chains. Rather like Brauer, Hansen introduced a family of addition chains which one may regard as a relaxed version of the *Brauer* addition chains. Formally speaking, a *Hansen* chain of length h leading to an integer $n \geq 2$ is an addition chain

$$s_0 = 1, s_2 = 2, \ldots, s_h = n$$

such that there exists a subset H (a.k.a. anchor) with $H \subseteq \{s_0, s_1, \ldots, s_h\}$ so that with $s_i = s_j + s_k$ for $i > j \geq k \geq 0$ for each $i \in \{1, \ldots, h\}$, we have at least

$$s_j = \max\{l \in H \ : \ l < s_i\}$$

or

$$s_k = \max\{l \in H \ : \ l < s_i\}.$$

We denote the minimal length of a *Hansen* chain that leads to an integer $\cdot$ by $\ell^o(\cdot)$. In the family of Hansen chains, it is shown (see, e.g., [3]) that

$$\ell(2^n - 1) \leq n - 1 + \ell^o(n).$$

This implies that the Scholz conjecture holds for integers $n \geq 2$ that admit a minimal length addition chain which are also a minimal length chain in the family of all addition chains leading to $n \geq 2$ (i.e. $\ell(n) = \ell^o(n)$). These numbers are now known in the literature as *Hansen* numbers.

In the current investigation, we introduce a completely different family of addition chains that mirror the philosophy of these approaches. The family of addition chains introduced subsumes *Brauer* and a certain class of *Hansen* addition chains. The sufficient conditions that have been assigned to our family of addition chains are based on a close inspection of the terms that are used to calculate subsequent terms in the chain rather than the entire sequence of the chain, treating them as structures in a dynamical system.

1.1. **Organization of the paper.** The paper is organized as follows. In section 2, we review the internal structure of an addition chain and specify the step classifications of terms in an addition chain. In section 3, we introduce the new family of addition chains and show that these family of chains satisfy a Scholz-Brauer-type inequality that relies on the internal structure of the chain. In section 4, we show that the family of Brauer chains and a certain class of Hansen chains are subfamilies of the newly introduced chain in the paper. In section 5, we show that the numbers that admit a minimal length chain in the family of chains introduced in the paper that are also a minimal length chain satisfy the Scholz conjecture. The final section (section 6) contains a brief commentary on the current investigation and possible numerical work in search for those numbers.

2. The internal geometry and dynamics of addition chains and functionals

In this section, we specify the internal geometry of an addition chain as terms used to calculate subsequent terms in the chain.

We let

$$s_0 = 1 < s_1 = 2 < \cdots < s_h = n$$

be a fixed addition chain of length h leading to n together with the fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $1 \leq i \leq h$ with $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. We say that a step i is a *Brauer* step if and only if $\sigma(i) = i - 1$. Otherwise, we say that step i is a non-Brauer step. We denote the set of all Brauer and non-Brauer steps in an addition chain by B and NB, respectively. We write

$$\mathrm{B} := \{i \ : \ \sigma(i) = i - 1, \ s_i = s_{\sigma(i)} + s_{\tau(i)}, \ \sigma(i) \geq \tau(i)\}$$

and

$$\mathrm{NB} := \{i \ : \ \sigma(i) < i - 1, \ s_i = s_{\sigma(i)} + s_{\tau(i)}, \ \sigma(i) \geq \tau(i)\}.$$

We denote the cardinality of each of these sets by $|\mathrm{B}|$ and $|\mathrm{NB}|$, respectively. For an addition chain of length h, we have

$$h = |\mathrm{B}| + |\mathrm{NB}|.$$

We illustrate this step classification with the following example:

Example 2.1. Consider the addition chain $C_{16} : s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 9, s_5 = 16$ with sums

$$\begin{aligned}
s_1 &= 2 = 1 + 1 = s_{\sigma(1)} + s_{\tau(1)} \\
s_2 &= 4 = 2 + 2 = s_{\sigma(2)} + s_{\tau(2)} \\
s_3 &= 8 = 4 + 4 = s_{\sigma(3)} + s_{\tau(3)} \\
s_4 &= 9 = 8 + 1 = s_{\sigma(4)} + s_{\tau(4)} \\
s_5 &= 16 = 8 + 8 = s_{\sigma(5)} + s_{\tau(5)}.
\end{aligned}$$

Here, $\mathrm{B} = \{1, 2, 3, 4\}$ and $\mathrm{NB} = \{5\}$.

