The Mathematical Structure of the Cosmological Special Theory of Relativity: Coordinate Transformations, Group Structure, and Invariants

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Abstract

The Cosmological Special Theory of Relativity (CSTR) relates a local, unexpanded spatial coordinate to the physically expanded coordinate of a Robertson-Walker universe through a cosmological expansion function $\tau(t_0)$ of the cosmological time t_0, and re-derives the Lorentz transformation, relativistic wave equations, and electrodynamics within the resulting cosmological inertial frame *(Yi, 2020, 2025). This paper examines the purely mathematical structure of that construction: the explicit form of the cosmological Lorentz transformation and its matrix representation; whether the set of such transformations closes under composition to form a genuine group, both at a single fixed cosmological time and across different cosmological times; the four-vector and tensor formalism appropriate to the cosmological inertial frame; the invariant and non-invariant quantities of the theory; and the status of energy and momentum conservation under Noether's theorem given the explicit time dependence introduced by $\tau(t_0)$. We show that at fixed cosmological time the cosmological Lorentz transformations reduce, after a coordinate relabeling, exactly to the ordinary Lorentz group, but that composing a boost with the passive evolution of $\tau(t_0)$ between two different cosmological times does not, in general, close within the class of cosmological Lorentz transformations, so that the full set of transformations used across the CSTR literature is more naturally described as a groupoid than as a Lie group. We relate this structure to the mainstream distinction between comoving and physical coordinates and to the conformal (Weyl) group of transformations, and discuss the implications for the open composition question raised in companion reviews of CSTR (A Critical Review, 2025).*

Keywords: cosmological special theory of relativity; Lorentz group; groupoid; conformal transformations; Noether's theorem; invariants

1. Introduction

The Cosmological Special Theory of Relativity (CSTR) is built from a single operational rule: a local, unexpanded spatial coordinate x'^i is related to the physically expanded coordinate x^i of a spatially flat Robertson-Walker universe through a cosmological expansion function $\tau(t_0)$ of the cosmological time t_0, $x^i = \tau(t_0)x'^i$, and this relation is substituted directly into the coordinate transformations and differential operators of ordinary special relativity (Yi, 2020). Companion reviews of this program have examined its consequences for relativistic quantum mechanics, field quantization, and general relativity (Yi, 2025), and have each, in their discussion sections, flagged a common open question: whether the cosmological Lorentz transformation composes correctly under successive boosts applied at different cosmological times. This paper takes up that question directly, by examining the mathematical structure of the cosmological Lorentz transformation as a transformation group (or, as we argue, groupoid) in its own right, independently of any specific wave equation or field theory built on top of it.

The mathematical question addressed here is largely independent of which specific wave equation or field theory is under consideration: it concerns only the coordinate transformation of Equations (2)–(3) below and the space of local and physical coordinates on which it acts. Because every companion result – the Klein-Gordon and Dirac equations, the Klein-Gordon-Maxwell theory, and the Cosmological General Theory of Relativity's black-hole solutions – is built by substituting the same coordinate relation into a pre-existing flat-space or curved-space equation, a precise characterization of the transformation itself, and of exactly which mathematical structures survive the substitution and which do not, is a necessary complement to those companion results: it clarifies which of their conclusions follow from the transformation law alone and which depend additionally on the specific dynamics of the equation being transformed.

Section 2 recalls the cosmological inertial frame and the cosmological Lorentz transformation in matrix form. Section 3 examines the group-theoretic structure of the transformation, distinguishing the fixed-epoch case (in which it reduces to the ordinary Lorentz group) from the multi-epoch case (in which composition with the intervening evolution of $\tau(t_0)$ generally fails to close). Section 4 develops the four-vector and tensor formalism appropriate to the cosmological inertial frame. Section 5 catalogs the invariant and non-invariant quantities of the theory. Section 6 applies Noether's theorem to the time- and space-translation symmetries of the construction. Section 7 relates this structure to the mainstream comoving-versus-physical-coordinate distinction and to the conformal group, and discusses the implications for

CSTR's overall mathematical consistency.

2. The Cosmological Inertial Frame

The spatially flat Robertson-Walker line element underlying CSTR is

$$ds^2 = -c^2 dt_0^2 + \tau(t_0)^2 \left[dx'^2 + dy'^2 + dz'^2 \right], \quad (1)$$

with x'^i the local (unexpanded) Cartesian coordinate and $\tau(t_0)$ the cosmological expansion function, normalized so that $\tau(t_0^{\rm obs}) = 1$ at the present epoch (Yi, 2020). The physically expanded coordinate is $x^i = \tau(t_0)x'^i$ (Wave Function Type A), and the cosmological redshift is identified through $1 + z = \tau(t_0^{\rm obs})/\tau(t_0^{\rm em})$.

2.1. The Cosmological Lorentz Transformation

For two cosmological inertial frames in relative motion with velocity v along the shared x-axis, evaluated at a single, common cosmological time t_0 (so that $\tau(t_0)$ takes a single fixed value for both frames), CSTR retains the Lorentz form of the transformation with the local coordinate replaced by its expanded counterpart (Yi, 2020),

$$c\, t_0' = \gamma \left(c\, t_0 - \frac{v}{c}\, \tau(t_0)\, x' \right), \quad (2)$$

$$\tau(t_0)\, x'' = \gamma \left(\tau(t_0)\, x' - v\, t_0 \right), \quad (3)$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$, and x'' the spatial coordinate in the boosted frame. Writing $X \equiv \tau(t_0)x'$ for the physical coordinate at the shared instant t_0, Equations (2)–(3) are simply the ordinary Lorentz transformation acting on (ct_0, X),

$$\begin{pmatrix} ct_0' \\ X' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c \\ -\gamma v/c & \gamma \end{pmatrix} \begin{pmatrix} ct_0 \\ X \end{pmatrix}, \quad (4)$$

with $X' = \tau(t_0)x''$. Equation (4) makes explicit that, *at a single fixed cosmological time*, the cosmological Lorentz transformation is nothing more than the ordinary Lorentz transformation expressed in a relabeled spatial coordinate: the physical content of Equations (2)–(3) is identical to ordinary special relativity, and $\tau(t_0)$ enters only as a fixed, overall scale factor relating x' to X at that instant.

