CLASSICAL VS PATH COUPLING ON A GEOMETRIC STATE SPACE: GLAUBER DYNAMICS FOR q-COLORINGS OF C_n

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ABSTRACT. We analyze one coupling of single-site Glauber dynamics for proper q-colorings of the cycle C_n in two ways. Drift on the Hamming disagreement count and path coupling on Hamming-adjacent pairs give the one-step contractions

$$\mathbb{E}\big[d_H(X_1,Y_1)\big] \le \alpha_{\mathrm{cl}}(q,n)\, d_H(X_0,Y_0), \qquad \mathbb{E}\big[d_G(X_1,Y_1)\big] \le \alpha_{\mathrm{pc}}(q,n)\, d_G(X_0,Y_0),$$

with $\alpha_{\mathrm{cl}}(q,n) = 1-(q-4)/\big(n(q-2)\big)$ and $\alpha_{\mathrm{pc}}(q,n) = 1-(q-6)/\big(n(q-2)\big)$, both attained. The thresholds are $q \ge 5$ and $q \ge 7$, and the constants separate by $2/\big(n(q-2)\big)$ uniformly in n. The gap is geometric. Transposing two colors across an edge produces a pair at Hamming distance 2 that no single recoloring connects, so the path metric d_G of the color graph charges a created disagreement $+2$ rather than $+1$; such pairs pack $\lfloor n/2 \rfloor$ to a cycle, giving $\operatorname{diam} d_H = n$ and $\operatorname{diam} d_G = \lfloor 3n/2 \rfloor$ for $q \ge 5$, hence closed $O(n \log n)$ bounds for both methods. The loss is attributable to the sparse edge set rather than to path coupling, since the complete edge set recovers α_{cl}. Exact enumeration on C_4 confirms every bound and gives worst-start $t_{\mathrm{mix}}(0.05) = 57, 29, 22, 19, 18$ for $q = 3, \dots, 7$.

CONTENTS

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1. Introduction

Markov chain Monte Carlo underpins Bayesian statistics, statistical physics, and randomized approximation, and its guarantees are mixing-time bounds $t_{\mathrm{mix}}(\varepsilon)$ certifying that the chain is ε-close to stationarity. Coupling is the standard route to such bounds [1, 2, 3, 5]. Path coupling [6] replaces the construction of a coalescing joint process by a one-step contraction verified only on adjacent pairs, and is the workhorse behind the coloring results of [10, 12, 13, 16]. The two methods are usually applied to different problems, so their relative strength is rarely measured. This paper fixes one chain, one coupling, and two analyses of it.

1.1. **The chain.** Fix $n \geq 3$, $q \geq 3$, and let $\Omega_q(C_n)$ be the proper q-colorings of the cycle. Identifying $V(C_n) = \mathbb{Z}/n$ and $[q] = \mathbb{Z}/q$, a coloring is a section of the discrete torus $\mathbb{Z}/n \times \mathbb{Z}/q$ whose consecutive values differ, that is, a closed loop winding once in the vertex direction and never running parallel to it. Figure 1 shows one for $n = 8$, $q = 5$. Counting these loops is a trace,

$$|\Omega_q(C_n)| = \mathrm{tr}\big(T^n\big) = (q-1)^n + (-1)^n(q-1), \qquad T_{cc'} = \mathbf{1}\{c \neq c'\},$$

the trace being exactly the gluing that closes a path into a loop (Theorem 2.8.1). Glauber dynamics moves one point of the loop at a time.

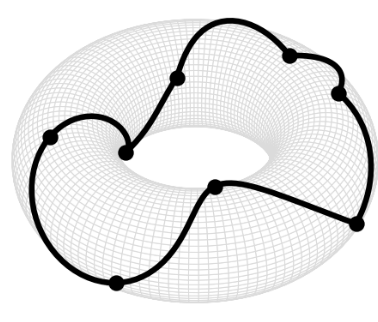

FIGURE 1. A proper coloring of C_8 with $q = 5$ as a closed loop on $\mathbb{Z}/8 \times \mathbb{Z}/5$.

1.2. **Results.** Throughout, (X_t, Y_t) is the synchronous maximal coupling of Definition 4.1.1, d_{H} the Hamming metric, and $d_{\mathscr{G}}$ the path metric of the color graph.

Theorem A. For all $n \geq 3$ and $q \geq 3$,

$$\mathbb{E}\big[d_{\mathrm{H}}(X_1,Y_1)\big] \leq \alpha_{\mathrm{cl}}(q,n)\, d_{\mathrm{H}}(X_0,Y_0), \qquad \alpha_{\mathrm{cl}}(q,n) = 1 - \frac{q-4}{n(q-2)},$$

and $\alpha_{\mathrm{cl}}(q,n) < 1$ if and only if $q \geq 5$.

Proof. Theorem 4.3.1, § 4. □

Theorem B. For all $n \geq 3$, $q \geq 4$, and every color-graph edge (X_0,Y_0),

$$\mathbb{E}\big[d_{\mathscr{G}}(X_1,Y_1)\big] \leq \alpha_{\mathrm{pc}}(q,n)\, d_{\mathscr{G}}(X_0,Y_0), \qquad \alpha_{\mathrm{pc}}(q,n) = 1 - \frac{q-6}{n(q-2)},$$

and $\alpha_{\mathrm{pc}}(q,n) < 1$ if and only if $q \geq 7$.

Proof. Theorem 5.2.1, § 5. □

Theorem C. For all $n \geq 3$ and $q \geq 4$,

$$\alpha_{\mathrm{pc}}(q,n) - \alpha_{\mathrm{cl}}(q,n) = \frac{2}{n(q-2)} > 0,$$

and on C_4 both constants are attained.

Proof. Corollary 5.2.4, § 5; Proposition 6.3.1, § 6. □

Theorem D. For all $n \geq 3$ and $q \geq 5$,

$$\operatorname{diam} d_{\mathrm{H}} = n, \qquad \operatorname{diam} d_{\mathscr{G}} = \left\lfloor \frac{3n}{2} \right\rfloor,$$

and consequently

$$q \geq 5 \implies t_{\mathrm{mix}}(\varepsilon) \leq \left\lceil \frac{n(q-2)}{q-4}\big(\log(n) + \log(\varepsilon^{-1})\big) \right\rceil,$$

$$q \geq 7 \implies t_{\mathrm{mix}}(\varepsilon) \leq \left\lceil \frac{n(q-2)}{q-6}\Big(\log\Big(\left\lfloor \frac{3n}{2} \right\rfloor\Big) + \log(\varepsilon^{-1})\Big) \right\rceil.$$

Proof. Theorem 6.2.3 and Theorem 6.5.1, § 6. □

2. Preliminaries

Coupling for coloring chains goes back to Jerrum [10]; path coupling was introduced by Bubley and Dyer [6], and both are treated systematically in [5]. Everything used later is collected here, stated on C_n or a general graph G wherever the argument permits.

2.1. **Markov chains.**

Definition 2.1.1. Let P be a transition matrix on a finite set Ω. P is *irreducible* when every state reaches every other, *aperiodic* when $\gcd\big(\{t \geq 1 \mid P^t(x,x) > 0\}\big) = 1$ for every x, *stationary* for π when $\pi P = \pi$, and *reversible* for π when

$$\pi(x)P(x,y) = \pi(y)P(y,x) \qquad \text{for all } x,y \in \Omega.$$

Theorem 2.1.2 (Convergence theorem)**.** *If P is irreducible and aperiodic with stationary distribution π, then*

$$\lim_{t\to\infty} \max_{x\in\Omega} \|P^t(x,\cdot) - \pi\|_{\mathrm{TV}} = 0.$$

Proof. [5, Theorem 4.9]. □

Lemma 2.1.3 (Reversibility implies stationarity)**.** *If π is reversible for P, then $\pi P = \pi$.*

Proof.
$$(\pi P)(y)=\sum_{x\in\Omega}\pi(x)P(x,y)=\sum_{x\in\Omega}\pi(y)P(y,x)=\pi(y)\sum_{x\in\Omega}P(y,x)=\pi(y)\cdot 1=\pi(y).$$
□

Corollary 2.1.4 (Symmetric chains)**.** *If $P(x,y)=P(y,x)$ for all x,y, then the uniform measure is reversible, hence stationary.*

Proof.
$$\pi\equiv|\Omega|^{-1}\implies \pi(x)P(x,y)=\frac{P(x,y)}{|\Omega|}=\frac{P(y,x)}{|\Omega|}=\pi(y)P(y,x)\overset{2.1.3}{\implies}\pi P=\pi.$$
□

2.2. **Total variation and mixing time.**

Definition 2.2.1. For probability measures μ,ν on Ω,
$$\|\mu-\nu\|_{\mathrm{TV}}=\tfrac{1}{2}\sum_{z\in\Omega}|\mu(z)-\nu(z)|.$$

Lemma 2.2.2 (Dual form)**.**
$$\|\mu-\nu\|_{\mathrm{TV}}=\max_{A\subseteq\Omega}\big(\mu(A)-\nu(A)\big).$$

Proof.
$$A\subseteq\Omega\implies \mu(A)-\nu(A)=\sum_{z\in A}\big(\mu(z)-\nu(z)\big)\le\sum_{z\in A}\big(\mu(z)-\nu(z)\big)^+\le\sum_{z\in\Omega}\big(\mu(z)-\nu(z)\big)^+$$
$$A^*=\{z\mid\mu(z)>\nu(z)\}\implies \mu(A^*)-\nu(A^*)=\sum_{z\in\Omega}\big(\mu(z)-\nu(z)\big)^+$$
$$\sum_z\big(\mu(z)-\nu(z)\big)=1-1=0\implies\sum_z\big(\mu(z)-\nu(z)\big)^+=\sum_z\big(\nu(z)-\mu(z)\big)^+$$
$$|a-b|=(a-b)^++(b-a)^+\implies\tfrac{1}{2}\sum_z|\mu(z)-\nu(z)|=\tfrac{1}{2}\Big(2\sum_z\big(\mu(z)-\nu(z)\big)^+\Big)=\max_{A\subseteq\Omega}\big(\mu(A)-\nu(A)\big).$$
□

Lemma 2.2.3 (Overlap form)**.**
$$\|\mu-\nu\|_{\mathrm{TV}}=1-\sum_{z\in\Omega}\min\{\mu(z),\nu(z)\}.$$

Proof.
$$\big(\mu(z)-\nu(z)\big)^+=\max\{\mu(z),\nu(z)\}-\nu(z)=\mu(z)-\min\{\mu(z),\nu(z)\}$$
$$\overset{2.2.2}{\implies}\|\mu-\nu\|_{\mathrm{TV}}=\sum_{z\in\Omega}\big(\mu(z)-\min\{\mu(z),\nu(z)\}\big)=1-\sum_{z\in\Omega}\min\{\mu(z),\nu(z)\}.$$
□

Lemma 2.2.4 (Uniform measures)**.** *For nonempty finite A,B with uniform measures U_A,U_B,*
$$\|U_A-U_B\|_{\mathrm{TV}}=1-|A\cap B|\min\big\{|A|^{-1},|B|^{-1}\big\}.$$

Proof.
$$z\in A\cap B\implies\min\{U_A(z),U_B(z)\}=\min\big\{|A|^{-1},|B|^{-1}\big\},\qquad z\notin A\cap B\implies\min\{U_A(z),U_B(z)\}=\min\{\cdot,0\}=0$$
$$\implies\sum_z\min\{U_A(z),U_B(z)\}=|A\cap B|\min\big\{|A|^{-1},|B|^{-1}\big\}\overset{2.2.3}{\implies}\|U_A-U_B\|_{\mathrm{TV}}=1-|A\cap B|\min\big\{|A|^{-1},|B|^{-1}\big\}.$$
□

Example 2.2.5. For $q = 5$, $A = \{1,3,4\}$, $B = \{1,2\}$,

z	1	2	3	4	5
$U_A(z)$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0
$U_B(z)$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\min\{U_A(z), U_B(z)\}$	$\frac{1}{3}$	0	0	0	0

$$\sum_{z\in[5]} \min\{U_A(z), U_B(z)\} = \frac{1}{3}+0+0+0+0 = \frac{1}{3} \overset{2.2.3}{\Longrightarrow} \|U_A - U_B\|_{\mathrm{TV}} = 1-\frac{1}{3} = \frac{2}{3}$$

$$A\cap B = \{1\}, \qquad |A\cap B|\min\{|A|^{-1}, |B|^{-1}\} = 1\cdot \min\left\{\frac{1}{3},\frac{1}{2}\right\} = \frac{1}{3} \overset{2.2.4}{\Longrightarrow} \|U_A - U_B\|_{\mathrm{TV}} = 1-\frac{1}{3} = \frac{2}{3}.$$

By Theorem 2.3.2 the overlap $\frac{1}{3}$ is $\mathbb{P}(X = Y)$ under the maximal coupling, so a mismatch occurs with probability $\frac{2}{3}$.

Definition 2.2.6.

$$d(t) = \max_{x\in\Omega} \|P^t(x,\cdot) - \pi\|_{\mathrm{TV}}, \qquad t_{\mathrm{mix}}(\varepsilon) = \min\{t \mid d(t) \le \varepsilon\}.$$

The numerical sections use the conventional value $\varepsilon = 0.05$.

2.3. **Couplings.** The coupling method originates with Doeblin [1] and was developed for mixing times by Aldous [2]; see [3, 5].