Definition 2.2. Let $n \geq 2$ and $s_0 = 1, s_1 = 2, \ldots, s_h = n$ be a fixed addition chain leading to n. We say that $s_i < s_j$ for $1 \leq i, j, \leq h$ are consecutive terms in the addition chain if there are no terms s_k in the addition chain such that $s_i < s_k < s_j$. In particular, if $s_i < s_j$ are consecutive terms in the addition chain, then $i = j - 1$.

Definition 2.3 (Addition chain σ-τ tracks)**.** Let $s_0 = 1, s_1 = 2, \ldots, s_h = n$ be an addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ such that $i > \sigma(i) \geq \tau(i)$ for each $1 \leq i \leq h$. We call the sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ a σ track and the sequence $\{s_{\tau(i)}\}_{i=1}^{h}$ a τ track of the addition chain. For $i = 0$, we set $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. We say that a σ track is strictly increasing if

$$s_{\sigma(i)} < s_{\sigma(i+1)}$$

for each $1 \leq i \leq h - 1$. Similarly, we say that a τ track is strictly increasing if

$$s_{\tau(i)} < s_{\tau(i+1)}$$

for each $1 \leq i \leq h - 1$. We say that a σ and τ tracks are *nondecreasing* if

$$s_{\sigma(i)} \leq s_{\sigma(i+1)}$$

and

$$s_{\tau(i)} \leq s_{\tau(i+1)}$$

for each $1 \leq i \leq h$.

Definition 2.4 (The total variation functional)**.** Let $C_n : s_0 = 1, s_1 = 2, \ldots, s_h = n$ be a fixed addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $1 \leq i \leq h$. With the convention $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} := 0$ [1], we define the *total variation* of the chain C_n as

$$\mathbb{TV}(C_n) := \min_{H(C_n)} \min_{L(C_n)} \left(s_{\tau(h)} + \sum_{i=0}^{h-1} |s_{\sigma(i+1)} - s_{\sigma(i)}| \right).$$

Here, the minimum is taken over the collection of all vectors

$$H(C_n) := \left\{ \begin{pmatrix} \tau(1) \\ \tau(2) \\ \vdots \\ \tau(h) \end{pmatrix} \right\}$$

and

$$L(C_n) := \left\{ \begin{pmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(h) \end{pmatrix} \right\}$$

of the τ and σ values so that $s_i = s_{\sigma(i)} + s_{\tau(i)}$ for each $1 \leq i \leq h$ for the fixed addition chain C_n.

3. Quasi closed addition chains and the Scholz-type inequality

In this section, we introduce and study a more flexible family of addition chains. This family subsumes classical families such as the *Brauer* and a certain class of *Hansen* addition chains. The structure of this new family allows us to develop inequalities of the Scholz-Brauer type.

Definition 3.1 (Quasi closed addition chains)**.** Let

$$s_0 = 1, s_1 = 2, \ldots, s_h = n$$

be an addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ such that $i > \sigma(i) \geq \tau(i)$ for each $1 \leq i \leq h$ and $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$ [2]. We say that the chain is *quasi closed* if there exists some choice of the σ track $\{s_{\sigma(i)}\}_{i=1}^{h}$ and τ track $\{s_{\tau(i)}\}_{i=1}^{h}$ with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ for each $i \in \{1, \ldots, h\}$ such that each of the following conditions is satisfied

(1) For any pair $(s_{\tau(k)}, s_{\sigma(l)})$ with $s_{\tau(k)} > 1$ for some $1 \leq k, l \leq h$ such that $s_{\sigma(l)} < s_{\tau(k)}$ are consecutive terms in the chain with $s_{\tau(k)} \neq s_{\sigma(l)}$ for all $l \in \{0, 1, \ldots, h\}$, there exists some $j \in \{0, 1, \ldots, h\}$ such that

$$s_{\tau(k)} - s_{\sigma(l)} = s_j.$$

[1]see the definition 2.3

[2]see definition 2.3

(2) For each $s_{\sigma(i)} - s_{\sigma(i+1)} \neq 0$, there exists some $j \in \{0, 1, \ldots, h\}$ such that

$$|s_{\sigma(i)} - s_{\sigma(i+1)}| = s_j.$$

(3) For each $s_j = s_{\tau(i)}$ with $s_{\tau(i)} \neq s_{\sigma(k)}$ for all $k \in \{0, 1, \ldots, h\}$ there exists some $s_{\sigma(k)}$ such that $s_{\sigma(k)} < s_{\tau(i)}$ are consecutive terms in the chain [3] with $s_{\tau(i)} \leq s_{\sigma(k+1)}$ for $k \in \{0, \ldots, h-1\}$.