2.2. Time Dilation and Simultaneity at Fixed Epoch

Because Equation (4) is exactly the ordinary Lorentz transformation acting on (ct_0, X), the familiar kinematic consequences of special relativity carry over unchanged at any single cosmological epoch: two events simultaneous in one cosmological inertial frame ($\Delta t_0 = 0$) are, in general, not simultaneous in a frame boosted relative to it ($\Delta t_0' = -\gamma(v/c^2)\Delta X \neq 0$ for $\Delta X \neq 0$), and a clock at rest in the boosted frame is observed to run slow by the ordinary factor γ. Because $\tau(t_0)$ takes a single, common value for both frames at the instant considered, none of these effects acquire any additional cosmological modification: relativity of simultaneity and time dilation in CSTR are, at fixed epoch, identical in every respect to their counterparts in ordinary special relativity, and it is only when comparing measurements made at different cosmological epochs – the subject of Section 3 below – that genuinely new structure appears.

Length contraction follows the same pattern: a rod of proper length L_0' (physical length $\tau(t_0)L_0'$) at rest in a cosmological inertial frame is measured, in a frame boosted with velocity v relative to it *at the same epoch*, to have physical length $\tau(t_0)L_0'/\gamma$, exactly the ordinary Lorentz contraction applied to the physical length $\tau(t_0)L_0'$. Because both the rest length and the contracted length are evaluated at the same $\tau(t_0)$, the contraction factor $1/\gamma$ is unaffected by the expansion function, and only the overall scale $\tau(t_0)L_0'$ common to both lengths carries any cosmological signature. This is a further, purely kinematic illustration of the general theme of this paper: every relativistic effect that can be computed within a single epoch reduces to its ordinary special-relativistic form once expressed in the physical coordinate X, with $\tau(t_0)$ entering only as a common, and hence physically unobservable in any single such measurement, overall scale.

3. Group-Theoretic Structure

This section is the technical core of the paper. We first establish, at the level of the transformation defined in Section 2, that restricting attention to a single cosmological epoch reproduces the ordinary Lorentz group exactly (Sections 3.5 and 4 below verify this both abstractly and by explicit matrix computation, and extend it to rotations and discrete symmetries). We then turn to the genuinely new question raised by the cosmological embedding: what happens when the transformation is used to relate frames, or a single frame's description of a particle, across two different cosmological epochs, as is implicitly required whenever a companion review computes a process (an infalling geodesic, a scattering amplitude, a propagating wave) that unfolds over a cosmologically non-negligible interval of t_0.

3.1. Closure at Fixed Cosmological Time

Because Equation (4) is exactly the ordinary Lorentz boost matrix $\Lambda(v)$ acting on (ct_0, X), the set of cosmological Lorentz transformations evaluated at a single fixed t_0 inherits the group structure of the ordinary Lorentz group directly: composing two boosts $\Lambda(v_2)\Lambda(v_1)$ along the same axis gives a third boost $\Lambda(v_1 \oplus v_2)$ with $v_1 \oplus v_2 = (v_1 + v_2)/(1 + v_1 v_2/c^2)$ the relativistic velocity-addition formula, the identity transformation corresponds to $v = 0$, and the inverse of $\Lambda(v)$ is $\Lambda(-v)$. All four group axioms – closure, associativity, identity, and inverses – therefore hold at fixed t_0, and the restriction of the cosmological Lorentz transformations to a single cosmological epoch is isomorphic, as a group, to the ordinary one-dimensional Lorentz boost group $SO(1,1)$ (or, allowing boosts in an arbitrary direction together with spatial rotations of the local coordinate, to the full homogeneous Lorentz group $SO(3,1)$).

3.2. Explicit Verification via Matrix Multiplication

It is worth verifying this closure explicitly at the level of the matrices themselves. Writing $\Lambda(v)$ for the boost matrix of Equation (4) with rapidity $\phi = \mathrm{artanh}(v/c)$, so that $\Lambda(\phi) =$

$\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$, the product of two such matrices is

$$\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2), \tag{5}$$

using the hyperbolic angle-addition identities, exactly as for ordinary Lorentz boosts, since the rapidities simply add. Because Equation (5) involves only the matrix Λ acting on (ct_0, X) at the single, shared value of $\tau(t_0)$ common to both boosts (as both are evaluated at the same epoch by assumption), no additional $\tau(t_0)$-dependent factor enters the product, and closure, associativity ($[\Lambda(\phi_3)\Lambda(\phi_2)]\Lambda(\phi_1) = \Lambda(\phi_3)[\Lambda(\phi_2)\Lambda(\phi_1)] = \Lambda(\phi_1 + \phi_2 + \phi_3)$), the identity ($\Lambda(0) = I$), and inverses ($(\Lambda(\phi)^{-1} = \Lambda(-\phi)$) all follow immediately from the corresponding properties of addition on the real line $\mathbb{R}$ of rapidities. This confirms, at the level of explicit matrix computation, that the fixed-epoch cosmological Lorentz transformations form a one-parameter abelian subgroup isomorphic to $(\mathbb{R}, +)$, and hence to $SO(1,1)$, for boosts restricted to a single spatial axis.

3.3. Rotations of the Local Coordinate

Spatial rotations of the local coordinate, $x'^i \to R^i{}_j x'^j$ for $R \in SO(3)$, leave $\tau(t_0)$ and hence the physical coordinate $X^i = \tau(t_0)x'^i$ transforming in exactly the same way, $X^i \to R^i{}_j X^j$, since $\tau(t_0)$ is a scalar with respect to spatial rotations. Combined with the boosts of the preceding subsections, this extends the fixed-epoch closure result from the one-dimensional boost group $SO(1,1)$ to the full six-parameter homogeneous Lorentz group $SO(3,1)$, generated by three rotations and three boosts, all acting on (ct_0, X^i) in the ordinary way at any single fixed epoch.

A further, well-known subtlety of the full $(3+1)$-dimensional Lorentz group is that the composition of two non-collinear boosts is not itself a pure boost, but a boost combined with a spatial rotation (the Thomas- Wigner rotation). Because Section 3.5, for simplicity, restricts attention to boosts along a single shared axis, this subtlety does not affect the one-dimensional analysis given there, but it is worth noting that it persists, essentially unmodified, within the fixed-epoch sector of CSTR: two non-collinear boosts applied at the same cosmological time t_0 compose into a boost-plus-rotation exactly as in ordinary special relativity, since, as established above, the fixed-epoch transformations are the ordinary Lorentz group acting on (ct_0, X^i). The multi-epoch composition question of Section 3.5 is therefore a genuinely new complication introduced by the cosmological embedding, layered on top of, rather than replacing, the ordinary Thomas-Wigner phenomenon already present in the fixed-epoch sector.

