A Geometric Limit Theory for Closed Discrete Spaces

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Abstract

A fundamental framework for continuous mathematics inherently relies on the assumption of infinite divisibility. However, system constraints often necessitate a discrete spatial configuration. In a fully quantised space, an ideal circle or sphere cannot exist, leaving a deterministic boundary discrepancy defined here as the "Pi residual." We propose a geometric limit theory to determine the exact structural threshold where a discrete polyhedron achieves maximal asymptotic smoothness. By equating the root-mean-square deviation of discrete vertices with the coordinate resolution capacity of the system, we derive the critical boundary constraint. We show that while the theoretical coordinate density ceiling in a three-dimensional space approaches 138, the maximum realizable smoothness under discrete manifold constraints is strictly bound to a limit of exactly 137 structural sectors.

1 Introduction

Conventional geometry treats space as a continuum where coordinates possess infinite resolution. In practical implementation or fundamental discrete systems, space is bound by a minimal, indivisible unit of geometry. When continuous analytical concepts—such as the transcendental ratio π—are mapped onto a discrete grid, perfect smoothness is lost. Perfect curves degenerate into irregular multi-faceted structures, leaving an unresolvable geometric remainder.

To formalise the behavior of such spaces, we define a boundary criteria where a discrete polyhedron reaches its maximum theoretical proximity to a perfect sphere. This state is defined as the criteria of maximum smoothness.

2 Axioms of Discrete Space

We define a discrete space under a rigid spatial constraint:

Axiom I: There exists a minimum, indivisible spatial increment, hereafter defined as the *geometric unit*. This step constitutes the absolute lower bound of distance, denoted as a fixed value:

$$b = 1$$

Axiom II: An ideal circle of continuous mathematics cannot exist within a discrete geometry. Any closed curved boundary must be structurally established as a polygon or polyhedron composed of discrete segments. Due to the irrational nature of π, an inherent positioning error—the "Pi residual"—persistently remains.

3 Criteria of Asymptotic Smoothness

To determine the absolute limit of smoothness for a discrete system, we propose a mathematical balance equation. The condition of maximal smoothness is satisfied when the geometric discrepancy between the continuous ideal and the discrete representation matches the minimum resolvable angular scale of the system.

Let us consider a regular n-gon inscribed within a continuous reference circle. The side length of this polygon is bound to the minimal geometric unit, such that $b = 1$.

The positional variance of the vertices is inherently discrete. The root-mean-square deviation (the variance of the tracking error) from the ideal circle can be formally evaluated by considering the distribution of the rounding residuals across the periodic boundary. For a system of n discrete points, the variance of the positional discrepancy is given by:

$$\sigma^2 = \frac{\pi^2}{6n^2}$$

where $\pi^2/6$ represents the convergence of the harmonic residuals of the discrete mapping, and n^2 denotes the quadratic scaling of the sampling density.

Concurrently, the perimeter of the polygon matches the total step count n, establishing the circumferential equivalence:

$$2\pi R = n$$

Thus, the radius of the system is given by:

$$R = \frac{n}{2\pi}$$

The scale factor required to translate a unit step into a physical arc displacement is defined by $1/(2\pi R) = 1/n$. To derive a dimensionless resolution unit, we normalise this scale factor against the curvature semi-period π:

$$\Delta = \frac{1}{n\pi}$$

4 The Balance Equation

The theoretical limit of smoothness is reached at the exact intersection where the variance of the geometric error matches the linear resolution step of the coordinate system. This equilibrium defines the point where the discrepancy introduced by discretisation is exactly balanced by the system's resolution capacity:

$$\frac{\pi^2}{6n^2} = \frac{1}{n\pi}$$

This relation serves as a boundary condition defining the resolution equilibrium point. Solving for the sampling density n yields:

$$\frac{\pi^2}{6} = \frac{n}{\pi} \implies n = \frac{\pi^3}{6}$$

Evaluating this numerically where $\pi^3 \approx 31.00627668$:

$$n \approx 5.16771278$$

Hence, n represents the maximum asymptotic density of discrete coordinate points distributed along a single directional axis.

5 Extension to Three-Dimensional Volume

Real space operates across three independent orthogonal axes: X, Y, and Z. Assuming the maximum sampling density n is uniformly maintained across each independent coordinate vector, the total number of volumetric sectors (pyramidal components originating from the sphere center) is determined by the cubic product of the axial limits:

$$N_{theory} = n \cdot n \cdot n = n^3$$

We utilise the cubic exponent n^3 because the calculation evaluates the density of volumetric sectors formed by the intersecting radial planes. Each facet of the resulting discrete polyhedron forms the base of an interior pyramid.

Substituting the single-axis limit $n = \pi^3/6$ into the volumetric density equation:

$$N_{theory} = \left(\frac{\pi^3}{6}\right)^3 = \frac{\pi^9}{216}$$

Applying numerical computation where $\pi^9 \approx 29809.099$, the absolute continuous asymptote evaluates to:

$$N_{theory} \approx 138.005$$

This threshold defines the theoretical upper bound where the residual discrepancy conceptually approaches zero. However, to preserve the discrete integrity of the manifold, the system cannot exist at the absolute continuous ceiling.

6 The Quantum Deficit Principle

By definition, a discrete space must maintain its fundamental property of non-zero indivisibility. If a system reaches the absolute continuous limit, its discrete characteristics vanish. Therefore, any structurally stable object within a quantised space must exist exactly one discrete step below this theoretical maximum smoothness to maintain the integrity of the discrete manifold.

Applying a strict floor reduction from the calculated asymptotic ceiling:

$$N_{real} = 137$$

7 Conclusion

We have demonstrated a deterministic geometric limit for discrete environments. By establishing a balance equation between spatial error variance and angular resolution, we showed that three-dimensional discrete space cannot achieve infinite smoothness. The system converges on a maximum realizable volumetric sector capacity of exactly 137, establishing a precise boundary constraint for quantised spatial manifolds.