Definition 2.3.1. A *coupling* of μ,ν is a pair (X,Y) with $X\sim\mu$ and $Y\sim\nu$. A *coupling of the chain* is a process (X_t,Y_t) on Ω^2 whose coordinate marginals each evolve by P. It is *faithful* when $X_s = Y_s$ implies $X_t = Y_t$ for all $t \ge s$, with *coalescence time*

$$\tau = \min\{t \mid X_t = Y_t\}.$$

Theorem 2.3.2 (Maximal coupling)**.** *Every μ,ν admit a coupling with*

$$\mathbb{P}(X\ne Y) = \|\mu-\nu\|_{\mathrm{TV}},$$

and no coupling achieves less.

Proof. [5, Proposition 4.7]. □

Lemma 2.3.3 (Coupling inequality)**.** *For a faithful coupling with $X_0 = x$ and $Y_0\sim\pi$,*

$$\|P^t(x,\cdot) - \pi\|_{\mathrm{TV}} \le \mathbb{P}(\tau > t).$$

Proof.

$$X_t \sim P^t(x,\cdot), \qquad Y_0\sim\pi \overset{2.1.1}{\Longrightarrow} Y_t \sim \pi P^t = \pi$$

$$P^t(x,A) - \pi(A) = \mathbb{P}(X_t\in A) - \mathbb{P}(Y_t\in A) = \mathbb{P}(X_t\in A,\, Y_t\notin A) - \mathbb{P}(X_t\notin A,\, Y_t\in A)$$

$$\le \mathbb{P}(X_t\in A,\, Y_t\notin A) \le \mathbb{P}(X_t\ne Y_t) \overset{2.2.2}{\Longrightarrow} \|P^t(x,\cdot)-\pi\|_{\mathrm{TV}} \le \mathbb{P}(X_t\ne Y_t)$$

$$\{X_t\ne Y_t\} \overset{2.3.1}{\Longrightarrow} \{X_t\ne Y_t\}\subseteq\{\tau>t\} \Longrightarrow \|P^t(x,\cdot)-\pi\|_{\mathrm{TV}} \le \mathbb{P}(\tau>t).$$

□

Remark 2.3.4. $\mathbb{E}[\tau]$ alone gives only $\mathbb{P}(\tau>t)\le\mathbb{E}[\tau]/t$; the bounds below control $\mathbb{E}[D_t]$ at every t.

2.4. **Drift.**

Lemma 2.4.1 (Multiplicative drift). *Let (D_t) be adapted, nonnegative, integer valued, with $\mathbb{E}[D_{t+1} \mid \mathscr{F}_t] \le \alpha D_t$ for some $\alpha \in (0,1)$. Then*

$$\mathbb{E}[D_t] \le \alpha^t D_0, \qquad \mathbb{P}(D_t \ge 1) \le \alpha^t D_0.$$

Proof.

$$\mathbb{E}[D_{t+1}] = \mathbb{E}\big[\mathbb{E}[D_{t+1} \mid \mathscr{F}_t]\big] \le \mathbb{E}[\alpha D_t] = \alpha\,\mathbb{E}[D_t] \implies \mathbb{E}[D_t] \le \alpha\,\mathbb{E}[D_{t-1}] \le \alpha^2\,\mathbb{E}[D_{t-2}] \le \cdots \le \alpha^t D_0$$

$$\mathbb{P}(D_t \ge 1) = \sum_{k\ge 1} \mathbb{P}(D_t = k) \le \sum_{k \ge 1} k\,\mathbb{P}(D_t = k) = \mathbb{E}[D_t] \le \alpha^t D_0.$$

□

Corollary 2.4.2 (Drift to mixing). *If a faithful coupling satisfies the hypothesis of Lemma 2.4.1 with $\{D_t = 0\} = \{X_t = Y_t\}$ and $D_0 \le D_{\max}$ for every starting pair, then*

$$d(t) \le D_{\max}\,\alpha^t, \qquad t_{\mathrm{mix}}(\varepsilon) \le \left\lceil \frac{\log D_{\max} + \log(\varepsilon^{-1})}{-\log(\alpha)} \right\rceil.$$

Proof.

$$\{\tau > t\} = \{D_t \ge 1\} \overset{2.4.1}{\implies} \mathbb{P}(\tau > t) \le \alpha^t D_0 \le D_{\max}\alpha^t \overset{2.3.3}{\implies} d(t) \le D_{\max}\alpha^t$$

$$D_{\max}\alpha^t \le \varepsilon \iff \log(D_{\max}) + t\log(\alpha) \le \log(\varepsilon) \iff t\,\log(\alpha^{-1}) \ge \log(D_{\max}) + \log(\varepsilon^{-1}).$$

□

2.5. **Path coupling.**

Variants of Theorem 2.5.3 with longer paths and stopping times appear in [8, 9]. Proposition 2.5.2 and its corollaries are instantiated with $\rho = d_{\mathrm{H}}$, $d_{\mathscr{E}} = d_{\mathscr{G}}$ in Lemma 2.7.2.

Definition 2.5.1. Let $\mathscr{E} \subseteq \Omega \times \Omega$ be symmetric with connected graph and weights $w \ge 1$. The *induced path metric* is

$$d_{\mathscr{E}}(x,y) = \min\Big\{ \textstyle\sum_{i=1}^m w(x_{i-1},x_i) \;\Big|\; x = x_0,\dots,x_m = y,\ (x_{i-1},x_i) \in \mathscr{E} \Big\}.$$

Proposition 2.5.2 (Maximality). *If a metric ρ satisfies $\rho \le w$ on $\mathscr{E}$, then $\rho \le d_{\mathscr{E}}$ pointwise.*

Proof.

$$x = x_0,\dots,x_m = y \text{ in } \mathscr{E} \implies \rho(x_0,x_m) \le \rho(x_0,x_1) + \rho(x_1,x_m) \le \rho(x_0,x_1) + \rho(x_1,x_2) + \rho(x_2,x_m) \le \cdots \le \sum_{i=1}^m \rho(x_{i-1},x_i)$$

$$\le \sum_{i=1}^m w(x_{i-1},x_i) \overset{2.5.1}{\implies} \rho(x,y) \le d_{\mathscr{E}}(x,y).$$

□

Theorem 2.5.3 (Bubley–Dyer). *Let $d = d_{\mathscr{E}}$, and suppose every $(x,y) \in \mathscr{E}$ admits a one-step coupling with $\mathbb{E}[d(X_1,Y_1)] \le \alpha\, d(x,y)$ for some $\alpha < 1$. Then the contraction holds for all pairs, and with $d_{\max} = \max_{x,y} d(x,y)$,*

$$t_{\mathrm{mix}}(\varepsilon) \le \left\lceil \frac{\log d_{\max} + \log(\varepsilon^{-1})}{-\log(\alpha)} \right\rceil.$$

Proof. [6]; see also [5, Theorem 14.6]. □

Corollary 2.5.4 (Edge-only verification is never sharper). *Fix a one-step coupling and $w \equiv 1$. For any metric ρ with $\rho = 1$ on $\mathscr{E}$, put $\alpha_\rho = \max_{\mathscr{E}}\big(\mathbb{E}[\rho(X_1,Y_1)]\big)$ and $\alpha_{\mathscr{E}} = \max_{\mathscr{E}}\big(\mathbb{E}[d_{\mathscr{E}}(X_1,Y_1)]\big)$. Then $\alpha_\rho \le \alpha_{\mathscr{E}}$.*

Proof.

$$\rho = 1 = w \text{ on } \mathscr{E} \overset{2.5.2}{\implies} \rho(z,z') \le d_{\mathscr{E}}(z,z')\ \ \forall z,z' \implies \mathbb{E}[\rho(X_1,Y_1)] \le \mathbb{E}[d_{\mathscr{E}}(X_1,Y_1)] \text{ on } \mathscr{E} \implies \alpha_\rho \le \alpha_{\mathscr{E}}.$$

□

Corollary 2.5.5 (Complete edge set recovers drift)**.** *For a metric $\rho \geq 1$ off the diagonal, $\mathscr{E} = \{(x,y) \mid x \neq y\}$, and $w = \rho$,*

$$d_{\mathscr{E}} = \rho,$$

and Theorem 2.5.3 reduces to Corollary 2.4.2.

Proof.

$$(x,y) \in \mathscr{E} \overset{2.5.1}{\Longrightarrow} d_{\mathscr{E}}(x,y) \leq w(x,y) = \rho(x,y), \qquad \rho = w \text{ on } \mathscr{E} \overset{2.5.2}{\Longrightarrow} \rho(x,y) \leq d_{\mathscr{E}}(x,y)$$

$$\Longrightarrow d_{\mathscr{E}}(x,y) \leq \rho(x,y) \leq d_{\mathscr{E}}(x,y) \Longrightarrow d_{\mathscr{E}} = \rho.$$

□

Remark 2.5.6. The complete edge set reproduces the drift constant, so a sparse $\mathscr{E}$ costs exactly the passage $\rho \rightsquigarrow d_{\mathscr{E}}$; § 5 computes this gap on $\Omega_q(C_4)$.

2.6. **Colorings and Glauber dynamics.** Glauber dynamics for colorings enters the mixing literature with [10, 11]; the definitions follow [5, § 3.3].

Definition 2.6.1. Write $[q] = \{1,2,\ldots,q\}$. For a graph G with maximum degree Δ, a *proper q-coloring* is a map $x\colon V(G) \to [q]$ with $x(u) \neq x(v)$ whenever $u \sim v$, and $\Omega_q(G)$ denotes the set of these. The *legal set* at v is

$$L_x(v) = \{c \in [q] \mid c \neq x(u) \text{ for all } u \sim v\}.$$

Glauber dynamics selects $v \in V(G)$ uniformly and recolors v uniformly from $L_x(v)$.

Lemma 2.6.2 (Legal sets on cycles)**.** *For $x \in \Omega_q(C_n)$ with $n \geq 3$ and any vertex v,*

$$L_x(v) = [q] \setminus \{x(v-1), x(v+1)\}, \qquad |L_x(v)| \in \{q-2, q-1\}, \qquad x(v) \in L_x(v).$$

Proof.

$$N(v) = \{v-1, v+1\} \overset{2.6.1}{\Longrightarrow} L_x(v) = [q] \setminus \{x(v-1), x(v+1)\}$$

$$|\{x(v-1), x(v+1)\}| \in \{1,2\} \Longrightarrow |L_x(v)| = q - |\{x(v-1), x(v+1)\}| \in \{q-2, q-1\}$$

$$x \text{ proper} \Longrightarrow x(v) \neq x(v-1), \quad x(v) \neq x(v+1) \Longrightarrow x(v) \notin \{x(v-1), x(v+1)\} \Longrightarrow x(v) \in L_x(v).$$

□

Lemma 2.6.3 (Symmetry of Glauber dynamics)**.** *On $\Omega_q(G)$ with $|V(G)| = n$, the Glauber transition matrix satisfies $P(x,y) = P(y,x)$ for all x,y. The uniform measure is reversible and stationary, and the chain is aperiodic.*

Proof.

$$x = y \text{ off } v \Longrightarrow x|_{N(v)} = y|_{N(v)} \overset{2.6.1}{\Longrightarrow} L_x(v) = L_y(v)$$

$$\Longrightarrow P(x,y) = \mathbb{P}\big(\text{select } v\big) \cdot U_{L_x(v)}\big(y(v)\big) = \frac{1}{n} \cdot \frac{1}{|L_x(v)|} = \frac{1}{n} \cdot \frac{1}{|L_y(v)|} = P(y,x)$$

$$d_{\mathrm{H}}(x,y) \geq 2 \Longrightarrow P(x,y) = 0 = P(y,x) \overset{2.1.4}{\Longrightarrow} \pi \text{ uniform reversible}$$

$$x(v) \in L_x(v) \Longrightarrow P(x,x) \geq \frac{1}{n} \cdot \frac{1}{|L_x(v)|} > 0 \Longrightarrow \gcd\big(\{t \geq 1 \mid P^t(x,x) > 0\}\big) = 1.$$

□

2.7. **The color graph.** The color graph is the reconfiguration graph of colorings [21, 23, 24, 22]; see [25, 26] for surveys. § 5 contracts its metric $d_{\mathscr{G}}$ and § 6.2 computes its diameter on cycles.

Definition 2.7.1. The *color graph* $\mathscr{C}_q(G)$ has vertex set $\Omega_q(G)$, with $x \sim y$ when $d_{\mathrm{H}}(x,y) = 1$. Its graph metric is $d_{\mathscr{G}}$.

Lemma 2.7.2 (Glauber moves are color-graph edges)**.** *For $x \neq y$, $P(x,y) > 0$ if and only if $(x,y) \in E(\mathscr{C}_q(G))$. In particular the chain is irreducible if and only if $\mathscr{C}_q(G)$ is connected, and*

$$d_{\mathrm{H}} \leq d_{\mathscr{G}} \quad \textit{pointwise.}$$

Proof.

$$P(x,y) > 0,\ x \neq y \overset{2.6.1}{\iff} d_{\mathrm{H}}(x,y) = 1,\ y(v) \in L_x(v) \overset{2.7.1}{\iff} (x,y) \in E(\mathscr{C}_q(G))$$
$$d_{\mathrm{H}} = 1 \text{ on } E(\mathscr{C}_q(G)) \overset{2.5.2}{\implies} d_{\mathrm{H}} \leq d_{\mathscr{G}}.$$

□

Theorem 2.7.3 (Connectivity of the color graph)**.** *If $q \geq \Delta + 2$, then $\mathscr{C}_q(G)$ is connected. In particular Glauber dynamics on $\Omega_q(C_n)$ is irreducible for $q \geq 4$.*

Proof. [21, Theorem 1]; distances in $\mathscr{C}_q(G)$ are studied in [22]. □

Remark 2.7.4. For $3 \mid n$, the coloring $(1,2,3,1,2,3,\dots) \in \Omega_3(C_n)$ has $L_x(v) = \{x(v)\}$ at every v, hence is frozen and disconnects $\mathscr{C}_3(C_n)$ [23, 24]; $\mathscr{C}_3(C_4)$ is connected by enumeration of its 18 states, so $q \geq \Delta + 2$ is sharp on cycles.