3.4. Discrete Symmetries

Beyond the continuous boosts and rotations considered above, the discrete parity ($x'^i \to -x'^i$) and time-reversal ($t_0 \to -t_0$) transformations are worth commenting on separately. Parity, being a purely spatial operation applied at a single fixed t_0, commutes with $\tau(t_0)$ exactly as rotations do, so that the ordinary parity properties of the wave equations considered in the companion quantum-theoretic review (vector versus pseudovector behavior of spin, and so on) are unaffected by the cosmological embedding, at fixed epoch. Time reversal is more delicate, because $\tau(t_0)$ is a monotonically evolving function of t_0 in an expanding universe and is therefore not, in general, symmetric under $t_0 \to -t_0$; formally implementing time reversal in CSTR would require also specifying how $\tau(t_0)$ itself is to be continued to negative or reversed time arguments, a step not addressed in the papers reviewed here. This is a further, discrete-symmetry counterpart of the continuous time-translation breaking identified in Section 6 below: both reflect the same underlying fact, that $\tau(t_0)$ singles out a preferred direction and rate of flow of cosmological time that is simply absent from the flat, static background of ordinary special relativity.

3.5. Composition Across Different Cosmological Times

The situation changes once two operations are composed at *different* cosmological times. Consider a boost $\Lambda(v_1)$ applied at cosmological time $t_0 = t_1$, followed by the passive evolution of the background from t_1 to a later time $t_0 = t_2$, followed by a second boost $\Lambda(v_2)$ applied at t_2. Because the local coordinate x' of a comoving observer does not change during the passive evolution step (only the relation $X = \tau(t_0)x'$ between it and the physical coordinate changes, as τ evolves from $\tau(t_1)$ to $\tau(t_2)$), this three-step sequence is equivalent to inserting a purely spatial rescaling,

$$X \longrightarrow \frac{\tau(t_2)}{\tau(t_1)}\, X, \tag{6}$$

between the two boosts, with t_0 left unrescaled. Unlike the boost matrix $\Lambda(v)$, the rescaling of Equation (6) acts only on the spatial coordinate and not on ct_0, and therefore does *not* preserve the Minkowski interval $-c^2 dt_0^2 + dX^2$: a genuine Lorentz transformation must rescale time and space together (as $\Lambda(v)$ does) or, if a pure scaling is included at all, must scale ct_0 and X by the same factor, as in the dilatations of the conformal group. Equation (6), by contrast, is exactly the kind of anisotropic, space-only rescaling that is *not* an isometry of Minkowski space. Consequently, the composite transformation $\Lambda(v_2) \cdot [\text{rescale by } \tau(t_2)/\tau(t_1)] \cdot \Lambda(v_1)$ is not, in general, expressible as a single cosmological Lorentz transformation $\Lambda(v_1 \oplus v_2)$ of the form of Equation (4) at any single epoch: the class of transformations used across different cosmological times in the CSTR literature does not close under composition in the same way that the fixed-epoch transformations of Section 3.5 do.

Explicitly, writing $\rho \equiv \tau(t_2)/\tau(t_1)$ for the rescaling factor and $S(\rho) = \mathrm{diag}(1, \rho)$ for the matrix implementing Equation (6) on (ct_0, X), the full composite transformation is $M = \Lambda(\phi_2)\, S(\rho)\, \Lambda(\phi_1)$. Carrying out this matrix product gives

$$M = \begin{pmatrix} \cosh\phi_2\cosh\phi_1 - \rho\sinh\phi_2\sinh\phi_1 & * \\ * & * \end{pmatrix}, \tag{7}$$

where each entry marked $*$ is similarly a mixture of $\cosh\phi_i$, $\sinh\phi_i$, and ρ (we suppress the full expressions here for brevity). For $\rho = 1$ (equivalently, $t_1 = t_2$, the fixed-epoch case), Equation (7) reduces to $\Lambda(\phi_1 + \phi_2)$ by Equation (5), as expected. For $\rho \neq 1$, however, the $(1,1)$ entry of M above is $\cosh\phi_2 \cosh\phi_1 - \rho \sinh\phi_2 \sinh\phi_1$, which does not equal $\cosh(\phi_1 + \phi_2) = \cosh\phi_2 \cosh\phi_1 + \sinh\phi_2 \sinh\phi_1$ for any choice of combined rapidity unless $\rho = 1$: no single boost $\Lambda(\phi_{12})$ reproduces M for general $\rho \neq 1$, confirming algebraically that the multi-epoch composite transformation is not a pure boost, and, since $S(\rho)$ for $\rho \neq 1$ is not itself a Lorentz transformation either (it fails to preserve $-c^2 dt_0^2 + dX^2$, as noted above), M does not belong to the Lorentz group $SO(1,1)$ at all for $\rho \neq 1$.

3.6. Groupoid Structure

The appropriate mathematical structure for a collection of transformations that compose only when their domains and codomains match, but do not otherwise form a single group, is a *groupoid*: a category in which every morphism is invertible. Here, the objects are the cosmological epochs t_0, and the morphisms from t_1 to t_2 are pairs $(v, t_1 \to t_2)$ consisting of a boost velocity v together with the associated rescaling of Equation (6); composition is defined only for morphisms whose intermediate epochs match ($t_1 \to t_2$ followed by $t_2 \to t_3$), exactly as required by the groupoid axioms, and every such morphism has a well-defined inverse (apply $\Lambda(-v)$ and rescale by $\tau(t_1)/\tau(t_2)$). The restriction of this groupoid to morphisms with $t_1 = t_2$ (i.e., a single fixed epoch) recovers the ordinary Lorentz group of Section 3.5 as expected, since the groupoid's "isotropy group" at any fixed object t_0 is, by definition, the set of self-morphisms at that object. We suggest that this groupoid description, rather than a single global Lorentz group, is the mathematically precise way to understand the collection of coordinate transformations used throughout the CSTR literature, and that the open question – raised in companion reviews – of whether successive boosts at different cosmological times compose correctly is best answered by recognizing that they compose correctly *as groupoid morphisms*, but not as elements of a single, epoch-independent Lorentz group.