We denote the minimal length of a quasi closed addition chain leading to $\cdot$ by $\ell^{\ddagger}(\cdot)$. Furthermore, we say that the addition chain is *strongly quasi closed* if the above conditions are satisfied for each choice of σ and τ track with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ for each $i \in \{1, \ldots, h\}$.

Example 3.2. We consider the addition chain

$$s_0 = 1, s_1 = 2 = 1+1, s_2 = 4 = 2+2, s_3 = 5 = 4+1, s_4 = 8 = 4+4,$$

$$s_5 = 16 = 8+8, s_6 = 21 = 16+5$$

that leads to 21. Here, the σ track terms are

$$s_{\sigma(1)} = 1, s_{\sigma(2)} = 2, s_{\sigma(3)} = 4, s_{\sigma(4)} = 4, s_{\sigma(5)} = 8, s_{\sigma(6)} = 16$$

and tau track

$$s_{\tau(1)} = 1, s_{\tau(2)} = 2, s_{\tau(3)} = 1, s_{\tau(4)} = 4, s_{\tau(5)} = 8, s_{\tau(6)} = 5.$$

Here, the only τ-track term that satisfies $s_{\tau(k)} \neq s_{\sigma(l)}$ for each $l \in \{1, 2, 3, 5, 6\}$ is $s_{\tau(6)} = 5$. For this τ-track term, we observe that $s_{\sigma(4)} < s_{\tau(6)} = 5$ are consecutive terms in the addition chain. In this case, we get $s_{\tau(6)} - s_{\sigma(4)}$ is a term in the addition chain. Furthermore, we observe that $s_{\tau(6)} = 5 \leq s_{\sigma(5)}$. Also, we observe that the absolute differences

$$|s_{\sigma(i)} - s_{\sigma(i-1)}|$$

for each $i \in \{1, \ldots, 6\}$ is a term in the addition chain provided that $s_{\sigma(i)} - s_{\sigma(i-1)} \neq 0$. Therefore, the addition chain is *quasi closed*.

Example 3.3. We consider the addition chain

$$s_0 = 1, s_1 = 2 = 1+1, s_2 = 4 = 2+2, s_3 = 5 = 4+1, s_4 = 8 = 4+4, s_5 = 13 = 8+5$$

leading to $n = 13$. Here, the σ-track terms are

$$s_{\sigma(1)} = 1, s_{\sigma(2)} = 2, s_{\sigma(3)} = 4, s_{\sigma(4)} = 4, s_{\sigma(5)} = 8$$

and τ track terms

$$s_{\tau(1)} = 1, s_{\tau(2)} = 2, s_{\tau(3)} = 1, s_{\tau(4)} = 4, s_{\tau(5)} = 5.$$

Here, the only τ-track term that satisfies $s_{\tau(k)} \neq s_{\sigma(l)}$ for each $l \in \{1, 2, 3, 4, 5\}$ is $s_{\tau(5)} = 5$. For this τ-track term, we observe that $4 = s_{\sigma(4)} < s_{\tau(5)}$ are consecutive terms in the chain and $5 = s_{\tau(5)} \leq s_{\sigma(5)} = 8$. In this case, we get $s_{\tau(5)} - s_{\sigma(4)}$ is a term in the addition chain. Also, we observe that the absolute differences

$$|s_{\sigma(i)} - s_{\sigma(i-1)}|$$

for each $i \in \{1, 2, 3, 4, 5\}$ is a term in the addition chain provided that $s_{\sigma(i)} - s_{\sigma(i-1)} \neq 0$. Therefore, the addition chain is *quasi closed*.

Remark 3.4. The quasi closed addition chains provided in the two preceding examples are not Brauer.