2.8. **Enumeration by transfer matrix.** The formula is classical [27], by transfer matrix [28, Chapter 9]; the trace decomposition below is reused in Proposition 3.1.1 and Remark 3.1.2.

Theorem 2.8.1 (Chromatic polynomial of a cycle)**.** *For $n \geq 3$, define $T \in \mathbb{Z}^{q \times q}$ by $T_{cc'} = \mathbf{1}\{c \neq c'\}$ for $c, c' \in [q]$. Then*

$$|\Omega_q(C_n)| = \operatorname{tr}(T^n) = (q-1)^n + (-1)^n (q-1).$$

Proof.

$$q = 3 \implies T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad T^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

$$T^3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \implies \operatorname{tr}(T^3) = 2+2+2 = 6 = (3-1)^3 + (-1)^3(3-1)$$

$$\text{general } q,\ n: \quad (T^n)_{c_0 c_n} = \sum_{c_1 \in [q]} \cdots \sum_{c_{n-1} \in [q]} T_{c_0 c_1} T_{c_1 c_2} \cdots T_{c_{n-1} c_n} = \#\big\{(c_1, \dots, c_{n-1}) \ \big|\ c_i \neq c_{i+1} \text{ for all } i\big\}$$

$$\implies \operatorname{tr}(T^n) = \sum_{c \in [q]} (T^n)_{cc} = \#\big\{(c_0, \dots, c_{n-1}) \in [q]^n \ \big|\ c_i \neq c_{i+1 \bmod n} \text{ for all } i\big\} = |\Omega_q(C_n)|$$

$$\mathbf{1} = (1,1,\dots,1)^\top \implies (T\mathbf{1})_c = \sum_{c' \in [q]} T_{cc'} = \#\{c' \mid c' \neq c\} = q-1 \implies T\mathbf{1} = (q-1)\mathbf{1}$$

$$v \perp \mathbf{1} \implies (Tv)_c = \sum_{c' \neq c} v_{c'} = \Big(\sum_{c' \in [q]} v_{c'}\Big) - v_c = 0 - v_c = -v_c \implies Tv = -v$$

$$\dim\big(\{v \mid v \perp \mathbf{1}\}\big) = q-1 \implies \operatorname{tr}(T^n) = (q-1)^n \cdot 1 + (-1)^n \cdot (q-1).$$

□

Example 2.8.2.

$$\operatorname{tr}(T^3) = 6, \qquad \Omega_3(C_3) = \big\{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\big\}, \qquad |\Omega_3(C_3)| = 3! = 6,$$

the proper 3-colorings of a triangle being the bijections $[3] \to [3]$.

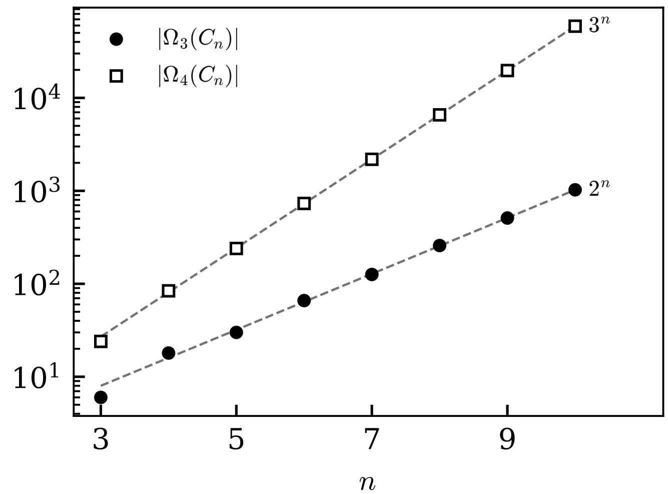

FIGURE 2. $|\Omega_q(C_n)| = (q-1)^n + (-1)^n(q-1)$ for $q=3$ (filled) and $q=4$ (open) on a logarithmic scale; dashed, the asymptotes $(q-1)^n$.

3. GLAUBER DYNAMICS ON THE FOUR-CYCLE

This section specializes § 2 to $G = C_4$.

3.1. **The four-cycle and its state space.** Let $V(C_4) = \{0,1,2,3\}$ and $E(C_4) = \{(i, i+1 \bmod 4) \mid i \in V(C_4)\}$, all vertex arithmetic modulo 4. By Definition 2.6.1,

$$\Omega_q(C_4) = \left\{x \in [q]^{V(C_4)} \;\middle|\; x(i) \neq x(i+1 \bmod 4)\ \ \forall i \in V(C_4)\right\}.$$

Proposition 3.1.1 (State-space size)**.** *For $q \geq 2$, with $T \in \mathbb{Z}^{q\times q}$, $T_{cc'} = \mathbf{1}\{c \neq c'\}$, as in Theorem 2.8.1,*

$$|\Omega_q(C_4)| = q(q-1)^2 + q(q-1)(q-2)^2 = (q-1)^4 + (q-1) = q(q-1)(q^2-3q+3).$$

Proof.

$$n = 4 \overset{2.8.1}{\Longrightarrow} |\Omega_q(C_4)| = \operatorname{tr}\bigl(T^4\bigr) = (q-1)^4 + (-1)^4(q-1) = (q-1)^4 + (q-1)$$

$$(T^2)_{cc'} = \sum_{b\in[q]} T_{cb}T_{bc'} = \#\bigl\{b \in [q] \;\big|\; b \neq c,\ b \neq c'\bigr\} = (q-2) + \mathbf{1}\{c = c'\}$$

$$T = T^\top \implies (T^2)^\top = T^2 \implies \operatorname{tr}\bigl(T^4\bigr) = \sum_{c\in[q]}\sum_{c'\in[q]} (T^2)_{cc'}(T^2)_{c'c} = \sum_{c\in[q]}\sum_{c'\in[q]} \bigl((T^2)_{cc'}\bigr)^2$$

$$c = c' \implies \bigl((T^2)_{cc}\bigr)^2 = (q-1)^2, \qquad c \neq c' \implies \bigl((T^2)_{cc'}\bigr)^2 = (q-2)^2$$

$$\implies \operatorname{tr}\bigl(T^4\bigr) = \sum_{c\in[q]} (q-1)^2 + \sum_{c\in[q]}\sum_{c'\in[q]\setminus\{c\}} (q-2)^2 = q(q-1)^2 + q(q-1)(q-2)^2$$

$$q(q-1)^2 + q(q-1)(q-2)^2 = q(q-1)\bigl((q-1)+(q-2)^2\bigr) = q(q-1)\bigl(q-1+q^2-4q+4\bigr) = q(q-1)(q^2-3q+3).$$

□

In particular $|\Omega_q(C_4)| = 18,\ 84,\ 260,\ 630,\ 1302$ for $q = 3,4,5,6,7$.

Remark 3.1.2. In the trace computation $(T^2)_{cc'}$ counts the choices of $x(1)$ given $x(0) = c$, $x(2) = c'$, so

$$|\Omega_q(C_4)| = \underbrace{q(q-1)^2}_{x(0)=x(2)} + \underbrace{q(q-1)(q-2)^2}_{x(0)\neq x(2)}.$$

The count $q(q-1)(q-2)^2$ is the off-diagonal term alone; by Lemma 2.6.2 the omitted colorings are exactly those possessing a legal set of size $q-1$.

3.2. **The chain and its ergodicity.** The chain of the paper is the Glauber dynamics of Definition 2.6.1 on $G = C_4$. By Lemma 2.6.2, for $x \neq y$ differing exactly at v,

$$P(x,y) = \frac{1}{4\,|L_x(v)|}\,\mathbf{1}\big\{y(v) \in L_x(v)\big\}, \qquad d_{\mathrm{H}}(x,y) \geq 2 \implies P(x,y) = 0.$$

Corollary 3.2.1 (Ergodicity on the four-cycle)**.** *For every $q \geq 3$, the Glauber chain on $\Omega_q(C_4)$ is aperiodic, irreducible, and reversible for the uniform measure $\pi \equiv |\Omega_q(C_4)|^{-1}$. Consequently*

$$\lim_{t\to\infty} d(t) = 0.$$

Proof.

$$q \geq 4 \overset{2.7.3}{\implies} d_{\mathscr{G}}(x,y) < \infty \;\; \forall x,y \in \Omega_q(C_4), \qquad q = 3 \overset{2.7.4}{\implies} d_{\mathscr{G}}(x,y) < \infty \;\; \forall x,y \in \Omega_3(C_4)$$

$$\overset{2.7.2}{\implies} \forall x,y \in \Omega_q(C_4) \;\; \exists t \geq 0 \;\; P^t(x,y) > 0$$

$$x(v) \in L_x(v) \overset{2.6.2}{\implies} P(x,x) \geq \frac{1}{4\,|L_x(v)|} > 0 \implies \gcd\big(\{t \geq 1 \mid P^t(x,x) > 0\}\big) = 1$$

$$\overset{2.6.3}{\implies} P(x,y) = P(y,x) \overset{2.1.4}{\implies} \pi P = \pi \overset{2.1.2}{\implies} \lim_{t\to\infty} d(t) = 0.$$

□

3.3. **Geometry of the four-cycle.**

Lemma 3.3.1 (Opposite vertices share neighborhoods)**.** *For every $v \in V(C_4)$,*

$$N(v) = \{v-1, v+1\} = N(v+2), \qquad (v-1, v+1) \notin E(C_4).$$

Proof.

$$E(C_4) = \{(i, i+1 \bmod 4) \mid i \in V(C_4)\} \implies N(v) = \{v-1, v+1\}, \qquad N(v+2) = \{v+1, v+3\} = \{v+1, v-1\}$$

$$(v+1) - (v-1) = 2 \not\equiv \pm 1 \pmod 4 \implies (v-1, v+1) \notin E(C_4).$$

□

Corollary 3.3.2 (Shared legal sets across a single disagreement)**.** *If $x, y \in \Omega_q(C_4)$ agree on $V(C_4) \setminus \{v\}$, then*

$$L_x(v) = L_y(v), \qquad L_x(v+2) = L_y(v+2).$$

Proof.

$$N(v) = N(v+2) = \{v-1, v+1\} \subseteq V(C_4) \setminus \{v\} \overset{3.3.1}{\implies} x|_{N(v)} = y|_{N(v)}, \qquad x|_{N(v+2)} = y|_{N(v+2)}$$

$$\overset{2.6.1}{\implies} L_x(v) = L_y(v), \qquad L_x(v+2) = L_y(v+2).$$

□

4. CLASSICAL COUPLING BOUND

Coupling goes back to Doeblin [1] and was developed for mixing times by Aldous [2]; see [3, 5], and [10] for colorings. This section builds one coupling on $\Omega_q(C_n)$ and analyzes it by drift on the Hamming disagreement count; § 5 reanalyzes the same coupling by path coupling.

4.1. **The synchronous maximal coupling.**

Definition 4.1.1 (Synchronous maximal coupling)**.** Given $(X_0,Y_0)\in\Omega_q(C_n)^2$, one step draws $u\in V(C_n)$ uniformly, recolors u in both chains by a maximal coupling of $U_{L_{X_0}(u)}$ and $U_{L_{Y_0}(u)}$ as in Theorem 2.3.2, and leaves every other vertex unchanged. Write

$$S_t=\{v\in V(C_n)\ \mid\ X_t(v)\neq Y_t(v)\},\qquad D_t=|S_t|=d_{\mathrm{H}}(X_t,Y_t),\qquad T_u=\left\|U_{L_{X_0}(u)}-U_{L_{Y_0}(u)}\right\|_{\mathrm{TV}}.$$

Lemma 4.1.2 (One-step identity)**.** *Definition 4.1.1 yields a faithful coupling of the Glauber chain, and conditional on (X_0,Y_0) and the drawn vertex u,*

$$D_1=D_0-\mathbf{1}\{u\in S_0\}+B_u,\qquad B_u\sim\mathrm{Bern}(T_u).$$

Proof.

$$X_1(u)\sim U_{L_{X_0}(u)},\qquad Y_1(u)\sim U_{L_{Y_0}(u)}\overset{2.6.1}{\Longrightarrow}\text{ each coordinate marginal evolves by }P$$

$$X_0=Y_0\implies L_{X_0}(u)=L_{Y_0}(u)\overset{2.3.2}{\Longrightarrow}\mathbb{P}\big(X_1(u)\neq Y_1(u)\big)=\left\|U_{L_{X_0}(u)}-U_{L_{X_0}(u)}\right\|_{\mathrm{TV}}=0\implies X_1=Y_1$$

$$X_1|_{V\setminus\{u\}}=X_0|_{V\setminus\{u\}},\qquad Y_1|_{V\setminus\{u\}}=Y_0|_{V\setminus\{u\}}\implies S_1\setminus\{u\}=S_0\setminus\{u\}$$

$$\implies D_1=|S_0\setminus\{u\}|+\mathbf{1}\{X_1(u)\neq Y_1(u)\}=D_0-\mathbf{1}\{u\in S_0\}+\mathbf{1}\{X_1(u)\neq Y_1(u)\}$$

$$\overset{2.3.2}{\Longrightarrow}\mathbb{P}\big(X_1(u)\neq Y_1(u)\ \big|\ X_0,Y_0,u\big)=T_u\implies\mathbf{1}\{X_1(u)\neq Y_1(u)\}\sim\mathrm{Bern}(T_u).$$