3.7. Infinitesimal Generators

It is instructive to examine the infinitesimal (Lie algebra) version of the structure identified above. The generator of boosts, obtained from $\Lambda(\phi)$ by differentiating at $\phi = 0$, is the ordinary Lorentz boost generator $K = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, satisfying $\Lambda(\phi) = \exp(-\phi K)$ as usual. The generator of the rescaling of Equation (6), obtained by differentiating $S(\rho)$ with respect to $\ln\rho$ at $\rho = 1$, is $D = \mathrm{diag}(0, 1)$, so that $S(\rho) = \exp[(\ln\rho) D]$. Whereas K alone generates the one-dimensional Lorentz algebra $\mathfrak{so}(1,1)$, which closes under the Lie bracket trivially (being one-dimensional, $[K, K] = 0$), including D enlarges the algebra: the commutator $[K, D] = KD - DK$ is easily computed to be

$$[K, D] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq 0, \tag{8}$$

so that K and D do not commute and instead generate, together with $[K, D]$ itself, a larger three-dimensional Lie algebra closed under the bracket, of the general type generated by boosts together with dilatations in two-dimensional Minkowski space. This confirms at the infinitesimal level that including the rescaling of Equation (6) alongside the boosts of Section 3.5 does not merely add an extra parameter to the Lorentz group $SO(1,1)$: it generates a strictly larger group, of which the fixed-epoch Lorentz transformations form only a one-dimensional subgroup. The failure of the multi-epoch composite transformation of Equation (7) to reduce to a pure boost is the group-level manifestation of this non-commutativity at the Lie-algebra level.

4. Four-Vectors and Tensors in the Cosmological Frame

Having established in Section 3 that the fixed-epoch cosmological Lorentz transformations coincide exactly with the ordinary Lorentz group acting on the physical coordinate $X^i = \tau(t_0) x'^i$, it follows immediately that any object built covariantly from $X^\mu = (ct_0, X^i)$ at a single fixed epoch transforms as an ordinary Lorentz tensor. The purpose of this section is to make this explicit for the specific four-vectors and tensors used throughout the companion reviews – four-velocity, four-momentum, the stress-energy tensor, and the electromagnetic field tensor – and, in each case, to identify precisely where the multi-epoch groupoid structure of Section 3 reappears once quantities at different cosmological times are compared.

4.1. The Cosmological Four-Velocity

For a particle following a worldline $x'^i(t_0)$ in the local coordinate, the ordinary definition of four-velocity as the derivative of position with respect to proper time $d\tau_{\text{proper}} = \sqrt{1 - v^2/c^2}\, dt_0$ generalizes to

$$u^\mu = \frac{dx^\mu}{d\tau_{\text{proper}}} = \gamma \left(c,\ \tau(t_0)\dot{x}'^i + \dot{\tau}(t_0) x'^i \right), \tag{9}$$

where $\dot{} \equiv d/dt_0$ and the spatial components pick up an additional term $\dot{\tau}(t_0)x'^i$ beyond the ordinary $\tau(t_0)\dot{x}'^i$ term, reflecting the fact that the physical position $X^i = \tau(t_0)x'^i$ changes both because the particle moves in the local coordinate and because the expansion function itself evolves. This additional term is the CSTR counterpart of the Hubble-flow velocity $H(t_0)X^i$ familiar from ordinary Robertson-Walker kinematics, with $H(t_0) = \dot{\tau}(t_0)/\tau(t_0)$ the Hubble parameter.

4.2. The Cosmological Four-Acceleration

Differentiating Equation (9) once more with respect to proper time gives the cosmological four-acceleration,

$$w^\mu = \frac{du^\mu}{d\tau_{\text{proper}}} \propto \Big(0,\ \tau(t_0)\ddot{x}'^i + 2\dot{\tau}(t_0)\dot{x}'^i + \ddot{\tau}(t_0)x'^i \Big) + O(\dot{\gamma}), \tag{10}$$

where the three spatial terms correspond, respectively, to the ordinary local-coordinate acceleration, a Coriolis-like cross term between the particle's local velocity and the rate of change of the expansion function, and a term proportional to the second time derivative $\ddot{\tau}(t_0)$ that vanishes identically for a particle at rest in the local coordinate ($\dot{x}'^i = \ddot{x}'^i = 0$) only if $\ddot{\tau}(t_0) = 0$ as well. This shows that even a comoving particle, at rest in the local coordinate at all times, is not, in general, force-free in the sense of having vanishing four-acceleration once $\tau(t_0)$ has nonzero second derivative (equivalently, once the background is accelerating or decelerating rather than expanding at a constant rate); this is the CSTR counterpart of the well-known fact that comoving observers in an accelerating Robertson-Walker background are geodesic (force-free) with respect to the full curved-spacetime connection even though their local-coordinate acceleration, computed naively as if in flat space, appears nonzero.

4.3. The Cosmological Four-Momentum and Stress-Energy Tensor

The four-momentum $p^\mu = mu^\mu$ built from Equation (9) transforms, at fixed cosmological time, as an ordinary Lorentz four-vector under the boosts of Section 3, since $\tau(t_0)$ is a constant scalar at any single instant and does not interfere with the tensorial transformation law. Across different cosmological times, by contrast, p^μ acquires an explicit, non-tensorial t_0-dependence through $\tau(t_0)$ and $\dot{\tau}(t_0)$ in Equation (9), exactly paralleling the groupoid, rather than group, structure identified in Section 3. The stress-energy tensor $T^{\mu\nu}$ of a field built on the cosmological inertial frame (for instance, the Klein-Gordon or Klein-Gordon-Maxwell stress-energy tensor of the companion quantum-theoretic review) transforms as an ordinary rank-2 tensor under fixed-epoch Lorentz boosts, for the same reason, and it is only the explicit $\tau(t_0)$-dependence built into the field equations themselves – not any breakdown of tensorial transformation law at fixed t_0 – that distinguishes the cosmological theory from ordinary flat-space field theory at any single instant.

The covariant conservation law $\partial_\mu T^{\mu\nu} = 0$ satisfied by an ordinary flat-space stress-energy tensor is modified, in the cosmological inertial frame, by the explicit $\tau(t_0)$-dependence of the spatial derivative operator (the substitution rule $\partial/\partial x^i \rightarrow \tau(t_0)^{-1}\partial/\partial x'^i$ recalled from the companion reviews), so that the conservation law becomes $\partial_{t_0} T^{00} + \tau(t_0)^{-1}\partial'_i T^{0i} = 0$ for the energy-density component, with an additional source term proportional to $\dot{\tau}(t_0)$ appearing when the full spatial components are included, in direct analogy with the non-conservation of the perfect-fluid energy density in the ordinary Robertson-Walker Friedmann equations (where the analogous source term is the familiar $-3H(t_0)(\rho + p/c^2)$ dilution term). This is a further, field-theoretic instance of the same Noether-theoretic pattern developed systematically in Section 6 for the point-particle case: whatever is conserved locally, at fixed t_0, in ordinary flat-space field theory acquires an explicit $H(t_0)$-proportional source term once promoted to the cosmological inertial frame.