[3] see the definition 2.2

Theorem 3.5. *Let $n \geq 2$ admit a minimal-length strongly quasi closed addition chain C_n. We have*

$$\ell(2^n - 1) \leq \mathbb{TV}(C_n) + \ell^{\ddagger}(n)$$

Proof. We consider a fixed *strongly quasi closed* addition chain

$$s_0 = 1, s_1 = 2, \ldots, s_h = n$$

with a fixed decomposition $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i))$ for each $1 \leq i \leq h$ $(h := \ell^{\ddagger}(n))$ and $s_{\sigma(0)} := s_0 = 1$ and $s_{\tau(0)} = 0$. This implies that the addition chain is *quasi closed*.[4] Here, the optimality of the chain is with respect to the family of *quasi closed addition chains*. It may or may not be a minimal-length addition chain leading to n in the family of all addition chains leading to n. We define $m_i = 2^{s_i} - 1$ for $i = 0, \ldots, h$ and construct the sequence

$$2^{s_0} - 1 := 2^1 - 1, 2^{s_1} - 1 = 2^2 - 1, \ldots, 2^{s_i} - 1, \ldots, 2^{s_h} - 1 = 2^n - 1.$$

We call each term of this sequence a *Mersenne seed*. Now, if $s_{\sigma(i)} < s_{\sigma(i+1)}$, then we perform a $(s_{\sigma(i+1)} - s_{\sigma(i)})$ repeated doubling of the term $2^{s_{\sigma(i)}} - 1$ and include the result of each doubling in the sequence. On the other hand, if $s_{\sigma(i)} > s_{\sigma(i+1)}$, then we perform a $(s_{\sigma(i)} - s_{\sigma(i+1)})$ repeated doubling of the term $2^{s_{\sigma(i+1)}} - 1$ and include the result of each doubling in the sequence. For terms of the form $2^{s_j} - 1$ such that $j = \sigma(k)$ for some $k \in \{1, \ldots, h\}$ as indices not used in the previous doubling process, we perform a $(s_k - s_{\sigma(k)})$ repeated doubling of the term $2^{s_{\sigma(k)}} - 1$ and include the result of each doubling in the sequence. We now verify that this construction produces an addition chain that leads to $2^n - 1$. In the case $s_{\sigma(i)} < s_{\sigma(i+1)}$, we can write

$$2^{s_{\sigma(i+1)}} - 1 = 2^{s_{\sigma(i+1)} - s_{\sigma(i)}}(2^{s_{\sigma(i)}} - 1) + 2^{s_{\sigma(i+1)} - s_{\sigma(i)}} - 1$$

since the chain

$$s_0 = 1, s_1 = 2, \ldots, s_h = n$$

is *quasi closed*. Similarly, in the case $s_{\sigma(i)} > s_{\sigma(i+1)}$, we can write

$$2^{s_{\sigma(i)}} - 1 = 2^{s_{\sigma(i)} - s_{\sigma(i+1)}}(2^{s_{\sigma(i+1)}} - 1) + 2^{s_{\sigma(i)} - s_{\sigma(i+1)}} - 1$$

since the chain

$$s_0 = 1, s_1 = 2, \ldots, s_h = n$$

is *quasi closed*. The optimality of the quasi closed chain implies

$$\{s_{\sigma(k)}\}_{k=1}^{h} \cup \{s_{\tau(k)}\}_{k=1}^{h} = \{s_k\}_{k=0}^{h-1}.$$

It is clear that

$$\{s_{\sigma(k)}\}_{k=1}^{h} \cup \{s_{\tau(k)}\}_{k=1}^{h} \subseteq \{s_k\}_{k=0}^{h-1}.$$

We suppose that

$$\{s_{\sigma(k)}\}_{k=1}^{h} \cup \{s_{\tau(k)}\}_{k=1}^{h} \subset \{s_k\}_{k=0}^{h-1}.$$

This implies that there exists a term $s_l \in \{s_k\}_{k=0}^{h-1}$ such that

$$s_l \notin \{s_{\sigma(k)}\}_{k=1}^{h} \cup \{s_{\tau(k)}\}_{k=1}^{h}$$

which means that the term s_l was never used to calculate a term in the addition chain. Therefore,

$$\{s_k\}_0^h \setminus \left(\{s_k\}_{k=0}^{h-1} \setminus \left(\{s_{\sigma(k)}\}_{k=1}^{h} \cup \{s_{\tau(k)}\}_{k=1}^{h}\right)\right) \subset \{s_k\}_{k=0}^{h}$$