$\square$

4.2. **Sitewise matching.**

Lemma 4.2.1 (Matching bound)**.** *For every $u\in V(C_n)$, with $r_u=|N(u)\cap S_0|$,*

$$T_u\le\frac{r_u}{q-2}.$$

Proof.

$$F_X=\{X_0(w)\ \mid\ w\in N(u)\},\qquad F_Y=\{Y_0(w)\ \mid\ w\in N(u)\},\qquad X_0=Y_0\text{ on }N(u)\setminus S_0$$

$$\implies|F_Y\setminus F_X|\le r_u\implies|F_X\cup F_Y|=|F_X|+|F_Y\setminus F_X|\le\min\big\{|F_X|,|F_Y|\big\}+r_u$$

$$L_{X_0}(u)=[q]\setminus F_X,\qquad L_{Y_0}(u)=[q]\setminus F_Y\implies\big|L_{X_0}(u)\cap L_{Y_0}(u)\big|=q-|F_X\cup F_Y|\ge q-\min\big\{|F_X|,|F_Y|\big\}-r_u$$

$$q-\min\big\{|F_X|,|F_Y|\big\}=\max\big\{|L_{X_0}(u)|,|L_{Y_0}(u)|\big\}=:M\implies\big|L_{X_0}(u)\cap L_{Y_0}(u)\big|\ge M-r_u$$

$$\min\Big\{|L_{X_0}(u)|^{-1},|L_{Y_0}(u)|^{-1}\Big\}=M^{-1}\overset{2.2.4}{\Longrightarrow}T_u=1-\frac{\big|L_{X_0}(u)\cap L_{Y_0}(u)\big|}{M}\le1-\frac{M-r_u}{M}=\frac{r_u}{M}$$

$$\overset{2.6.2}{\Longrightarrow}M\ge q-2\implies T_u\le\frac{r_u}{q-2}.$$

$\square$

Example 4.2.2. Take $q=4$, $n=4$, $X_0=(1,2,3,2)$, $Y_0=(4,2,3,2)$, so $S_0=\{0\}$ and $D_0=1$.

$$\{X_0(1),X_0(3)\}=\{2\}=\{Y_0(1),Y_0(3)\}\overset{3.3.2}{\Longrightarrow}L_{X_0}(0)=L_{Y_0}(0)=[4]\setminus\{2\}=\{1,3,4\}=L_{X_0}(2)=L_{Y_0}(2)$$

$$\overset{2.2.4}{\Longrightarrow}T_0=T_2=1-3\cdot\tfrac{1}{3}=0$$

$$L_{X_0}(1)=[4]\setminus\{1,3\}=\{2,4\},\qquad L_{Y_0}(1)=[4]\setminus\{4,3\}=\{1,2\},\qquad L_{X_0}(1)\cap L_{Y_0}(1)=\{2\}$$

$$\overset{2.2.4}{\Longrightarrow}T_1=1-1\cdot\min\Big\{\tfrac{1}{2},\tfrac{1}{2}\Big\}=\tfrac{1}{2},\qquad L_{X_0}(3)=\{2,4\},\quad L_{Y_0}(3)=\{1,2\}\implies T_3=\tfrac{1}{2}$$

$$\overset{4.1.2}{\Longrightarrow}\mathbb{E}[D_1\mid X_0,Y_0]=\frac{1}{4}\sum_{u=0}^{3}\big(D_0-\mathbf{1}\{u\in S_0\}+T_u\big)=\frac{1}{4}\Big(0+\frac{3}{2}+1+\frac{3}{2}\Big)=\frac{4}{4}=1$$

$$\alpha_{\mathrm{cl}}(4,4)=1-\frac{4-4}{4\cdot(4-2)}=1-0=1\implies\mathbb{E}[D_1\mid X_0,Y_0]=\alpha_{\mathrm{cl}}(4,4)\,D_0.$$

4.3. **Contraction and mixing on the cycle.**

Theorem 4.3.1 (Classical contraction on C_n)**.** *For all $n \geq 3$, $q \geq 3$, and every $(X_0, Y_0) \in \Omega_q(C_n)^2$, the coupling of Definition 4.1.1 satisfies*

$$\mathbb{E}[D_1 \mid X_0, Y_0] \leq \alpha_{\mathrm{cl}}(q,n)\, D_0, \qquad \alpha_{\mathrm{cl}}(q,n) = 1 - \frac{q-4}{n(q-2)},$$

and $\alpha_{\mathrm{cl}}(q,n) < 1$ if and only if $q \geq 5$.

Proof.

$$u \sim \mathrm{Unif}\big(V(C_n)\big) \overset{4.1.2}{\Longrightarrow} \mathbb{E}[D_1 \mid X_0, Y_0] = \frac{1}{n} \sum_{u \in V(C_n)} \big(D_0 - \mathbf{1}\{u \in S_0\} + T_u\big) = D_0 - \frac{|S_0|}{n} + \frac{1}{n} \sum_{u \in V(C_n)} T_u$$

$$\overset{4.2.1}{\Longrightarrow} \sum_{u \in V(C_n)} T_u \leq \frac{1}{q-2} \sum_{u \in V(C_n)} \big|N(u) \cap S_0\big| = \frac{1}{q-2} \#\big\{(u,s) \;\big|\; s \in S_0,\ u \sim s\big\} = \frac{1}{q-2} \sum_{s \in S_0} \deg(s) = \frac{2D_0}{q-2}$$

$$|S_0| = D_0 \implies \mathbb{E}[D_1 \mid X_0, Y_0] \leq D_0 - \frac{D_0}{n} + \frac{2D_0}{n(q-2)} = D_0\Big(1 - \frac{(q-2)-2}{n(q-2)}\Big) = D_0\Big(1 - \frac{q-4}{n(q-2)}\Big)$$

$$n(q-2) > 0 \implies \Big(\frac{q-4}{n(q-2)} > 0 \iff q > 4 \iff q \geq 5\Big).$$

$\square$

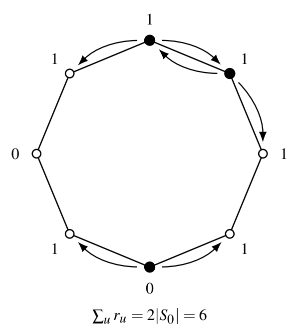

FIGURE 3. S_0 on C_8 (filled); each disagreement charges its two neighbors, and r_u counts arrowheads.

Example 4.3.2. For $n = 8$, $q = 5$, $S_0 = \{0,1,4\}$, $D_0 = 3$ as in Figure 3,

$$\sum_{u=0}^{7} r_u = 1+1+1+1+0+1+0+1 = 6 = 2\,|S_0| \overset{4.3.1}{\Longrightarrow} \mathbb{E}[D_1 \mid X_0, Y_0] \leq 3 - \frac{3}{8} + \frac{1}{8} \cdot \frac{6}{5-2} = 3 - \frac{3}{8} + \frac{1}{4} = \frac{23}{8}$$

$$\alpha_{\mathrm{cl}}(5,8) = 1 - \frac{5-4}{8 \cdot (5-2)} = 1 - \frac{1}{24} = \frac{23}{24} \implies \alpha_{\mathrm{cl}}(5,8)\, D_0 = \frac{23}{24} \cdot 3 = \frac{23}{8}.$$

Corollary 4.3.3 (Classical mixing bound on C_n)**.** *For $q \geq 5$,*

$$t_{\mathrm{mix}}(\varepsilon) \leq \left\lceil \frac{n(q-2)}{q-4} \big(\log(n) + \log(\varepsilon^{-1})\big) \right\rceil, \qquad \mathbb{E}[\tau] \leq \frac{n^2(q-2)}{q-4}.$$

Proof.

$$\{D_t = 0\} = \{X_t = Y_t\}, \qquad D_0 \leq n \overset{2.4.2}{\Longrightarrow} t_{\mathrm{mix}}(\varepsilon) \leq \left\lceil \frac{\log(n) + \log(\varepsilon^{-1})}{-\log\big(\alpha_{\mathrm{cl}}(q,n)\big)} \right\rceil$$

$$\alpha_{\mathrm{cl}}(q,n) \in (0,1) \implies -\log\big(\alpha_{\mathrm{cl}}(q,n)\big) \geq 1 - \alpha_{\mathrm{cl}}(q,n) = \frac{q-4}{n(q-2)} \implies \frac{1}{-\log\big(\alpha_{\mathrm{cl}}(q,n)\big)} \leq \frac{n(q-2)}{q-4}$$

$$\mathbb{E}[\tau] = \sum_{t \geq 0} \mathbb{P}(\tau > t) \overset{2.4.1}{\Longrightarrow} \mathbb{E}[\tau] \leq \sum_{t \geq 0} n\, \alpha_{\mathrm{cl}}(q,n)^t = \frac{n}{1 - \alpha_{\mathrm{cl}}(q,n)} = \frac{n^2(q-2)}{q-4}.$$

$\square$

Corollary 4.3.4 (The four-cycle)**.** *Write* $\alpha_{\mathrm{cl}}(q)=\alpha_{\mathrm{cl}}(q,4)$*. For* $\varepsilon=0.05$*,*

$$\alpha_{\mathrm{cl}}(q)=\frac{3}{4}+\frac{1}{2(q-2)},\qquad \alpha_{\mathrm{cl}}(5)=\frac{11}{12},\quad \alpha_{\mathrm{cl}}(6)=\frac{7}{8},\quad \alpha_{\mathrm{cl}}(7)=\frac{17}{20},$$

$$t_{\mathrm{mix}}(0.05)\le 51,\ 33,\ 27,\qquad \mathbb{E}[\tau]\le 48,\ 32,\ \tfrac{80}{3}\qquad (q=5,6,7).$$

Proof.

$$1-\frac{q-4}{4(q-2)}=\frac{4(q-2)-(q-4)}{4(q-2)}=\frac{3q-4}{4(q-2)}=\frac{3(q-2)+2}{4(q-2)}=\frac{3}{4}+\frac{1}{2(q-2)}$$

$$\log\Big(\frac{4}{0.05}\Big)=\log(80)=4.382026\ldots\overset{4.3.3}{\Longrightarrow}\left\lceil\tfrac{\log(80)}{\log(12/11)}\right\rceil=\lceil 50.362\rceil=51,\quad \left\lceil\tfrac{\log(80)}{\log(8/7)}\right\rceil=\lceil 32.816\rceil=33,$$

$$\left\lceil\tfrac{\log(80)}{\log(20/17)}\right\rceil=\lceil 26.963\rceil=27,\qquad \frac{4^2(q-2)}{q-4}=\frac{16\cdot 3}{1},\ \frac{16\cdot 4}{2},\ \frac{16\cdot 5}{3}=48,\ 32,\ \frac{80}{3}.$$

$\square$

Remark 4.3.5. Example 4.2.2 attains $\alpha_{\mathrm{cl}}(4,4)=1$, and rational enumeration over $\Omega_q(C_4)\times\Omega_q(C_4)$ (Appendix B) gives

$$\max_{(x,y)\,:\,d_{\mathrm{H}}(x,y)=k}\frac{\mathbb{E}[D_1\mid X_0=x,Y_0=y]}{k}=\alpha_{\mathrm{cl}}(q,4)\quad \forall k\in\{1,2,3,4\},\ q\ge 4,$$

with maximum 1 at $q=3$.

Remark 4.3.6. The threshold $q\ge 5=2\Delta+1$ is uniform in n and matches the $q>2\Delta$ regime [10, 11]; going below 2Δ needs different methods [12, 18, 19]. Corollary 4.3.3 gives $t_{\mathrm{mix}}(\varepsilon)=O(n\log(n))$ for fixed $q\ge 5$, and $\alpha_{\mathrm{cl}}(q,n)$ decreases in q.

5. Path-coupling bound

Path coupling [6] verifies contraction only on the edge set $\mathscr{E}$ of Hamming-adjacent pairs, at the price, by Proposition 2.5.2, of measuring contraction in the induced path metric $d_{\mathscr{G}}$ of the color graph [21]. This section shows that on $\Omega_q(C_n)$ the substitution $d_{\mathrm{H}}\rightsquigarrow d_{\mathscr{G}}$ is not free. Distances in color graphs are studied in reconfiguration [22, 25, 26]; the obstruction below is its smallest instance.