4.4. The Electromagnetic Field Tensor

The electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ of the companion Klein-Gordon-Maxwell review provides a further illustration of the same pattern. At fixed cosmological time, $F_{\mu\nu}$ transforms as an ordinary antisymmetric rank-2 tensor under the boosts of Section 3, so that the familiar mixing of electric and magnetic field components under a boost, $\mathbf{E}'_\parallel = \mathbf{E}_\parallel$, $\mathbf{B}'_\parallel = \mathbf{B}_\parallel$, $\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B})$, $\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \mathbf{v} \times \mathbf{E}/c^2)$, carries over unchanged, with $\mathbf{E}$ and $\mathbf{B}$ understood as the fields expressed in the physical coordinate at that instant. The two quadratic field invariants, $F_{\mu\nu}F^{\mu\nu} \propto B^2 - E^2/c^2$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} \propto \mathbf{E} \cdot \mathbf{B}$, are correspondingly invariant under fixed-epoch boosts, exactly as in ordinary electrodynamics; only when comparing field configurations at two different cosmological times does the explicit $\tau(t_0)$-dependence built into the covariant derivative of the companion Klein-Gordon-Maxwell theory introduce any epoch dependence into these invariants, and even then only through the source terms and equations of motion, not through any modification of the algebraic tensor-transformation law itself.

5. Invariant and Non-Invariant Quantities

5.1. Invariance of the Speed of Light

Before cataloging the invariants built from the theory's dynamical quantities, it is worth isolating the single invariance property on which every other result of this section rests: the speed of light c itself. Because the boost matrix of Equation (4) is exactly the ordinary Lorentz boost acting on (ct_0, X), the light cone $-c^2dt_0^2 + dX^2 = 0$ is preserved under any fixed-epoch boost by construction, and c is accordingly invariant, and equal to its ordinary flat-space value, at every cosmological epoch. This is what justifies treating c as a fixed constant throughout every equation of this paper and its companion reviews, rather than as a further quantity whose epoch-dependence would need to be tracked separately: the invariance of c is built into the definition of the cosmological inertial frame at the outset, via the unmodified appearance of c in Equation (1), and is not itself a consequence of, or threatened by, the multi-epoch groupoid structure of Section 3, since that structure concerns only the relation between spatial coordinates at different epochs and never rescales t_0 itself.

5.2. Quantities Invariant at Fixed Cosmological Time

At fixed t_0, the interval ds^2 built from Equation (4), the rest mass m, and the Klein-Gordon dispersion relation $E^2 = c^2p'^2 + m^2c^4$ of the companion quantum-theoretic review are all invariant under the boosts of Section 3, exactly as in ordinary special relativity, since these boosts are, as shown in Section 3, literally the ordinary Lorentz transformations acting on the physical coordinate X.

The rest mass m plays the role, at fixed epoch, of the quadratic Casimir invariant of the fixed-epoch Lorentz group $SO(1,1)$ (or $SO(3,1)$, including rotations), $p^\mu p_\mu = -m^2c^2$, exactly as in ordinary special relativity: since p^μ transforms as an ordinary Lorentz four-vector at fixed t_0 (Section 4), the quadratic form $p^\mu p_\mu$ built from the fixed-epoch Minkowski metric is automatically invariant under every element of the fixed-epoch group, by the same argument used for the ordinary Poincaré group's Casimir invariants.

5.3. Quantities Not Invariant Across Cosmological Times

Across different cosmological times, several quantities that would be invariant in ordinary special relativity acquire an explicit $\tau(t_0)$-dependence. The physical wavelength of a light signal, $\lambda \propto \tau(t_0)$ (the cosmological redshift of Section 2), is the clearest example: it is invariant under boosts at fixed t_0 but scales with $\tau(t_0)$ between different epochs. Similarly, as shown in Section 4, the spatial components of the four-momentum are not simply related by a Lorentz transformation between their values at two different cosmological times, since the additional Hubble-flow term of Equation (9) intervenes.

As a concrete illustration, consider a photon emitted with local wavelength λ' at cosmological time t_0^{em}. Because the physical wavelength is $\lambda = \tau(t_0)\lambda'$, and λ' itself is fixed by the emission process (an atomic transition, for instance, whose local-coordinate wavelength does not depend on when the transition occurs, by the adiabatic argument of Section 6), the physical wavelength observed at a later time t_0^{obs} is $\lambda_{\mathrm{obs}} = \tau(t_0^{\mathrm{obs}})\lambda'$, while the physical wavelength at emission was $\lambda_{\mathrm{em}} = \tau(t_0^{\mathrm{em}})\lambda'$; their ratio $\lambda_{\mathrm{obs}}/\lambda_{\mathrm{em}} = \tau(t_0^{\mathrm{obs}})/\tau(t_0^{\mathrm{em}}) = 1 + z$ recovers the redshift relation of Section 2 directly from the transformation properties of the local versus physical coordinate, without any additional physical input beyond the coordinate relation $X = \tau(t_0)x'$ itself. This confirms that the cosmological redshift, far from being an independent postulate of the theory, follows directly from the same coordinate relation that defines the cosmological inertial frame in Section 2.

5.4. The Interval Across Different Cosmological Times

A related and more subtle point concerns the interval ds^2 itself between two events at different cosmological times. Within a single epoch, $ds^2 = -c^2dt_0^2 + dX^2$ is invariant under the fixed-epoch boosts of Section 3, exactly as the ordinary Minkowski interval is invariant under the Poincaré group. But the very notion of "the same interval" evaluated at two different epochs is ambiguous in Robertson-Walker geometry: the physical distance $dX = \tau(t_0)dx'$ between two comoving observers at fixed local coordinate separation dx' is itself a function of t_0, so that ds^2 computed at $t_0 = t_1$ and at $t_0 = t_2$ for the same pair of comoving local-coordinate points differ simply because $\tau(t_1) \neq \tau(t_2)$, independently of any boost. This is not a failure of invariance in the ordinary sense – no symmetry transformation is being applied – but rather a reflection of the fact that the spacetime itself is not static, and it underlies why the groupoid (rather than group) structure of Section 3 is the appropriate framework: the interval is a well-defined invariant only within, and not across, the fixed-epoch slices that serve as the objects of that groupoid.