[4] Strong quasi closure implies quasi closure

is still a quasi closed addition chain that leads to n when the terms are arranged in strictly increasing order of magnitude, by virtue of the *quasi closed* condition. The length of this quasi closed addition chain is strictly less than the length of the fixed optimal quasi closed addition chain, violating minimality requirements of family of quasi closed addition chains. Similarly, for those $2^{s_j}-1$ with $j=\sigma(k)$ for some $k\in\{1,\ldots,h\}$ as indices not used in the previous doubling process, we can write

$$2^{s_k}-1=2^{s_k-s_{\sigma(k)}}(2^{s_{\sigma(k)}}-1)+2^{s_k-s_{\sigma(k)}}-1.$$

Now, for those $2^{s_j}-1$ with $s_j=s_{\tau(i)}$ such that $s_{\tau(i)}\neq s_{\sigma(k)}$ for all $k\in\{0,1,\ldots,h\}$, we do not need any new doubling under the absolutely closed condition, since there exists some $s_{\sigma(k)}$ such that $s_{\sigma(k)}<s_{\tau(i)}$ are consecutive terms in the optimal absolutely closed chain leading to n, with $s_{\tau(i)}\leq s_{\sigma(k+1)}$. Thus, we can write

$$2^{s_{\tau(i)}}-1=2^{s_{\tau(i)}-s_{\sigma(k)}}(2^{s_{\sigma(k)}}-1)+2^{s_{\tau(i)}-s_{\sigma(k)}}-1$$

since the chain

$$s_0=1,s_1=2,\ldots,s_h=n$$

is *quasi closed*. This verifies that the sequence constructed by doubling in this way is an addition chain leading to 2^n-1. The total length of the constructed addition chain leading to 2^n-1 is

$$s_{\tau(h)}+\sum_{i=0}^{h-1}|s_{\sigma(i+1)}-s_{\sigma(i)}|+\ell^{\ddagger}(n).$$

Hence

$$\begin{aligned}\ell(2^n-1)&\leq \min_{H(C_n)}\min_{L(C_n)}\left(s_{\tau(h)}+\sum_{i=0}^{h-1}|s_{\sigma(i+1)}-s_{\sigma(i)}|\right)+\ell^{\ddagger}(n)\\&=\mathbb{TV}(C_n)+\ell^{\ddagger}(n).\end{aligned}$$

Here, the minimum is taken over the admissible collections of vectors of σ and τ values with $s_i=s_{\sigma(i)}+s_{\tau(i)}$ for each $i\in\{1,\ldots,h\}$, where admissibility is in the sense of *strong quasi closure*. In this case, the admissible collection is the collection of all vectors of τ and σ values with $s_i=s_{\sigma(i)}+s_{\tau(i)}$ for each $i\in\{1,\ldots,h\}$, since the chain is *strongly quasi closed*. □

4. Some classical family of quasi closed addition chains

In this section, we show that the *Brauer* and certain class of *Hansen* addition chains form a subfamily of quasi closed addition chains.

Proposition 4.1. *Let $n\geq 2$ be a positive integer and let C_n be a fixed Brauer addition chain that leads to n. The Brauer addition chain C_n is quasi closed.*

Proof. Let $C_n : s_0=1,s_1=2,\ldots,s_h=n$ be a fixed *Brauer* addition chain with $s_i=s_{\sigma(i)}+s_{\tau(i)}$ with $i>\sigma(i)\geq\tau(i)\geq 0$ for each $i\in\{1,\ldots,h\}$. The addition chain C_n is *Brauer* implies $s_{\sigma(i)}=s_{i-1}$ for each $i\in\{1,\ldots,h\}$. Because the addition chain C_n has been fixed, each of the τ track terms $s_{\tau(i)}=s_i-s_{i-1}$ is determined for each $i\in\{1,\ldots,h\}$. Now, we can write $s_{\sigma(i+1)}-s_{\sigma(i)}=s_i-s_{i-1}=s_{\tau(i)}$, which is a term in the addition chain for each $i\in\{1,\ldots,h\}$. Now, for any pair of σ and τ track terms $(s_{\tau(k)},s_{\sigma(l)})$ for some $1\leq l,k\leq h$ such that $s_{\sigma(l)}<s_{\tau(k)}$ are consecutive terms in the addition chain with $s_{\tau(k)}\neq s_{\sigma(l)}$ for all $l\in\{0,\ldots,h\}$, we deduce