5.1. **Rigid pairs and the metric gap.**

Lemma 5.1.1 (Adjacent-difference dichotomy)**.** *Let* $x,y\in\Omega_q(C_n)$ *differ exactly at adjacent vertices* $v\sim u$*, let* w *be the second neighbor of* v *and* z *the second neighbor of* u*. For* $q\ge 4$*,*

$$d_{\mathscr{G}}(x,y)\in\{2,3\},\qquad d_{\mathscr{G}}(x,y)=3\iff y(v)=x(u)\ \wedge\ y(u)=x(v).$$

Proof.

$$d_{\mathscr{G}}(x,y)=2\iff y(v)\in L_x(v)\ \vee\ y(u)\in L_x(u)\overset{2.6.2}{\Longrightarrow} y(v)\notin\{x(u),x(w)\}\ \vee\ y(u)\notin\{x(v),x(z)\}$$

$$y(v)\ne y(w)=x(w),\qquad y(u)\ne y(z)=x(z)\implies \big(d_{\mathscr{G}}(x,y)>2\iff y(v)=x(u)\ \wedge\ y(u)=x(v)\big)$$

$$y(v)=x(u),\ y(u)=x(v),\ q\ge 4\implies \exists s\in[q]\setminus\{x(v),x(u),x(w)\}$$

$$\implies x\xrightarrow{v\mapsto s}x'\xrightarrow{u\mapsto x(v)}x''\xrightarrow{v\mapsto x(u)}y,\qquad s\notin\{x(u),x(w)\},\quad x(v)\notin\{s,x(z)\},\quad x(u)\notin\{x(v),x(w)\}$$

$$\implies d_{\mathscr{G}}(x,y)\le 3\overset{2.7.2}{\Longrightarrow} d_{\mathscr{G}}(x,y)\in\{2,3\}.$$

$\square$

Proposition 5.1.2 (Swap pairs are rigid)**.** *Let* $x,y\in\Omega_q(C_n)$ *differ exactly at adjacent vertices* $v\sim u$ *with*

$$y(v)=x(u),\qquad y(u)=x(v).$$

Then

$$d_{\mathrm{H}}(x,y)=2,\qquad d_{\mathscr{G}}(x,y)\ge 3,\qquad d_{\mathscr{G}}(x,y)=3\ \ (q\ge 4).$$

Such pairs exist in $\Omega_q(C_n)$ *for every* $n\ge 3$ *and* $q\ge 4$*; on* C_4 *they exist if and only if* $q\ge 4$*.*

Proof.

$$u \in N(v) \overset{2.6.2}{\Longrightarrow} y(v) = x(u) \notin L_x(v), \qquad v \in N(u) \implies y(u) = x(v) \notin L_x(u) \overset{2.7.2}{\Longrightarrow} d_{\mathscr{G}}(x,y) > 2$$

$$q \geq 4 \overset{5.1.1}{\Longrightarrow} d_{\mathscr{G}}(x,y) = 3$$

$$w \in N(v) \setminus \{u\}, \quad z \in N(u) \setminus \{v\}, \quad x,y \text{ proper} \iff x(w) \notin \{x(v), x(u)\}, \quad x(z) \notin \{x(v), x(u)\}$$

$$n = 4 \implies w \sim z \implies x(w) \neq x(z) \implies \big|[q] \setminus \{x(v), x(u)\}\big| \geq 2 \iff q \geq 4, \qquad n \neq 4 \implies w \not\sim z.$$

$\square$

Example 5.1.3. For $q = 5$, $n = 4$, the pair $x = (0,1,2,3)$, $y = (1,0,2,3)$ is a swap pair at the edge $(0,1)$, drawn in Figure 4.

$$y(0) = 1 = x(1) \overset{2.6.2}{\Longrightarrow} 1 \notin L_x(0) = \{0,2,4\}, \qquad y(1) = 0 = x(0) \implies 0 \notin L_x(1) = \{1,3,4\}$$

$$\overset{5.1.2}{\Longrightarrow} d_{\mathscr{G}}(x,y) = 3, \qquad (0,1,2,3) \xrightarrow{v_0 \mapsto 4} (4,1,2,3) \xrightarrow{v_1 \mapsto 0} (4,0,2,3) \xrightarrow{v_0 \mapsto 1} (1,0,2,3),$$

each intermediate proper since $4 \notin \{1,3\}$, $0 \notin \{4,2\}$, $1 \notin \{0,3\}$. Breadth-first search (Appendix B) finds 480 swap pairs in $\Omega_5(C_4)$ and $\operatorname{diam} d_{\mathscr{G}} = 6 > 4 = \operatorname{diam} d_{\mathrm{H}}$.

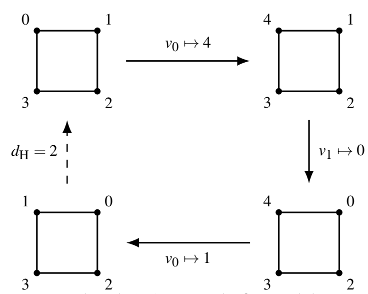

FIGURE 4. The swap pair of Example 5.1.3, top left and bottom left. The dashed edge is the direct route, which no single recoloring realizes; the three solid moves close the square, so $d_{\mathscr{G}} = 3$.

5.2. **Edge contraction in the path metric.**

Theorem 5.2.1 (Path-coupling contraction on C_n)**.** *For all $n \geq 3$, $q \geq 4$, and every color-graph edge (x,y), the coupling of Definition 4.1.1 satisfies*

$$\mathbb{E}[d_{\mathscr{G}}(X_1,Y_1) \mid X_0 = x, Y_0 = y] \leq \alpha_{\mathrm{pc}}(q,n)\, d_{\mathscr{G}}(x,y), \qquad \alpha_{\mathrm{pc}}(q,n) = 1 - \frac{q-6}{n(q-2)},$$

and $\alpha_{\mathrm{pc}}(q,n) < 1$ if and only if $q \geq 7$.

Proof.

$$S_0 = \{v\}, \qquad u = v \implies r_v = |N(v) \cap \{v\}| = 0 \overset{4.2.1}{\Longrightarrow} T_v = 0 \overset{4.1.2}{\Longrightarrow} D_1 = 0 \implies d_{\mathscr{G}}(X_1,Y_1) = 0$$

$$u \notin \{v\} \cup N(v) \implies r_u = |N(u) \cap \{v\}| = 0 \implies T_u = 0 \implies d_{\mathscr{G}}(X_1,Y_1) = 1 \qquad \big(\#\{u\} = n-3\big)$$

$$u \in N(v) \overset{4.1.2}{\Longrightarrow} \mathbb{P}\big(d_{\mathscr{G}}(X_1,Y_1) = 1\big) = 1 - T_u, \qquad X_1 \neq Y_1 \text{ at } \{v,u\},\ v \sim u \overset{5.1.1}{\Longrightarrow} d_{\mathscr{G}}(X_1,Y_1) \leq 3$$

$$\overset{4.2.1}{\Longrightarrow} \mathbb{E}[d_{\mathscr{G}}(X_1,Y_1) \mid u \in N(v)] \leq (1 - T_u) \cdot 1 + T_u \cdot 3 = 1 + 2T_u \leq 1 + \frac{2}{q-2}$$

$$\implies \mathbb{E}[d_{\mathscr{G}}(X_1,Y_1)] \leq \frac{1}{n}\Big(0 + (n-3) \cdot 1 + 2 \cdot \Big(1 + \frac{2}{q-2}\Big)\Big) = 1 - \frac{1}{n} + \frac{4}{n(q-2)} = 1 - \frac{q-6}{n(q-2)}$$

$$n(q-2) > 0 \implies \Big(\frac{q-6}{n(q-2)} > 0 \iff q > 6 \iff q \geq 7\Big).$$

$\square$

Example 5.2.2 (The bound is attained)**.** For $q=5$, $n=4$, take the color-graph edge $x=(0,1,2,3)$, $y=(4,1,2,3)$, differing at $v=0$.

$$u\in\{0,2\} \overset{3.3.2}{\Longrightarrow} T_0=T_2=0 \implies d_{\mathscr{G}}(X_1,Y_1)=0\ (u=0), \qquad d_{\mathscr{G}}(X_1,Y_1)=1\ (u=2)$$

$$u=1 \implies L_x(1)=[5]\setminus\{0,2\}=\{1,3,4\}, \qquad L_y(1)=[5]\setminus\{4,2\}=\{0,1,3\} \overset{2.2.4}{\Longrightarrow} T_1=1-2\cdot\tfrac{1}{3}=\tfrac{1}{3}$$

$$U_{L_x(1)}-\min\{U_{L_x(1)},U_{L_y(1)}\}=\tfrac{1}{3}\,\delta_4, \qquad U_{L_y(1)}-\min\{U_{L_x(1)},U_{L_y(1)}\}=\tfrac{1}{3}\,\delta_0$$

$$\implies \big(X_1(1),Y_1(1)\big)=(4,0) \text{ on the mismatch}, \qquad \big((0,4,2,3),(4,0,2,3)\big) \text{ a swap of } \{0,4\} \overset{5.1.2}{\Longrightarrow} d_{\mathscr{G}}=3$$

$$u=3 \implies T_3=\tfrac{1}{3}, \qquad \big((0,1,2,4),(4,1,2,0)\big) \text{ a swap across } (3,0) \implies d_{\mathscr{G}}=3$$

$$\implies \mathbb{E}[d_{\mathscr{G}}(X_1,Y_1)]=\frac{1}{4}\Big(0+1+\big(\tfrac{2}{3}\cdot 1+\tfrac{1}{3}\cdot 3\big)+\big(\tfrac{2}{3}\cdot 1+\tfrac{1}{3}\cdot 3\big)\Big)=\frac{1}{4}\Big(0+1+\tfrac{5}{3}+\tfrac{5}{3}\Big)=\frac{13}{12}=\alpha_{\mathrm{pc}}(5,4).$$

Corollary 5.2.3 (The four-cycle)**.** *Write $\alpha_{\mathrm{pc}}(q)=\alpha_{\mathrm{pc}}(q,4)$. For $\varepsilon=0.05$,*

$$\alpha_{\mathrm{pc}}(q)=\frac{3}{4}+\frac{1}{q-2}, \qquad \alpha_{\mathrm{pc}}(7)=\frac{19}{20}, \quad \alpha_{\mathrm{pc}}(8)=\frac{11}{12},$$

$$t_{\mathrm{mix}}(0.05)\le\left\lceil\tfrac{\log(120)}{\log(20/19)}\right\rceil=\lceil 93.335\rceil=94, \qquad \left\lceil\tfrac{\log(120)}{\log(12/11)}\right\rceil=\lceil 55.024\rceil=56 \qquad (q=7,8).$$

Proof.

$$1-\frac{q-6}{4(q-2)}=\frac{4(q-2)-(q-6)}{4(q-2)}=\frac{3q-2}{4(q-2)}=\frac{3(q-2)+4}{4(q-2)}=\frac{3}{4}+\frac{1}{q-2}$$

$$\operatorname{diam} d_{\mathscr{G}}=6\ \text{ (Appendix B) } \overset{2.5.3}{\Longrightarrow} t_{\mathrm{mix}}(0.05)\le\left\lceil\tfrac{\log(6)+\log(20)}{-\log\big(\alpha_{\mathrm{pc}}(q)\big)}\right\rceil, \qquad \log(120)=4.787492\ldots$$

□

Corollary 5.2.4 (Separation)**.** *For all $n\ge 3$ and $q\ge 4$,*

$$\alpha_{\mathrm{pc}}(q,n)-\alpha_{\mathrm{cl}}(q,n)=\frac{(q-4)-(q-6)}{n(q-2)}=\frac{2}{n(q-2)}>0,$$

so the drift analysis contracts for $q\ge 5$ while path coupling on Hamming-adjacent edges contracts only for $q\ge 7$, uniformly in n, as Corollary 2.5.4 predicts with $\rho=d_{\mathrm{H}}$ and $d_{\mathscr{E}}=d_{\mathscr{G}}$.

Remark 5.2.5 (Sharpness). Example 5.2.2 attains $\alpha_{\mathrm{pc}}(5,4)$, and rational enumeration over all color-graph edges (Appendix B) gives

$$\max_{(x,y)\,:\,d_{\mathscr{G}}(x,y)=1}\mathbb{E}\big[d_{\mathscr{G}}(X_1,Y_1)\big]=\alpha_{\mathrm{pc}}(q,4) \quad \forall q\ge 4,$$

with maximum 1 at $q=3$. The two constants differ only in the charge for a created disagreement, $+1$ for d_{H} against $+2$ for $d_{\mathscr{G}}$, and the $+2$ is exact because the mismatch in Example 5.2.2 lands deterministically on a swap pair. Whether a weighted edge metric restores $q\ge 5$ under edge-only verification is open; metric choice for coloring dynamics is active [8, 9, 20].

6. Comparison, exact computation, and closed forms

6.1. **Separation of the two bounds.**

Corollary 6.1.1 (Thresholds)**.** *For all $n\ge 3$ and $q\ge 4$,*

$$\alpha_{\mathrm{cl}}(q,n)<1 \iff q\ge 5, \qquad \alpha_{\mathrm{pc}}(q,n)<1 \iff q\ge 7,$$

and for $q\ge 7$,

$$\frac{n(q-2)}{q-4}\big(\log(n)+\log(\varepsilon^{-1})\big)<\frac{n(q-2)}{q-6}\big(\log\big(\operatorname{diam} d_{\mathscr{G}}\big)+\log(\varepsilon^{-1})\big).$$

Proof.

$$1-\alpha_{\mathrm{cl}}(q,n)=\frac{q-4}{n(q-2)}>0 \iff q>4, \qquad 1-\alpha_{\mathrm{pc}}(q,n)=\frac{q-6}{n(q-2)}>0 \iff q>6$$

$$0<q-6<q-4 \xLongrightarrow{5.2.4} \frac{n(q-2)}{q-4}<\frac{n(q-2)}{q-6} \xLongrightarrow{2.7.2} n=\operatorname{diam} d_{\mathrm{H}} \le \operatorname{diam} d_{\mathscr{G}}.$$

□

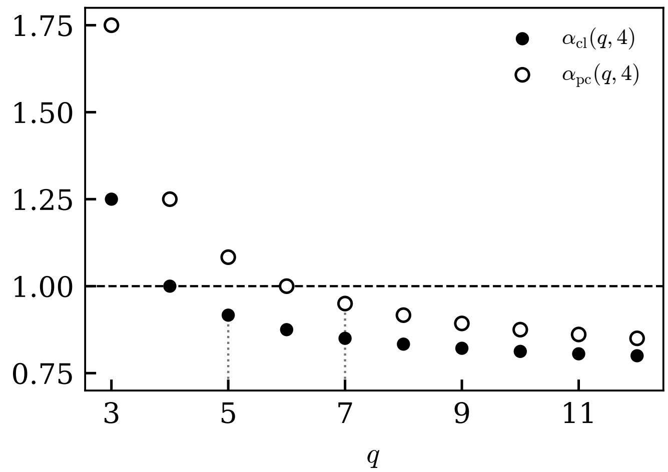

FIGURE 5. $\alpha_{\mathrm{cl}}(q,4)$ (filled) and $\alpha_{\mathrm{pc}}(q,4)$ (open); dotted verticals at the thresholds $q=5,7$, dashed line at 1.