Table 1 summarizes this distinction for the principal quantities considered throughout the CSTR literature.

Quantity	Status
Rest mass m	Invariant (fixed and across epochs)
Interval ds^2 at fixed t_0	Invariant under boosts
Dispersion relation $E^2 = c^2p'^2 + m^2c^4$	Invariant under boosts at fixed t_0
Physical wavelength λ	Scales as $\tau(t_0)$ across epochs
Spatial four-momentum p^i	Not simply related across epochs (Sec. 4)
Yukawa/Coulomb interaction range (physical)	Invariant across epochs (companion reviews)

Table 1: Invariance status of principal quantities under fixed-epoch boosts versus across different cosmological epochs.

6. Noether's Theorem and Conservation Laws

6.1. The Point-Particle Action

Noether's theorem is most transparently applied starting from an action principle. The point-particle action appropriate to the cosmological inertial frame is the ordinary relativistic action evaluated along the worldline $x'^i(t_0)$, using the proper-time element implied by Equation (1),

$$\begin{aligned} S &= -mc\int d\tau_{\mathrm{proper}} \\ &= -mc\int dt_0\,\sqrt{1-\frac{\tau(t_0)^2\dot{x}'^2}{c^2}} \\ &\quad + mc^2\int dt_0\,\frac{\dot{\tau}(t_0)^2x'^2}{2c^2} + \cdots, \end{aligned} \tag{11}$$

where the displayed expansion separates the ordinary special-relativistic Lagrangian (first term) from corrections induced by the time-dependence of $\tau(t_0)$ (second and higher terms, here shown only to leading order for a slowly expanding background). Because the integrand of Equation (11) depends explicitly on t_0 through $\tau(t_0)$, the action is not invariant under time translations of t_0 except in the limit that $\tau(t_0)$ is treated as constant, which is precisely the origin of the broken time-translation symmetry discussed below.

6.2. Spatial Translation Invariance

The Robertson-Walker line element of Equation (1) is invariant under spatial translations of the local coordinate, $x'^i \to x'^i + a^i$ for constant a^i, at any fixed t_0, since $\tau(t_0)$ does not depend on x'^i. By Noether's theorem, this continuous symmetry implies the conservation of the spatial momentum conjugate to x'^i – the quantity $\mathbf{p}'$ appearing throughout the companion quantum-theoretic review – at fixed cosmological time, exactly as ordinary spatial translation invariance implies momentum conservation in flat-space field theory. Explicitly, applying Noether's theorem to the action of Equation (11), the conserved charge associated with $x'^i \to x'^i + a^i$ is $p'_i = \partial L/\partial \dot{x}'^i$, the ordinary canonical momentum conjugate to the local coordinate, independent of $\tau(t_0)$ or its time derivatives, since the translation symmetry itself does not involve $\tau(t_0)$ at all.

6.3. Broken Time-Translation Invariance

By contrast, Equation (1) is *not* invariant under translations of the cosmological time, $t_0 \to t_0 + b$ for constant b, because $\tau(t_0)$ depends explicitly on t_0. Noether's theorem therefore does not guarantee conservation of the energy conjugate to t_0 in the cosmological inertial frame, in direct parallel with the well-known statement in ordinary Robertson-Walker cosmology that photon energy is not conserved (but instead redshifts) precisely because the expanding background breaks time-translation symmetry. This is consistent with, and indeed the deeper reason behind, the redshift of physical wavelength noted in Table 1: the absence of a time-translation Killing vector for $\tau(t_0) \neq \mathrm{const}$ is the Noether-theoretic origin of every epoch-dependent quantity identified throughout this paper and its companion reviews.

Although the exact energy conjugate to t_0 is not conserved, its fractional rate of change is set entirely by the fractional rate of change of the expansion function itself: for a free particle or photon whose energy depends on t_0 only through $\tau(t_0)$, as in the redshift relation $1 + z = \tau(t_0^{\mathrm{obs}})/\tau(t_0^{\mathrm{em}})$ of Section 2, the logarithmic rate of energy loss $-d(\ln E)/dt_0$ is equal to the Hubble parameter $H(t_0) = \dot{\tau}(t_0)/\tau(t_0)$, independent of further details of the particle's mass or momentum. This is simply the Noether-theoretic restatement of the redshift relation already established in Section 2, and confirms that the single function $\tau(t_0)$ – and specifically its logarithmic time derivative $H(t_0)$ – controls the entire departure from exact energy conservation throughout the theory.

6.4. The Adiabatic Limit

When $\dot{\tau}(t_0)/\tau(t_0)$ is small compared with the relevant particle's rest energy $mc^2/\hbar$ (the adiabatic regime identified in the companion quantum-theoretic review), the breaking of time-translation symmetry is correspondingly weak, and energy is conserved to good approximation over time intervals short compared with the Hubble time $1/H(t_0)$. This adiabatic approximate conservation law is the Noether-theoretic counterpart of the fixed-epoch approximation used throughout the companion reviews of CSTR's quantum mechanics and general relativity to justify treating $\tau(t_0)$ as constant over non-cosmological timescales.

As a worked numerical illustration, the present-day Hubble time $1/H(t_0^{\mathrm{obs}})$ is of order 10^{10} years, while the inverse rest energy $\hbar/(mc^2)$ for even a very light particle such as the electron is of order 10^{-21} seconds; the adiabaticity parameter $\hbar H(t_0)/(mc^2)$ is therefore smaller than order unity by roughly 46 orders of magnitude for the electron, and correspondingly smaller still for any heavier particle. This quantifies, for the specific case of ordinary massive particles, just how good the adiabatic approximation invoked throughout the companion reviews actually is: the breaking of time-translation symmetry identified in this section, while formally present at the level of the action of Equation (11), is utterly negligible for any process on laboratory, solar-system, or even galactic timescales, and becomes significant only for processes that span a cosmologically significant fraction of the age of the universe.

7. Discussion

7.1. Relation to Comoving and Physical Coordinates

The distinction drawn in this paper between the local coordinate x'^i and the physical coordinate $x^i = \tau(t_0)x'^i$ parallels, at the level of pure formalism, the standard distinction in mainstream Robertson-Walker cosmology between comoving coordinates (fixed for a comoving observer) and physical or proper distance (which grows with the scale factor). What is specific to CSTR, and is not part of the mainstream formalism, is the further step of substituting this coordinate relation directly into the Lorentz transformation and the differential operators of relativistic wave equations, as in Equations (2)–(3), rather than treating the expanding background purely as a fixed geometric arena within which ordinary special-relativistic physics occurs locally. The groupoid structure identified in Section 3 is, in effect, a precise statement of the sense in which this further step is and is not compatible with the ordinary Lorentz group: compatible within a single epoch, but not across different epochs without additional structure.

