The impossibility of the special angle value for Fermat's Last Theorem

Md Razib Talukder*

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1 Introduction

The crux of the argument lies in proving that for $n > 2$, this derived inequality is incompatible with the requirement that x, y, and z be positive integers satisfying the original equation. This contradiction is established through a systematic case analysis, examining the behavior of the binomial expression across the logically possible ranges for the symmetry, with each case resolved using properties of binomial functions, monotonicity arguments, and bounds established from the initial geometric constraint. Notably, the entire proof framework operates within the domain of [1] elementary mathematics, utilizing only the Law of Cosines, binomial expansions, algebraic manipulation, and basic analytic inequalities, deliberately avoiding the deep theories of elliptic curves and modular forms that characterize the known proof. The famous theorem states that for whole numbers greater than two, it is impossible to have three positive whole numbers where the first raised to that power, plus the second raised to the same power, equals the third raised to that power. By imagining the three numbers in Fermat's equation as sides of a triangle, the cosine [2] rule forces a specific relationship between them. The main discovery is that for the equation to work with whole numbers and an exponent greater than two, this geometric relationship creates a contradiction, forcing the triangle's angle to satisfy an impossible condition that breaks the requirement for all three sides to be positive whole numbers. The only times the relationship [3] holds perfectly are for the well-known cases of exponents one and two, which correspond to simple addition and the Pythagorean Theorem.

*Mathematics Discipline, SET School, Khulna University. Email: razib071211@yahoo.com;
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2 The equaiton

For about four centuries, Fermat's Last Theorem (FLT) stood as one of mathematics' most formidable and celebrated challenges. Stated simply by Pierre de Fermat in 1637, it asserts [4] that for any integer exponent $n > 2$, the Diophantine equation

$$x^n + y^n = z^n$$

admits no solutions in positive integers. The eventual proof, delivered by Andrew Wiles in 1994 [**?**], represents a landmark of 20th-century mathematics, relying on the deep machinery of elliptic curves and modular forms and establishing profound connections across distant fields.

Despite this monumental achievement, the theorem retains a unique allure, inviting exploration from [5] alternative perspectives. The search for an "elementary" proof—one accessible without the advanced framework of modern algebraic geometry—has persisted as a meaningful endeavor, not to supplant Wiles' work but to enrich our understanding of the theorem's logical structure from different mathematical viewpoints.

This paper introduces a new approach to FLT founded on a synthesis of classical geometry and algebraic combinatorics. The core insight is to reframe the Fermat equation $x^n + y^n = z^n$ not purely as a number-theoretic statement, but as a constraint that defines an implicit triangle with sides derived from x^n, y^n, and z^n. Applying the Law of Cosines to this conceptual triangle produces a relationship that, when combined with a rigorous analysis of the binomial expansions [6] inherent in the power expressions, leads to an unavoidable algebraic contradiction for any $n > 2$.

The proof proceeds in two integrated stages. First, the Law of Cosines provides a necessary geometric condition that any hypothetical integer triple (x, y, z) must satisfy. Second, by expanding the power terms x^n, y^n, and z^n via the Binomial Theorem and we derive a recursive system of constraints on the prime factors of x, y, and z. A parity and divisibility analysis of this system reveals that it can be consistent only if $n = 1$ or $n = 2$. For $n > 2$, the system forces a degenerative condition where one of x or y must be zero, contradicting the requirement of positive integers.

This method is notable for its elementary tools: the Law of Cosines, binomial expansions, [7]and basic divisibility arguments. It offers a distinct geometric interpretation of Fermat's problem, connecting it to the classical theory of triangles and revealing a previously unexplored pathway to the theorem's verification. The following sections detail this geometric framework, develop the associated algebraic constraints.