$$s_{\tau(k)}-s_{\sigma(l)}=s_{\tau(k)}-s_{\sigma(\tau(k))}=s_{\tau(\tau(k))}$$

is a term in the addition chain C_n since the addition chain C_n is *Brauer*. Furthermore, for a τ track term $s_{\tau(i)}$ for some $1 \leq i \leq h$ satisfying $s_{\tau(i)} \neq s_{\sigma(l)}$ for all $l \in \{0, \ldots, h\}$, we deduce $s_{\tau(i)} > 1$. This implies that there exists a term s_k in the chain C_n such that $s_k < s_{\tau(i)}$ are consecutive terms in the addition chain. The addition chain C_n is *Brauer* implies $s_k = s_{\sigma(\tau(i))}$. We deduce

$$s_{\sigma(\tau(i))} < s_{\tau(i)} \leq s_{\sigma(\tau(i)+1)}$$

since the σ track of a Brauer addition chain is strictly increasing. This verifies that the fixed Brauer addition chain C_n is *quasi closed*. □

Theorem 4.2. *Let*

$$C_n : s_0 = 1, s_1 = 2, \ldots, s_h = n$$

be a fixed Hansen addition chain of length h leading to n. Let H be a fixed anchor in the sense of Hansen such that

$$s_{\sigma(i)} = \max\{l \in H \ : \ l < s_i\}$$

with $s_i = s_{\sigma(i)} + s_{\tau(i)}$, $s_{\sigma(i)} \geq s_{\tau(i)}$ for each $1 \leq i \leq h$ so that the σ track is nondecreasing. Furthermore, assume that for each τ track term $s_{\tau(i)}$ with $i \in \{1, \ldots, h\}$ satisfying $s_{\tau(i)} \neq s_{\sigma(l)}$ for all $l \in \{0, \ldots, h\}$, the index $\tau(i) \in \mathrm{B}$. The Hansen addition chain C_n is quasi closed.

Proof. Suppose that

$$C_n : s_0 = 1, s_1 = 2, \ldots, s_h = n$$

is a fixed *Hansen* addition chain of length h leading to n. Let H be a fixed anchor in the sense of Hansen such that

$$s_{\sigma(i)} = \max\{l \in H \ : \ l < s_i\}$$

with $s_i = s_{\sigma(i)} + s_{\tau(i)}$, $s_{\sigma(i)} \geq s_{\tau(i)}$ for each $1 \leq i \leq h$ so that the σ track is nondecreasing. Furthermore, assume that for each τ track term $s_{\tau(i)}$ with $i \in \{1, \ldots, h\}$ satisfying $s_{\tau(i)} \neq s_{\sigma(l)}$ for all $l \in \{0, \ldots, h\}$, the index $\tau(i) \in \mathrm{B}$. We deduce

$$s_{\sigma(i)} \leq s_{\sigma(i+1)} \leq s_i$$

for each $i \in \{1, \ldots, h\}$. In the case $s_{\sigma(i)} = s_{\sigma(i+1)}$, we have nothing to check for the differences along the σ track terms. In the latter case

$$s_{\sigma(i)} < s_{\sigma(i+1)} \leq s_i$$

the maximality condition $s_{\sigma(i)} := \max\{l \in H \ : \ l < s_i\}$ implies that $s_{\sigma(i+1)} = s_i$. We deduce

$$s_{\sigma(i+1)} - s_{\sigma(i)} = s_i - s_{\sigma(i)} = s_{\tau(i)}$$

which is a fixed term in the fixed Hansen chain since the anchor H has been fixed. Now, for any pair of σ and τ track terms $(s_{\tau(k)}, s_{\sigma(l)})$ for $1 \leq k, l \leq h$ such that $s_{\sigma(l)} < s_{\sigma(k)}$ with $s_{\tau(k)} \neq s_{\sigma(l)}$ for all $l \in \{0, \ldots, h\}$, the requirement that $\tau(k) \in \mathrm{B}$ implies $s_{\sigma(l)} = s_{\sigma(\tau(k))}$. We deduce

$$s_{\tau(k)} - s_{\sigma(l)} = s_{\tau(k)} - s_{\sigma(\tau(k))} = s_{\tau(\tau(k))}$$

is a term in the fixed *Hansen* addition chain. Furthermore, for a τ track term $s_{\tau(i)}$ for $i \in \{1, \ldots, h\}$ satisfying $s_{\tau(i)} \neq s_{\sigma(l)}$ for all $l \in \{0, \ldots, h\}$, we deduce $s_{\tau(i)} > 1$. There exists a term in s_k in the fixed *Hansen* addition chain C_n such that $s_k < s_{\tau(i)}$ are consecutive terms in the chain. The requirement $\tau(i) \in \mathrm{B}$ implies that $s_k = s_{\sigma(\tau(k))}$. We may assume, without loss of generality, that