This distinction can be sharpened further by comparing it with the standard general-relativistic notion of a local inertial (Fermi normal) frame, valid in a small neighborhood of a freely falling observer's worldline at a single instant. In the mainstream treatment, special relativity is recovered locally in exactly this sense: at any single event, one may choose coordinates in which the metric is Minkowski and its first derivatives vanish, and ordinary Lorentz transformations relate different such local frames at that event. CSTR's fixed-epoch Lorentz group of Section 3 plays a structurally similar role, but defined globally in space at a fixed cosmological time, rather than only in an infinitesimal neighborhood of a single event; this is consistent with the fact that Equation (1) is spatially homogeneous, so that $\tau(t_0)$ is the same constant at every spatial point at a given t_0, unlike a generic curved spacetime in which a local inertial frame is generally valid only infinitesimally. The mainstream Fermi-normal construction, by contrast, does not attempt to extend the local Lorentz group to relate frames at different proper times along the observer's worldline in the direct, algebraic way that CSTR's cosmological Lorentz transformation attempts to do across different values of t_0; it is precisely this extension that gives rise to the groupoid structure of Section 3 rather than a straightforward group.

7.2. Relation to the Conformal Group

The rescaling of Equation (6) would be an isometry of Minkowski space if it acted on ct_0 as well as on X – that is, if it were a genuine dilatation of the type appearing in the conformal (Weyl) group, under which the flat metric transforms as $\eta_{\mu\nu} \to \Omega^2\eta_{\mu\nu}$ for a constant Ω. CSTR's construction instead rescales only the spatial part of the metric, consistent with the Robertson-Walker line element of Equation (1), in which $\tau(t_0)^2$ multiplies dx'^2 but not dt_0^2. This is precisely why the rescaling of Equation (6) fails to be an isometry and why the composition question of Section 3.5 does not resolve within the ordinary Lorentz or conformal groups: the Robertson-Walker metric is conformally flat only after a further change to conformal time $\eta = \int dt_0/\tau(t_0)$ (a step not taken in the CSTR literature reviewed here), under which $ds^2 = \tau(\eta)^2[-c^2 d\eta^2 + dx'^2 + \ldots]$ takes a manifestly conformally flat form and the associated rescaling acts symmetrically on time and space. Reformulating CSTR in terms of conformal time, so that its coordinate transformations connect more directly to the conformal group, is a natural mathematical direction suggested by the analysis of this section but not pursued in the papers reviewed here.

To sketch the direction such a reformulation would take: writing $d\eta = dt_0/\tau(t_0)$, the Robertson-Walker interval becomes $ds^2 = \tau(\eta)^2[-c^2 d\eta^2 + dx'^2 + dy'^2 + dz'^2]$, and the ordinary conformal (Weyl) transformation $\eta_{\mu\nu} \to \Omega(\eta)^2 \eta_{\mu\nu}$ with $\Omega(\eta) = \tau(\eta)$ relates this metric directly to the flat Minkowski metric on (η, x'^i). A boost combined with a conformal rescaling of this type acts symmetrically on $c\,d\eta$ and dx'^i together, in contrast with the anisotropic rescaling of Equation (6), which acts on $X = \tau(t_0)x'^i$ while leaving ct_0 untouched. Because a symmetric rescaling of $(c\,d\eta, dx'^i)$ together *is* an isometry of the conformally rescaled metric (up to the overall conformal factor, which is common to every term and therefore does not affect ratios of intervals), reformulating the multi-epoch composition of Section 3.5 in terms of η rather than t_0 is a natural candidate for recovering a genuine group structure – specifically, the conformal group of two-dimensional Minkowski space – in place of the groupoid identified above. Carrying out this reformulation in full, including its consequences for the wave equations and general-relativistic solutions of the companion reviews, is left for future work.

7.3. Relation to Scale-Invariant and Weyl Gravity Theories

The appearance of a dilatation generator D alongside the boost generator K in Section 3 invites comparison with a distinct body of mainstream literature on scale-invariant and Weyl gravity theories, in which spacetime is equipped not only with a metric but with an additional local scale (or conformal) connection, and physical laws are required to be invariant under local rescalings of the metric, $g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu}$, compensated by a corresponding transformation of a Weyl gauge field. Such theories generically predict observable consequences of scale non-invariance (for instance, a running of particle masses) unless the Weyl gauge field is fixed to a pure-gauge configuration. CSTR's cosmological expansion function $\tau(t_0)$ plays a superficially similar role to a position-dependent conformal factor, but enters only through the fixed, spatially homogeneous background of Equation (1) rather than through a dynamical Weyl connection with its own field equation; CSTR is accordingly better understood as a theory with a fixed, prescribed conformal factor (the background expansion history) rather than as a genuinely scale-invariant or Weyl-gauge theory in the technical sense used in that literature. Making this distinction precise, and exploring whether promoting $\tau(t_0)$ to a dynamical Weyl-type field would resolve the groupoid-versus-group tension of Section 3 by providing exactly the compensating structure needed to make the rescaling of Equation (6) into a genuine (gauged) symmetry rather than an external, prescribed one, is a further mathematical direction suggested by, but not pursued in, the present analysis.

7.4. Open Questions

Four questions follow directly from the analysis of this paper: (i) whether reformulating CSTR in conformal time η, as suggested in Section 7 above, resolves the groupoid-versus-group tension identified in Section 3 by embedding the theory's transformations within the conformal group; (ii) whether a connection one-form or analogous structure can be defined on the groupoid of Section 3 to make its composition law as calculationally convenient as an ordinary Lie group multiplication law, which would be of practical use in extending the perturbative, fixed-epoch calculations of the companion quantum-theoretic and general-relativistic reviews to genuinely multi-epoch processes; (iii) whether the broken time-translation symmetry identified in Section 6 can be restored approximately, via a Noether charge associated with a suitably generalized (approximate) Killing vector, in a way that would justify the adiabatic energy conservation of Section 6 more rigorously than the heuristic argument given there; and (iv) whether promoting the cosmological expansion function $\tau(t_0)$ from a fixed, prescribed background quantity to a dynamical Weyl-type gauge field, as suggested in Section 7, would supply the compensating structure needed to make the multi-epoch rescaling of Equation (6) into a genuine symmetry. An independent critical review of CSTR has additionally raised broader questions about the framework's consistency with the diffeomorphism invariance of general relativity (A Critical Review, 2025); the groupoid structure identified in this paper is directly relevant to that broader question, since a transformation law that closes only locally (within a fixed epoch) is precisely the kind of structure that a diffeomorphism-invariant formulation would need to accommodate systematically, rather than treating as a special feature of the Lorentz sector alone.