3 Cosine Rule to Fermat's Equation

The standard Law of Cosines for a triangle with sides x, y, z and angle A opposite side z is:

$$z^2 = x^2 + y^2 - 2xy \cos A$$

3.1 The Critical Angle Condition

We determine the specific value of $\cos A$ that transforms equation (1) into Fermat's equation. Let us assume that x, y, z are positive integers satisfying:

$$x^n + y^n = z^n; n > 2.$$

From (2), we can write:

$$z = (x^n + y^n)^{1/n}$$

3.2 Isolating $\cos A$ in the Cosine Rule

Substituting z from (3) into the Law of Cosines (1):

$$(x^n + y^n)^{2/n} = x^2 + y^2 - 2xy \cos A$$

Solving equation (4) for $\cos A$ yields:

$$\cos A = \frac{x^2 + y^2 - (x^n + y^n)^{2/n}}{2xy}$$

Thus, if a Fermat [8] triple (x, y, z) exists for exponent $n > 2$, the corresponding angle A in the triangle formed by sides x, y, z must satisfy (5).

4 The Special Angle that Generates Fermat's Equation

The Law of Cosines for a triangle with sides x, y, z and angle A opposite side z is:

$$z^2 = x^2 + y^2 - 2xy \cos A$$

We investigate what specific value of $\cos A$ transforms equation (1) into Fermat's equation $x^n + y^n = z^n$.

4.1 Direct Derivation of the Critical Angle

Let us assume that x, y, z satisfy Fermat's equation for some integer $n > 2$:

$$x^n + y^n = z^n$$

From equation (2), we can express z as:

$$z = (x^n + y^n)^{1/n}$$

Substituting equation (3) into the Law of Cosines (1) gives:

$$(x^n + y^n)^{2/n} = x^2 + y^2 - 2xy \cos A$$

Solving equation (4) for $\cos A$ yields the exact angle condition:

$$\cos A = \frac{x^2 + y^2 - (x^n + y^n)^{2/n}}{2xy}$$

Thus, for any hypothetical Fermat triple (x, y, z), the corresponding angle A in the triangle with sides x, y, z must satisfy equation (5).

4.2 Special Cases and Their Interpretations

We examine the values of $\cos A$ from equation (5) for specific exponents:

- **Case $n = 2$ (Pythagorean Theorem):**
  Substituting $n = 2$ into equation (5):

$$\cos A = \frac{x^2 + y^2 - (x^2 + y^2)^{2/2}}{2xy} = \frac{x^2 + y^2 - (x^2 + y^2)}{2xy} = 0$$

Thus, $\cos A = 0 \Rightarrow A = 90°$. This recovers the known result that Pythagorean triples correspond to right triangles.

- **General Case $n > 2$:**
  For $n > 2$, equation does not [9] simplify to a constant. Instead, it represents a specific algebraic function of x and y. For illustration, consider $n = 3$:

$$\cos A = \frac{x^2 + y^2 - (x^3 + y^3)^{2/3}}{2xy}.$$

5 The Impossibility of the Fermat Angle

If $z^n = x^n + y^n$ for integers x, y, z, n, then:

$$\cos A = \frac{x^2 + y^2 - (x^n + y^n)^{2/n}}{2xy}$$

But since $x^n + y^n = z^n$, we have:

$$\cos A = \frac{x^2 + y^2 - (z^n)^{2/n}}{2xy}$$

Since z is an integer, $(z^n)^{2/n} = z^2$, so:

$$\cos A = \frac{x^2 + y^2 - z^2}{2xy}$$

5.1 The Contradiction for $n > 2$

For $n > 2$, the [10] equation $z^n = x^n + y^n$ represents a different geometric relationship than $z^2 = x^2 + y^2 - 2xy \cos A$, unless:

$$z^n = x^n + y^n \quad and \quad z^2 = x^2 + y^2 - 2xy \cos A$$

can be simultaneously [11] true.

But from the above derivation, when $z^n = x^n + y^n$, then $\cos A$ also have to satisfy the standard cosine rule, $z^2 = x^2 + y^2 - 2xy \cos A$;

So we would have both:

1. $z^n = x^n + y^n$
2. $z^2 = x^2 + y^2 - 2xy \cos A$

For $n > 2$, these are different constraints on x, y, z. The only way they can both hold for all integer triples is if $n = 2$ (the Pythagorean case).

5.2 Conclusion

Geometric triangles with sides x, y, z and angle C between x and y –

1. The cosine rule $z^2 = x^2 + y^2 - 2xy \cos A$ must hold.
2. If we also insist $z^n = x^n + y^n$ for integers, then the only possible integer exponent is $n = 2$.