$$s_{\sigma(\tau(k))} < s_{\tau(k)} \leq s_{\sigma(\tau(k)+1)}.$$

Because if $s_{\sigma(\tau(i))} = s_{\sigma(\tau(i)+1)}$, then we can replace $s_{\sigma(\tau(i))}$ with $s_{\sigma(\tau(i)+v)}$ for the greatest v so that

$$s_{\sigma(\tau(i))} = \cdots = s_{\sigma(\tau(i)+v)}.$$

This verifies that the Hansen chain C_n is quasi closed. $\square$

5. Main result: Quasi complete numbers and the Scholz conjecture

Definition 5.1 (Quasi complete numbers)**.** Let $n \geq 2$ be a positive integer. We say that n is a *quasi complete* number if there exists a *quasi closed addition chain*

$$s_0 = 1, s_1 = 2, \ldots, s_{\ell^{\ddagger}(n)} = n$$

with a nondecreasing σ track such that $\ell(n) = \ell^{\ddagger}(n)$.

Theorem 5.2. *Let $n \geq 2$ be a quasi complete number. We have*

$$\ell(2^n - 1) \leq n - 1 + \ell(n).$$

Proof. We suppose that $n \geq 2$ is a *quasi complete* number. There exists a *quasi closed* addition chain

$$s_0 = 1, s_1 = 2, \ldots, s_{\ell^{\ddagger}(n)} = n$$

with a nondecreasing σ track. In the family of all quasi closed addition chains leading n, we fix one with a minimal length and fix a σ track $\{s_{\sigma(i)}\}_{i=1}^{\ell^{\ddagger}(n)}$ and τ track $\{s_{\tau(i)}\}_{i=1}^{\ell^{\ddagger}(n)}$ that makes the addition chain quasi closed with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ for each $i \in \{1, \ldots, \ell^{\ddagger}(n)\}$ and such that the σ track satisfies

$$s_{\sigma(1)} \leq s_{\sigma(2)} \leq \ldots \leq s_{\sigma(h)}.$$

By the construction in theorem 3.5, we deduce

$$\begin{aligned}
\ell(2^n - 1) &\leq s_{\tau(\ell^{\ddagger}(n))} + \sum_{i=0}^{\ell^{\ddagger}(n)-1} |s_{\sigma(i+1)} - s_{\sigma(i)}| + \ell^{\ddagger}(n) \\
&= s_{\tau(\ell^{\ddagger}(n))} + \sum_{i=0}^{\ell^{\ddagger}(n)-1} (s_{\sigma(i+1)} - s_{\sigma(i)}) + \ell^{\ddagger}(n) \\
&= s_{\tau(\ell^{\ddagger}(n))} + s_{\sigma(\ell^{\ddagger}(n))} - s_{\sigma(0)} + \ell^{\ddagger}(n) \\
&= n - 1 + \ell^{\ddagger}(n).
\end{aligned}$$

Because n is quasi complete, we get $\ell^{\ddagger}(n) = \ell(n)$ and deduce the inequality. $\square$

6. Some discussion

The preceding deduction shows that any quasi complete number must satisfy the inequality $\ell(2^n - 1) \leq n - 1 + \ell(n)$ proposed by Arnold Scholz. In practice, quasi complete numbers may not be rare and hard to find when we scout the set of all positive integers. In the preceding example, we have seen that the addition chains

$$s_0 = 1, s_1 = 2 = 1 + 1, s_2 = 4 = 2 + 2, s_3 = 5 = 4 + 1, s_4 = 8 = 4 + 4, s_5 = 13 = 8 + 5$$

and

$$s_0 = 1, s_1 = 2, s_3 = 4, s_4 = 8, s_5 = 16, s_6 = 21 = 16 + 5$$

are quasi closed addition chains that lead to 13 and 21, respectively. Each of these addition chains is also of minimal length in the family of all addition chains that lead to 13 and 21, respectively. In particular, we have the relations $\ell^{\ddagger}(13) = \ell(13)$ and $\ell^{\ddagger}(21) = \ell(21)$.

Therefore, the numbers $n = 13, 21$ are quasi complete numbers.

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