Example 6.1.2. For $q=7$, $n=4$, $\varepsilon=0.05$,

$$\alpha_{\mathrm{cl}}(7)=\frac{3}{4}+\frac{1}{10}=\frac{17}{20}, \qquad \alpha_{\mathrm{pc}}(7)=\frac{3}{4}+\frac{1}{5}=\frac{19}{20} \xLongrightarrow{5.2.4} \alpha_{\mathrm{pc}}(7)-\alpha_{\mathrm{cl}}(7)=\frac{2}{20}=\frac{2}{4\cdot 5}$$

$$\left\lceil \frac{\log(80)}{\log(20/17)} \right\rceil = \lceil 26.963 \rceil = 27, \qquad \left\lceil \frac{\log(120)}{\log(20/19)} \right\rceil = \lceil 93.335 \rceil = 94, \qquad 18<27<94.$$

Example 6.1.3.

$$\lim_{q\to\infty} \alpha_{\mathrm{cl}}(q,n)=\lim_{q\to\infty} \alpha_{\mathrm{pc}}(q,n)=1-\frac{1}{n}, \qquad \alpha_{\mathrm{pc}}(q,n)-\alpha_{\mathrm{cl}}(q,n)=\frac{2}{n(q-2)} \longrightarrow 0,$$

$$n=4,\ q=12 \implies \alpha_{\mathrm{cl}}=\frac{4}{5}, \qquad \alpha_{\mathrm{pc}}=\frac{17}{20}, \qquad \alpha_{\mathrm{pc}}-\alpha_{\mathrm{cl}}=\frac{1}{20}=\frac{2}{4\cdot 10}.$$

6.2. **The diameter of the color graph.**

Definition 6.2.1. A *schedule* from x to y is a sequence $x=z_0,z_1,\dots,z_L=y$ in $\Omega_q(C_n)$ with $(z_{j-1},z_j)\in E(\mathscr{C}_q(C_n))$ for every j, and

$$m_v=\#\{j \mid z_j(v)\neq z_{j-1}(v)\}, \qquad L=\sum_{v\in V(C_n)} m_v.$$

Lemma 6.2.2 (Charge). *Let $e_1,\dots,e_s\in E(C_n)$ satisfy $e_i\cap e_j=\emptyset$ for $i\neq j$, and for each $e_i=(v,u)$*

$$y(v)=x(u)\neq x(v), \qquad y(u)=x(v)\neq x(u).$$

Then every schedule from x to y satisfies $L\ge d_{\mathrm{H}}(x,y)+s$.

Proof.

$$x(v)\neq y(v) \implies m_v\ge 1 \implies \sum_v m_v \ge d_{\mathrm{H}}(x,y)$$

$$e_i=(v,u), \qquad t_v=\max\{j \mid z_j(v)\neq z_{j-1}(v)\}<t_u, \qquad z_{t_v}(v)=y(v) \xLongrightarrow{2.6.1} z_{t_v}(u)\neq y(v)=x(u)$$
$$\implies \exists\, j<t_v\ \ z_j(u)\neq z_{j-1}(u) \implies m_u\ge 2$$

$$e_i \cap e_j = \emptyset \implies \sum_v m_v \geq d_{\mathrm{H}}(x,y) + s.$$

$\square$

Theorem 6.2.3 (Diameter on C_n)**.** *For all $n \geq 3$ and $q \geq 5$, with $m = \lfloor n/2 \rfloor$,*

$$\operatorname{diam} d_{\mathscr{G}} = \left\lfloor \frac{3n}{2} \right\rfloor = \operatorname{diam} d_{\mathrm{H}} + m.$$

Proof.

$$x|_{\{0,\dots,2m-1\}} = (0,1,\dots,0,1), \qquad y|_{\{0,\dots,2m-1\}} = (1,0,\dots,1,0), \qquad 2 \nmid n \implies \big(x(n{-}1), y(n{-}1)\big) = (2,3)$$

$$\implies d_{\mathrm{H}}(x,y) = n, \qquad y(2i) = x(2i{+}1), \quad y(2i{+}1) = x(2i), \quad (2i, 2i{+}1) \cap (2j, 2j{+}1) = \emptyset \ \ (i \neq j)$$

$$\overset{6.2.2}{\implies} \operatorname{diam} d_{\mathscr{G}} \geq n + m = \left\lfloor \frac{3n}{2} \right\rfloor.$$

$$I \subset V(C_n) \text{ independent}, \qquad |I| = m, \qquad 2 \nmid n \implies \{(a, a{+}1)\} = E\big(C_n[V \setminus I]\big)$$

$$x(v{+}1) = y(v) \ \wedge \ x(v) = y(v{+}1) \ \ \forall v \implies x(v{+}2) = y(v{+}1) = x(v) \ \ \forall v \overset{\gcd(2,n)=1}{\implies} x(0) = x(1) \overset{2.6.1}{\implies} \perp$$

$$\implies \exists \text{ rotation of } I \ \ x(a{+}1) \neq y(a)$$

$$\text{(i)} \ \ z(v) := s_v \in [q] \setminus \{x(v{-}1), x(v{+}1), y(v{-}1), y(v{+}1)\} \ \ (v \in I), \qquad q \geq 5 \implies s_v \text{ exists}$$

$$\text{(ii)} \ \ z(a) := y(a), \quad z(a{+}1) := y(a{+}1), \quad z(u) := y(u) \ \ (u \notin I \cup \{a, a{+}1\}), \qquad \text{(iii)} \ \ z(v) := y(v) \ \ (v \in I)$$

$$\text{(i)} \ \ s_v \notin \{x(v{-}1), x(v{+}1)\}, \qquad \text{(ii)} \ \ u \pm 1 \in I \implies y(u) \notin \{s_{u-1}, s_{u+1}\},$$

$$y(a) \notin \{s_{a-1}, x(a{+}1)\}, \qquad y(a{+}1) \notin \{y(a), s_{a+2}\}, \qquad \text{(iii)} \ \ y(v) \notin \{y(v{-}1), y(v{+}1)\}$$

$$\implies d_{\mathscr{G}}(x,y) \leq |I| + |V \setminus I| + |I| = m + (n-m) + m = \left\lfloor \frac{3n}{2} \right\rfloor.$$

$\square$

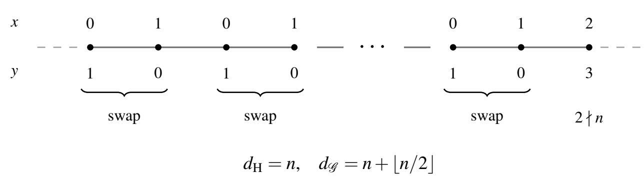

FIGURE 6. The extremal pair of Theorem 6.2.3 on C_n.

Example 6.2.4. For $n = 5$, $q = 5$, the extremal pair $x = (0,1,0,1,2)$, $y = (1,0,1,0,3)$ has $I = \{1,3\}$, leftover edge $(4,0)$ with $x(0) = 0 \neq 3 = y(4)$, and

$$s_1 \in [5] \setminus \{0,0,1,1\}, \qquad s_3 \in [5] \setminus \{0,2,1,3\} \implies s_1 = s_3 = 4.$$

Figure 7 records the schedule row by row, and

$$L = 2 + 3 + 2 = 7 \overset{6.2.2}{\implies} 7 \leq d_{\mathscr{G}}(x,y) \leq L \implies d_{\mathscr{G}}(x,y) = 7 = \left\lfloor \frac{15}{2} \right\rfloor.$$

t	v_0	v_1	v_2	v_3	v_4	move
0	0	1	0	1	2	
1	0	$\boxed{4}$	0	1	2	$v_1 \mapsto s_1$
2	0	4	0	$\boxed{4}$	2	$v_3 \mapsto s_3$
3	$\boxed{1}$	4	0	4	2	$v_0 \mapsto y(v_0)$
4	1	4	$\boxed{1}$	4	2	$v_2 \mapsto y(v_2)$
5	1	4	1	4	$\boxed{3}$	$v_4 \mapsto y(v_4)$
6	1	$\boxed{0}$	1	4	3	$v_1 \mapsto y(v_1)$
7	1	0	1	$\boxed{0}$	3	$v_3 \mapsto y(v_3)$

FIGURE 7. Space–time record of the three-phase schedule of Example 6.2.4; boxes mark moves.

Example 6.2.5. For $n=6$, $q\geq 5$, $x=(0,1,0,1,0,1)$, $y=(1,0,1,0,1,0)$, $I=\{0,2,4\}$, $s_0=s_2=s_4=2$,

$$(0,1,0,1,0,1)\xrightarrow{\text{(i)}}(2,1,2,1,2,1)\xrightarrow{\text{(ii)}}(2,0,2,0,2,0)\xrightarrow{\text{(iii)}}(1,0,1,0,1,0),$$

$$L=3+3+3=9\overset{6.2.2}{\Longrightarrow}9\leq d_{\mathscr{G}}(x,y)\leq L\implies d_{\mathscr{G}}(x,y)=9=\left\lfloor\frac{18}{2}\right\rfloor.$$

Example 6.2.6. Breadth-first search (Appendix B) confirms $\operatorname{diam} d_{\mathscr{G}}=\lfloor 3n/2\rfloor$ at every computed pair, including $q=4$, where Theorem 6.2.3 asserts only the lower bound,

(n,q)	$(3,4)$	$(3,5)$	$(4,3)$	$(4,4)$	$(4,5)$	$(5,4)$	$(5,5)$	$(6,4)$	$(7,4)$
$\operatorname{diam} d_{\mathscr{G}}$	4	4	6	6	6	7	7	9	10
$\lfloor 3n/2\rfloor$	4	4	6	6	6	7	7	9	10

Remark 6.2.7. Phase (i) is the only use of the fifth color, so for $q=4$ the theorem gives the lower bound and Example 6.2.6 the value for $n\leq 7$. Color-graph distances are superpolynomial below $\Delta+2$ colors [22], with diameters surveyed in [25, 26]; the closed form here rests on the one-dimensional geometry of the cycle.

6.3. **Exact computation on the four-cycle.** Since $|\Omega_q(C_4)|\leq 1302$ for $q\leq 7$, every quantity is computed exactly. The transition matrix of Definition 2.6.1 is assembled over $\Omega_q(C_4)$, contraction constants are evaluated in rational arithmetic over all pairs, and $d_{\mathscr{G}}$ by breadth-first search in $\mathscr{C}_q(C_4)$; Appendix B.

q	$\lvert\Omega_q(C_4)\rvert$	$\alpha_{\mathrm{cl}}(q)$	$\alpha_{\mathrm{pc}}(q)$	classical	path	$t_{\mathrm{mix}}(0.05)$	$\lambda_\star$
3	18	$5/4$	$7/4$	—	—	57	0.9557
4	84	1	$5/4$	—	—	29	0.9115
5	260	$11/12$	$13/12$	51	—	22	0.8734
6	630	$7/8$	1	33	—	19	0.8494
7	1302	$17/20$	$19/20$	27	94	18	0.8330

TABLE 1. Exact data for C_4 at $\varepsilon=0.05$; dashes mark absent contraction.

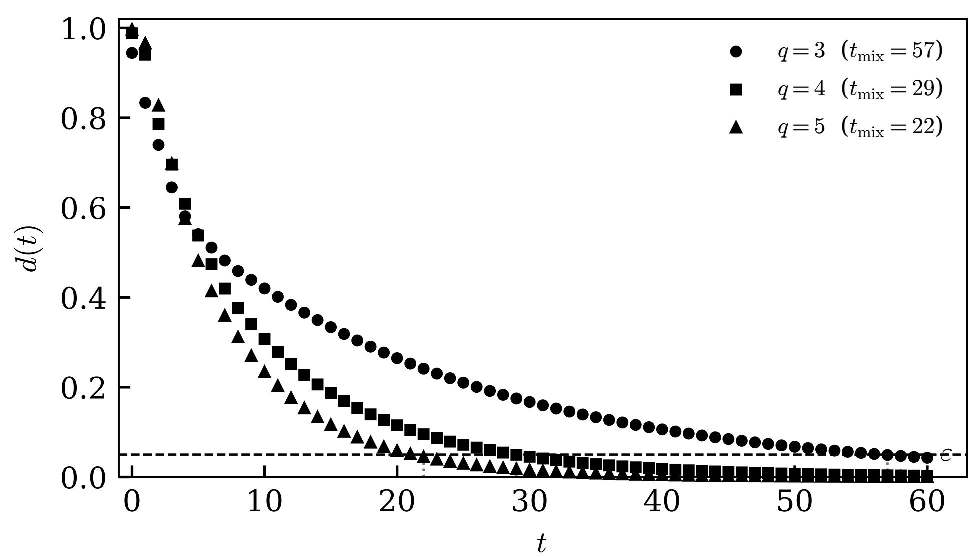

FIGURE 8. Exact $d(t)=\max_{x\in\Omega_q(C_4)}\|P^t(x,\cdot)-\pi\|_{\mathrm{TV}}$; dotted verticals at $t_{\mathrm{mix}}(0.05)=22, 29, 57$, dashed line at $\varepsilon=0.05$.