Table 2 summarizes the principal structural results of this paper, distinguishing what holds at fixed cosmological time from what requires further work across different epochs.

8. Conclusion

This paper has examined the purely mathematical structure of the Cosmological Special Theory of Relativity, independently of the specific wave equations and field theories built on top of it in companion reviews. We have shown that the cosmological Lorentz transformation, evaluated at a single fixed cosmological time, reduces after a coordinate relabeling to the ordinary Lorentz group, satisfying all four group axioms (Section 3); that composing a boost with the passive evolution of the expansion function $\tau(t_0)$ between two different cosmological times inserts an anisotropic spatial rescaling that is not an isometry of the Minkowski metric, so that

Structure	Status
Boosts/rotations at fixed t_0	Form $SO(3,1)$ exactly (Sec. 3)
Boosts across different t_0	Form a groupoid, not a group (Sec. 3)
Tensorial transformation of p^μ	Holds at fixed t_0 only (Sec. 4)
Casimir invariant $p^\mu p_\mu$	Invariant at fixed t_0 (Sec. 5)
Spatial momentum conservation	Holds exactly, any fixed t_0 (Sec. 6)
Energy conservation	Broken; adiabatic only (Sec. 6)
Embedding in conformal group	Open question (Sec. 7)

Table 2: Summary of the structural results of this paper.

the full collection of transformations used across the CSTR literature is more precisely described as a groupoid than as a single Lie group (Section 3); that four-vectors and tensors built on the cosmological inertial frame transform tensorially at fixed epoch but acquire explicit, non-tensorial $\tau(t_0)$-dependence across epochs (Section 4); and that this same structure is the Noether-theoretic origin of the broken time-translation symmetry, and hence of the absence of exact energy conservation, that underlies the cosmological redshift and the adiabatic approximations used throughout the companion reviews of CSTR (Section 6). Relating this groupoid structure to the conformal group via a conformal-time reformulation of CSTR (Section 7) is identified as the most promising direction for resolving the composition question flagged as open in companion reviews of this framework, and for placing CSTR's transformation law on a footing as mathematically precise as that of ordinary special relativity.

The overall picture that emerges is one in which CSTR is, at fixed cosmological time, mathematically indistinguishable from ordinary special relativity expressed in a rescaled spatial coordinate – a genuinely trivial statement in the sense that no new group-theoretic content is added within a single epoch – while the nontrivial mathematical content of the theory is concentrated entirely in how different epochs are related to one another. This is a useful clarification of the companion quantum-theoretic and general-relativistic reviews' recurring finding that physically measured quantities (interaction ranges, horizon radii, PPN parameters, and the like) are epoch-independent once expressed in physical coordinates: that finding is, in light of the present analysis, essentially guaranteed by the fixed-epoch group-theoretic triviality established in Section 3, since any quantity computed entirely within a single epoch's physics must be built from objects that transform under the ordinary Lorentz group alone. The genuinely open questions of the theory – and the place where a departure from ordinary special relativity could, in principle, be found – lie instead in the multi-epoch groupoid structure identified here, and in whether it can be upgraded to a genuine group via the conformal-time reformulation suggested in Section 7.

Framed this way, the mathematical structure of CSTR closely mirrors a familiar pattern from other areas of physics in which a symmetry is exact locally but only approximately or contingently extended globally – gauge symmetries that are exact pointwise but require a connection to compare fibers at different points, or local inertial frames that recover special relativity only infinitesimally within a curved spacetime. The specific contribution of this paper has been to make that pattern mathematically explicit for CSTR: to identify precisely which group axioms hold unconditionally (Section 3, fixed epoch), which quantities remain invariant or tensorial under exactly that unconditional symmetry (Sections 4–5), and which conservation law is lost, and at what quantitative rate, once the symmetry is only approximate (Section 6). Together with the companion quantum-theoretic, general-relativistic, and consolidated reviews of this framework, this analysis is intended to provide a complete and internally consistent mathematical foundation against which further extensions of the Cosmological Special Theory of Relativity can be checked.

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A. Glossary of Key Symbols

Symbol	Meaning
t_0	Cosmological time
$\tau(t_0)$	Cosmological expansion function (scale factor)
x^i, X	Physically expanded coordinate
x'^i	Local (unexpanded) coordinate
$\Lambda(v)$	Ordinary Lorentz boost matrix
u^μ, p^μ	Four-velocity, four-momentum
$H(t_0)$	Hubble parameter, $\dot{\tau}(t_0)/\tau(t_0)$
$SO(1,1)$, $SO(3,1)$	1D, (3+1)D Lorentz groups
η	Conformal time, $\int dt_0/\tau$
CSTR	Cosmological Special Theory of Relativity

Table 3: Glossary of symbols used in this paper.

B. Index of Principal Results

Table 4 indexes the coordinate-transformation and metric results underlying this paper against their original source; the group-theoretic, tensorial, invariance, and Noether-theoretic analysis of Sections 3–6 is the contribution of the present paper, applying standard tools of group theory and classical field theory to the transformation and metric of Yi (2020).

References

A Critical Review of Sangwha Yi's Cosmological Special Theory of Relativity, preprint, 2025

Result	**Original source**
Cosmological inertial frame, Wave Function Type A	Yi (2020)
Cosmological Lorentz transformation	Yi (2020)
Redshift identification $1 + z = \tau(t_0^{\rm obs})/\tau(t_0^{\rm em})$	Yi (2020)
Full consolidation	Yi (2025)

Table 4: Index of the transformation and metric results underlying this paper.

Yi, S. 2020, Cosmological Special Theory of Relativity, International Journal of Advanced Research in Physical Science, 7(11), 4–9

Yi, S. 2025, Cosmological Special Theory of Relativity, Cosmological Quantum Physics, Cosmological Uncertainty Principles, Cosmological General Theory of Relativity, SSRN Working Paper, DOI: 10.2139/ssrn.5340309