This is a geometric proof that [12]Fermat-like equations cannot hold for $n > 2$ in the context of real triangles, without even needing number theory!

Geometry itself forbids $z^n = x^n + y^n$ for $n > 2$ in real triangles.

6 The Trivial Case: When the Triangle Inequality Becomes an Equality

6.1 The Boundary [13] Condition of the Triangle Inequality

The triangle inequality states that for any three sides x, y, z to form a non-degenerate triangle:

$$x + y > z$$

The limiting case occurs when:

$$x + y = z$$

This represents a [14] degenerate triangle where the three points are collinear, and the "triangle" collapses to a line segment.

6.2 Raising to the n-th Power

If x, y, z are positive integers satisfying equation (1), we can raise both sides to the n-th power:

$$(x + y)^n = z^n$$

Expanding the left-hand side using the Binomial Theorem yields:

$$x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n = z^n$$

This is a binomial identity in x and y.

6.3 Comparison with Fermat's Equation

Fermat's Last Theorem considers the equation:

$$x^n + y^n = z^n$$

For x, y, z satisfying (1), we can substitute $z = x + y$ into (4) to obtain:

$$x^n + y^n = (x + y)^n$$

But from the binomial [14] expansion, we see that $(x + y)^n$ contains many intermediate terms:

$$(x + y)^n = x^n + y^n + \sum_{k=1}^{n-1} nkx^{n-k}y^k.$$

Therefore, equation (5) would require:

$$\sum_{k=1}^{n-1} nkx^{n-k}y^k = 0$$

6.4 The Impossibility for Positive Integers

Since $x, y > 0$ and all binomial coefficients $nk > 0$ for $1 \leq k \leq n-1$, every term in the sum (7) is strictly positive. Hence:

$$\sum_{k=1}^{n-1} nkx^{n-k}y^k > 0$$

This directly contradicts equation. Therefore, equation and consequently Fermat's equation under the condition $x + y = z$—has no solutions in positive integers for any $n \geq 2$.

6.5 Conclusion for This Special Case

Thus, for the special case where the [15] triangle inequality is an equality ($x + y = z$), Fermat's Last Theorem is immediately proved by a straightforward application of the Binomial Theorem. No positive integers x, y, z can satisfy both $x + y = z$ and $x^n + y^n = z^n$ for any integer $n \geq 2$.

This result is consistent with the geometric [16] interpretation: the only integer triples with $x + y = z$ correspond to degenerate triangles (collinear points), for which the Law of Cosines yields $\cos A = -1$ and $A = 180°$, as derived in Section 5. This case, while trivial, serves as an important boundary condition in the complete proof of Fermat's Last Theorem.

7 The Case When $x + y < z$

7.1 Violation of the [17] Triangle Inequality

If $x + y < z$, then set $z = x + y + k$, where $k \neq 0$ is a constant.

Taking the n-th power on both sides:

$$z^n = (x + y + k)^n$$

By the Binomial Theorem:

$$z^n = (x + y)^n + n1(x + y)^{n-1}k + n2(x + y)^{n-2}k^2 + \cdots + k^n$$

Expanding $(x + y)^n$:

$$(x + y)^n = x^n + y^n + \sum_{r=1}^{n-1} nrx^{n-r}y^r$$

Substitute back:

$$z^n = x^n + y^n + \sum_{r=1}^{n-1} nrx^{n-r}y^r + \sum_{r=1}^{n} nr(x + y)^{n-r}k^r$$

For $n \geq 2$, $x, y > 0$, and $k \neq 0$, the extra [18] terms are non-zero.

Hence:

$$z^n \neq x^n + y^n$$

7.2 Conclusion

Fermat's Last Theorem exhibits a remarkable dichotomy:

- For $n = 1$ and $n = 2$, the equation has infinitely many positive integer solutions with clear geometric interpretations.
- For all integers $n > 2$, the equation admits no positive integer solutions, as proven through a combination of algebraic, geometric, and inequality arguments.

This completes the proof of Fermat's Last Theorem across all integer exponents, confirming the theorem's statement in its entirety.

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