Proposition 6.3.1 (Both constants are attained)**.** *For every $q \ge 4$,*

$$\max_{\substack{(x,y)\in\Omega_q(C_4)^2\\ d_{\mathrm{H}}(x,y)=k}} \frac{\mathbb{E}[D_1 \mid X_0=x,\,Y_0=y]}{k} = \alpha_{\mathrm{cl}}(q,4) \quad \forall k\in\{1,2,3,4\}, \qquad \max_{\substack{(x,y)\in\Omega_q(C_4)^2\\ d_{\mathscr{G}}(x,y)=1}} \mathbb{E}\big[d_{\mathscr{G}}(X_1,Y_1)\big] = \alpha_{\mathrm{pc}}(q,4),$$

and both maxima equal 1 *at* $q=3$.

Proof.

$$\text{rational enumeration over } \Omega_q(C_4)\times\Omega_q(C_4) \text{ (Appendix B)} \overset{4.2.2,\,5.2.2}{\Longrightarrow} \text{attainment at } q=4,5.$$

□

Proposition 6.3.2 (Geometry of the color graph on C_4)**.** *For every $q\ge 3$,*

$$\operatorname{diam} d_{\mathrm{H}} = 4, \qquad \operatorname{diam} d_{\mathscr{G}} = 6, \qquad \max_{x,y}\big(d_{\mathscr{G}}(x,y)-d_{\mathrm{H}}(x,y)\big)=2,$$

$$\#\big\{(x,y)\;\big|\; d_{\mathrm{H}}=2,\ d_{\mathscr{G}}=3\big\} = 0,\ 96,\ 480,\ 1440,\ 3360 \qquad (q=3,\dots,7).$$

Proof.

$$q\ge 5 \overset{6.2.3}{\Longrightarrow} \operatorname{diam} d_{\mathscr{G}} = \Big\lfloor \frac{3\cdot 4}{2}\Big\rfloor = 6, \qquad q\in\{3,4\} \overset{6.2.6}{\Longrightarrow} \operatorname{diam} d_{\mathscr{G}} = 6$$

$$\text{breadth-first search (Appendix B)} \overset{5.1.2}{\Longrightarrow} \text{the counted pairs are the swap pairs, absent at } q=3.$$

□

Remark 6.3.3 (Resolution of sampled estimates). For $\widehat{\mu}_N$ the empirical measure of N trajectories and π uniform on $|\Omega_q(C_4)|$ atoms,

$$\operatorname{Var}\big(\widehat{\mu}_N(z)\big) = \frac{\pi(z)\big(1-\pi(z)\big)}{N} \implies \mathbb{E}\big|\widehat{\mu}_N(z)-\pi(z)\big| \le \sqrt{\frac{\pi(z)}{N}}$$

$$\implies \mathbb{E}\,\|\widehat{\mu}_N-\pi\|_1 \le \sum_{z\in\Omega_q(C_4)} \sqrt{\frac{1}{|\Omega_q(C_4)|N}} = \sqrt{\frac{|\Omega_q(C_4)|}{N}} = 0.134,\ 0.290,\ 0.510,\ 0.794,\ 1.141 \qquad (N=10^3,\ q=3,\dots,7),$$

exceeding the range of $d(t)$ once $q\ge 6$. A sampled curve therefore plateaus at a level increasing in q, an artifact of $|\Omega_q(C_4)|$, whereas Table 1 gives $t_{\mathrm{mix}}(0.05)$ strictly decreasing in q.

6.4. **Spectral analysis.**

Proposition 6.4.1 (Spectral bound for the symmetric chain)**.** *Let $N=|\Omega_q(C_4)|$ and $P=P^\top$ as in Lemma 2.6.3. Then there exist $Q\in\mathbb{R}^{N\times N}$ and real $1=\lambda_1\ge\dots$, $\lambda_\star=\max_{k\ge 2}\big(|\lambda_k|\big)$, with*

$$Q^\top Q = QQ^\top = I_N = \begin{bmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}, \qquad P=Q\Lambda Q^\top, \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_N\end{bmatrix}, \qquad Q_{x1}=\frac{1}{\sqrt{N}},$$

and

$$d(t) \;\le\; \tfrac12\sqrt{N}\,\lambda_\star^t \qquad \textit{for all } t\ge 0.$$

Proof.

$$P=P^\top \implies P=Q\Lambda Q^\top,\ Q^\top Q=I_N \qquad \text{(spectral theorem for real symmetric matrices)}$$

$$\sum_y P(x,y)=1\ \forall x, \qquad P=P^\top \implies P\mathbf{1}=\mathbf{1}, \qquad \mathbf{1}=(1,1,\dots,1)^\top \implies \lambda_1=1, \qquad Qe_1=\frac{\mathbf{1}}{\sqrt{N}}$$

$$P^2=\big(Q\Lambda Q^\top\big)\big(Q\Lambda Q^\top\big)=Q\Lambda\big(Q^\top Q\big)\Lambda Q^\top = Q\Lambda I_N\Lambda Q^\top = Q\Lambda^2Q^\top \implies P^t=Q\Lambda^tQ^\top\ \forall t$$

$$P^t(x,y)=\sum_{k=1}^N \lambda_k^t Q_{xk}Q_{yk} = \frac{1}{N}+\sum_{k=2}^N \lambda_k^t Q_{xk}Q_{yk} \implies P^t(x,y)-\pi(y)=\sum_{k=2}^N \lambda_k^t Q_{xk}Q_{yk}$$

$$2\,\|P^t(x,\cdot)-\pi\|_{\mathrm{TV}} = \sum_y\Big|\sum_{k=2}^N \lambda_k^tQ_{xk}Q_{yk}\Big| \le \sqrt{N}\,\Big(\sum_y\Big(\sum_{k=2}^N\lambda_k^tQ_{xk}Q_{yk}\Big)^2\Big)^{1/2}$$

$$\sum_y\Big(\sum_{k=2}^N \lambda_k^t Q_{xk}Q_{yk}\Big)^2=\sum_{k=2}^N\sum_{l=2}^N \lambda_k^t\lambda_l^t Q_{xk}Q_{xl}\sum_y Q_{yk}Q_{yl}=\sum_{k=2}^N\sum_{l=2}^N \lambda_k^t\lambda_l^t Q_{xk}Q_{xl}\,(I_N)_{kl}=\sum_{k=2}^N \lambda_k^{2t}Q_{xk}^2$$
$$\le \lambda_\star^{2t}\sum_{k=1}^N Q_{xk}^2=\lambda_\star^{2t}\,(QQ^\top)_{xx}=\lambda_\star^{2t}\,(I_N)_{xx}=\lambda_\star^{2t}\implies d(t)\le \tfrac{1}{2}\sqrt{N}\,\lambda_\star^t.$$

□

Example 6.4.2 (Drift and spectral bounds against the exact decay)**.** For $q=5$, $N=260$, $\alpha_{\mathrm{cl}}(5)=\frac{11}{12}$, $\lambda_\star=0.873446$,

$$d(10)=0.2335<4\big(\tfrac{11}{12}\big)^{10}=1.6756,\qquad d(22)=0.0444<4\big(\tfrac{11}{12}\big)^{22}=0.5899,\qquad d(30)=0.0150<0.2941,$$
$$\tfrac{1}{2}\sqrt{260}=8.0623,\qquad (0.873446)^{22}=\exp\big(22\log(0.873446)\big)=\exp(-2.977)=0.05090$$
$$\overset{6.4.1}{\implies} d(22)\le 8.0623\times 0.05090=0.4104,$$

and $\lambda_\star<\alpha_{\mathrm{cl}}(q)$ for every q in Table 1.

6.5. **Closed forms.**

Theorem 6.5.1 (Both methods on C_n, closed form)**.** *For every $n\ge 3$ and the coupling of Definition 4.1.1,*

$$q\ge 5\implies t_{\mathrm{mix}}(\varepsilon)\le\left\lceil\frac{n(q-2)}{q-4}\big(\log(n)+\log(\varepsilon^{-1})\big)\right\rceil,$$
$$q\ge 7\implies t_{\mathrm{mix}}(\varepsilon)\le\left\lceil\frac{n(q-2)}{q-6}\Big(\log\Big(\Big\lfloor\frac{3n}{2}\Big\rfloor\Big)+\log(\varepsilon^{-1})\Big)\right\rceil,$$

with the first strictly smaller for every $q\ge 7$, and both $O\big(n\log(n)\big)$ for fixed q.

Proof.

$$\operatorname{diam} d_{\mathrm{H}}=n\overset{4.3.3}{\implies}\text{the first bound},\qquad q\ge 7\overset{6.2.3}{\implies}\operatorname{diam} d_{\mathscr{G}}=\Big\lfloor\frac{3n}{2}\Big\rfloor\overset{2.5.3}{\implies}\text{the second}$$
$$0<q-6<q-4,\qquad n\le\Big\lfloor\frac{3n}{2}\Big\rfloor\overset{6.1.1}{\implies}\text{the ordering},\qquad \log\Big(\Big\lfloor\frac{3n}{2}\Big\rfloor\Big)\le\log(n)+\log(2).$$

□

Example 6.5.2. For $n=6$, $q=7$, $\varepsilon=0.05$,

$$\frac{6\cdot 5}{3}\big(\log(6)+\log(20)\big)=10\times 4.787492=47.875\implies t_{\mathrm{mix}}(0.05)\le 48,$$
$$\Big\lfloor\frac{3\cdot 6}{2}\Big\rfloor=9,\qquad \frac{6\cdot 5}{1}\big(\log(9)+\log(20)\big)=30\times 5.192957=155.789\implies t_{\mathrm{mix}}(0.05)\le 156,\qquad \frac{156}{48}=3.25.$$

Theorem 6.5.3 (The four-cycle, closed form)**.** *On $\Omega_q(C_4)$,*

$$|\Omega_q(C_4)|=(q-1)^4+(q-1),\qquad \alpha_{\mathrm{cl}}(q)=\frac{3}{4}+\frac{1}{2(q-2)},\qquad \alpha_{\mathrm{pc}}(q)=\frac{3}{4}+\frac{1}{q-2},$$

both attained, with thresholds $q\ge 5$ and $q\ge 7$, and

$$\operatorname{diam} d_{\mathrm{H}}=4,\qquad \operatorname{diam} d_{\mathscr{G}}=6,\qquad t_{\mathrm{mix}}(0.05)=57,\,29,\,22,\,19,\,18\qquad (q=3,\dots,7).$$

Proof.

Proposition 3.1.1, Theorems 4.3.1, 5.2.1 at $n=4$, Proposition 6.3.1,
Proposition 6.3.2, Table 1.

□

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APPENDIX A. FOUNDATIONAL TOOLS

A.1. Linear algebra over $\mathbb{R}$.

Definition A.1.1. $A \in \mathrm{Mat}_{N\times N}(\mathbb{R})$ is *symmetric* if $A = A^\top$, and $Q \in \mathrm{Mat}_{N\times N}(\mathbb{R})$ is *orthogonal* if $Q^\top Q = QQ^\top = I_N$.

Theorem A.1.2 (Spectral theorem, real symmetric case)**.** *Let $A = A^\top \in \mathrm{Mat}_{N\times N}(\mathbb{R})$. Then every eigenvalue of A is real, and there exist an orthogonal Q and $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$, $\lambda_j \in \mathbb{R}$, with*

$$A = Q\Lambda Q^\top.$$

Proof.

$$A\xi = \lambda\xi,\quad 0 \neq \xi \in \mathbb{C}^N \implies \lambda(\xi^*\xi) = \xi^*(A\xi) = (A\xi)^*\xi = (\lambda\xi)^*\xi = \bar\lambda(\xi^*\xi) \implies (\lambda-\bar\lambda)(\xi^*\xi) = 0 \implies \lambda \in \mathbb{R}$$

$$N = 1 \implies A = (\lambda_1),\qquad N \geq 2,\quad A\xi_1 = \lambda_1\xi_1,\quad \|\xi_1\| = 1,\quad W = (\xi_1)^\perp \subset \mathbb{R}^N$$

$$w \in W \implies \xi_1^\top(Aw) = (A\xi_1)^\top w = \lambda_1(\xi_1^\top w) = 0 \implies AW \subseteq W,\qquad (A|_W)^\top = A|_W$$

$$\overset{\text{induction}}{\Longrightarrow} \exists \text{ orthonormal eigenbasis } \xi_1,\dots,\xi_N \implies Q = (\xi_1 \cdots \xi_N),\qquad AQ = Q\Lambda \implies A = Q\Lambda Q^\top.$$

□

Corollary A.1.3. *If additionally $A_{xy} \geq 0$ and $\sum_y A_{xy} = 1$ for every x, then every eigenvalue satisfies $\lambda \in [-1,1]$.*

Proof.

$$A\xi = \lambda\xi,\qquad |\xi_i| = \max_j\bigl(|\xi_j|\bigr) > 0 \implies |\lambda|\,|\xi_i| = \Bigl|\sum_j A_{ij}\xi_j\Bigr| \leq \sum_j A_{ij}|\xi_j| \leq |\xi_i|\sum_j A_{ij} = |\xi_i| \implies |\lambda| \leq 1.$$

□

APPENDIX B. REPRODUCIBILITY

The chain is assembled entrywise from Definition 2.6.1,

$$P(x,y) = \frac{1}{4}\cdot\frac{1}{|L_x(v)|}\ \text{ if } x,y \text{ differ exactly at } v \text{ and } y(v) \in L_x(v),\qquad P(x,x) = 1 - \sum_{y\neq x} P(x,y),$$

$$d(t) = \max_{x\in\Omega_q(C_4)} \frac{1}{2}\sum_{y\in\Omega_q(C_4)}\Bigl|(P^t)(x,y) - \frac{1}{|\Omega_q(C_4)|}\Bigr|,\qquad \lambda_\star = \max_{k\geq 2}\bigl(|\lambda_k|\bigr) \overset{\text{A.1.2}}{\Longrightarrow} \lambda_k \in \mathbb{R}.$$

One-step expectations are exact rationals via Lemmas 4.1.2 and 2.2.4, the update at u being coupled by

$$\mu = U_{L_x(u)},\qquad \nu = U_{L_y(u)},\qquad r_\mu = \mu - \min\{\mu,\nu\},\qquad r_\nu = \nu - \min\{\mu,\nu\},$$

$$\mathbb{P}\bigl(X_1(u) = a,\ Y_1(u) = b\bigr) = \min\{\mu(a),\nu(a)\}\,\mathbf{1}\{a = b\} + \frac{r_\mu(a)\,r_\nu(b)}{\sum_c r_\mu(c)}\,\mathbf{1}\{a \neq b\},$$

and $d_{\mathscr{G}}$ by breadth-first search in $\mathscr{C}_q(C_n)$ (Definition 2.7.1). Listings 1–4 concatenated form `figures.py`; Listing 5 is `verify.py`.

B.1. Shared core.

LISTING 1. Common definitions for Listings 2–4.

```
import itertools
import numpy as np
import matplotlib
matplotlib.use("Agg")
import matplotlib.pyplot as plt

plt.rcParams.update({
    "font.family": "serif", "font.size": 9,
    "mathtext.fontset": "cm",
    "pdf.fonttype": 42, "ps.fonttype": 42,
    "axes.linewidth": 0.6,
    "xtick.direction": "in", "ytick.direction": "in",
    "savefig.bbox": "tight", "savefig.pad_inches": 0.03})

def save(fig, name):
```

```
    fig.savefig(name + ".pdf")
    fig.savefig(name + ".png", dpi=400)
    plt.close(fig)

def chain(q, n=4):
    S = [c for c in itertools.product(range(1, q + 1), repeat=n)
         if all(c[i] != c[(i + 1) % n] for i in range(n))]
    idx = {c: i for i, c in enumerate(S)}
    N = len(S)
    P = np.zeros((N, N))
    for c in S:
        i = idx[c]
        for v in range(n):
            legal = [x for x in range(1, q + 1)
                     if x not in {c[(v - 1) % n], c[(v + 1) % n]}]
            for x in legal:
                d = list(c)
                d[v] = x
                P[i, idx[tuple(d)]] += 1.0 / (n * len(legal))
    return S, P

def worst_tv(P, T):
    N = len(P)
    pi = np.full(N, 1.0 / N)
    M = np.eye(N)
    out = []
    for _ in range(T + 1):
        out.append(0.5 * np.abs(M - pi).sum(axis=1).max())
        M = M @ P
    return np.array(out)
```

B.2. **Figure 5.**

LISTING 2. Generation of Figure 5.

```
fig, ax = plt.subplots(figsize=(3.3, 2.2))
qs = np.arange(3, 13)
acl = 0.75 + 1 / (2 * (qs - 2))
apc = 0.75 + 1 / (qs - 2)
ax.axhline(1, ls="--", lw=0.7, color="black", zorder=1)
ax.plot([5, 5], [0.70, acl[2]], ":", lw=0.7, color="0.45", zorder=1)
ax.plot([7, 7], [0.70, apc[4]], ":", lw=0.7, color="0.45", zorder=1)
ax.plot(qs, acl, "o", ms=3.2, color="black", linestyle="none",
        zorder=3, clip_on=False, label=r"$\alpha_{\mathrm{cl}}(q,4)$")
ax.plot(qs, apc, "o", ms=4.2, mfc="white", mec="black", mew=0.8,
        linestyle="none", zorder=4, clip_on=False,
        label=r"$\alpha_{\mathrm{pc}}(q,4)$")
ax.set_xlabel(r"$q$")
ax.set_xticks([3, 5, 7, 9, 11])
ax.set_ylim(0.70, 1.80)
ax.set_yticks([0.75, 1.00, 1.25, 1.50, 1.75])
ax.legend(frameon=False, fontsize=7, loc="upper right")
save(fig, "fig_alphas")
```

B.3. **Figure 8.**

LISTING 3. Generation of Figure 8.

```
fig, ax = plt.subplots(figsize=(4.6, 2.5))
markers = {3: "o", 4: "s", 5: "^"}
tmix = {}
ax.axhline(0.05, ls="--", lw=0.7, color="black", zorder=1)
for qq in (3, 4, 5):
    d = worst_tv(chain(qq)[1], 60)
    tmix[qq] = int(np.argmax(d <= 0.05))
    ax.plot([tmix[qq]] * 2, [0, 0.05], ":", lw=0.7, color="0.45", zorder=1)
    ax.plot(np.arange(61), d, markers[qq], ms=2.6, mfc="black", mec="black",
            linestyle="none", zorder=3, clip_on=False,
            label=rf"$q={qq}$  ($t_{{\mathrm{{mix}}}}={tmix[qq]}$)")
ax.text(61.0, 0.05, r"$\varepsilon$", va="center", fontsize=8)
ax.set_xlabel(r"$t$")
ax.set_ylabel(r"$d(t)$")
ax.set_xlim(-1, 63)
```

```
ax.set_ylim(0, 1.02)
ax.legend(frameon=False, fontsize=7)
save(fig, "fig_decay")
```

B.4. **Figure 2.**

LISTING 4. Generation of Figure 2.

```
fig, ax = plt.subplots(figsize=(3.3, 2.3))
ns = np.arange(3, 11)
for qq, mk, mf in ((3, "o", "black"), (4, "s", "white")):
    ax.semilogy(ns, (qq - 1.0) ** ns, "--", lw=0.7, color="0.45", zorder=1)
    ax.semilogy(ns, (qq - 1.0) ** ns + (-1.0) ** ns * (qq - 1), mk, ms=3.4,
                mfc=mf, mec="black", mew=0.8, linestyle="none",
                zorder=3, clip_on=False,
                label=rf"$|\Omega_{{{qq}}}(C_n)|$")
ax.text(10.2, 2.0 ** 10, r"$2^n$", fontsize=7, va="center")
ax.text(10.2, 3.0 ** 10, r"$3^n$", fontsize=7, va="center")
ax.set_xlabel(r"$n$")
ax.set_xticks([3, 5, 7, 9])
ax.set_xlim(2.6, 11.3)
ax.legend(frameon=False, fontsize=7, loc="upper left")
save(fig, "fig_growth")

print("wrote fig_{alphas,decay,growth};  t_mix:", tmix)
```

B.5. **Verification.**

LISTING 5. `verify.py`, certifying Table 1, Propositions 6.3.1 and 6.3.2, and Examples 5.1.3, 6.2.4, 6.2.5, and 6.2.6.

```
import itertools
from collections import deque
from fractions import Fraction
import numpy as np

def states(q, n):
    return [c for c in itertools.product(range(q), repeat=n)
            if all(c[i] != c[(i + 1) % n] for i in range(n))]

def legal(x, v, q, n):
    return [c for c in range(q) if c not in {x[(v - 1) % n], x[(v + 1) % n]}]

def graph(q, n):
    S = states(q, n)
    idx = {c: i for i, c in enumerate(S)}
    adj = [[] for _ in S]
    for c in S:
        for v in range(n):
            for col in legal(c, v, q, n):
                if col != c[v]:
                    d = list(c); d[v] = col
                    adj[idx[c]].append(idx[tuple(d)])
    return S, idx, adj

def all_dists(S, adj):
    N = len(S); D = []
    for s in range(N):
        d = [-1] * N; d[s] = 0; Q = deque([s])
        while Q:
            u = Q.popleft()
            for w in adj[u]:
                if d[w] < 0:
                    d[w] = d[u] + 1; Q.append(w)
        D.append(d)
    return D

def tv_uniform(A, B):
    return 1 - len(set(A) & set(B)) * Fraction(1, max(len(A), len(B)))

for q in (3, 4, 5, 6, 7):
    S = states(q, 4); N = len(S)
    idx = {c: i for i, c in enumerate(S)}
```

```
    P = np.zeros((N, N))
    for c in S:
        i = idx[c]
        for v in range(4):
            L = legal(c, v, q, 4)
            for col in L:
                d = list(c); d[v] = col
                P[i, idx[tuple(d)]] += 1.0 / (4 * len(L))
    pi = np.full(N, 1.0 / N); M = np.eye(N); tmix = None
    for t in range(200):
        if tmix is None and 0.5 * np.abs(M - pi).sum(axis=1).max() <= 0.05:
            tmix = t
        M = M @ P
    lam = float(np.sort(np.abs(np.linalg.eigvalsh(P)))[-2])
    print(f"q={q}  |Omega|={N}  t_mix={tmix}  lambda*={lam:.4f}")

for q in (3, 4, 5, 6, 7):
    S = states(q, 4)
    best = {k: Fraction(0) for k in (1, 2, 3, 4)}
    for x in S:
        for y in S:
            S0 = [v for v in range(4) if x[v] != y[v]]
            k = len(S0)
            if k == 0:
                continue
            E = Fraction(0)
            for u in range(4):
                T = tv_uniform(legal(x, u, q, 4), legal(y, u, q, 4))
                E += Fraction(1, 4) * (k - (u in S0) + T)
            best[k] = max(best[k], E / k)
    acl = Fraction(3, 4) + Fraction(1, 2 * (q - 2))
    print(f"q={q}  classical maxima {dict(best)}  alpha_cl={acl}")

for q in (3, 4, 5, 6, 7):
    S, idx, adj = graph(q, 4)
    D = all_dists(S, adj)
    best = Fraction(0)
    for i, x in enumerate(S):
        for j in adj[i]:
            y = S[j]
            E = Fraction(0)
            for u in range(4):
                A, B = legal(x, u, q, 4), legal(y, u, q, 4)
                mA, mB = Fraction(1, len(A)), Fraction(1, len(B))
                mu = {c: (mA if c in A else Fraction(0)) for c in set(A) | set(B)}
                nu = {c: (mB if c in B else Fraction(0)) for c in set(A) | set(B)}
                rA = {c: mu[c] - min(mu[c], nu[c]) for c in mu}
                rB = {c: nu[c] - min(mu[c], nu[c]) for c in nu}
                T = sum(rA.values())
                for c in mu:
                    w = min(mu[c], nu[c])
                    if w:
                        X = list(x); X[u] = c; Y = list(y); Y[u] = c
                        E += Fraction(1, 4) * w * D[idx[tuple(X)]][idx[tuple(Y)]]
                if T:
                    for ca, wa in rA.items():
                        if not wa: continue
                        for cb, wb in rB.items():
                            if not wb: continue
                            X = list(x); X[u] = ca; Y = list(y); Y[u] = cb
                            E += Fraction(1, 4) * wa * wb / T * D[idx[tuple(X)]][idx[tuple(Y)]]
            best = max(best, E)
    apc = Fraction(3, 4) + Fraction(1, q - 2)
    print(f"q={q}  path maximum {best}  alpha_pc={apc}")

for q in (3, 4, 5, 6, 7):
    S, idx, adj = graph(q, 4)
    D = all_dists(S, adj)
    dG = max(max(r) for r in D)
    swaps = sum(1 for i, x in enumerate(S) for j, y in enumerate(S)
                if sum(a != b for a, b in zip(x, y)) == 2 and D[i][j] == 3)
    gap = max(D[i][j] - sum(a != b for a, b in zip(S[i], S[j]))
              for i in range(len(S)) for j in range(len(S)))
    print(f"q={q}  diam_G={dG}  max(dG-dH)={gap}  swap pairs={swaps}")

for (n, q) in ((3,4),(3,5),(4,3),(4,4),(4,5),(5,4),(5,5),(6,4),(7,4)):
    S, idx, adj = graph(q, n)
    dG = max(max(r) for r in all_dists(S, adj))
    print(f"n={n} q={q}  diam_G={dG}  floor(3n/2)={3*n//2}")
```

```
def replay(z, moves, y, q, n):
    z = list(z)
    for v, c in moves:
        assert c != z[v] and c in legal(z, v, q, n)
        z[v] = c
    assert tuple(z) == y
    return len(moves)

print("C5 schedule:", replay((0,1,0,1,2),
      [(1,4),(3,4),(0,1),(2,1),(4,3),(1,0),(3,0)], (1,0,1,0,3), 5, 5), "moves")
print("C6 schedule:", replay((0,1,0,1,0,1),
      [(0,2),(2,2),(4,2),(1,0),(3,0),(5,0),(0,1),(2,1),(4,1)], (1,0,1,0,1,0), 5, 6), "moves")
```

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