Cosmologically Embedded Black Holes: Schwarzschild and Kerr-Newman Solutions in the Cosmological General Theory of Relativity

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Abstract

The Cosmological General Theory of Relativity (CGTR) extends the Cosmological Special Theory of Relativity (CSTR) to curved spacetime by embedding a localized mass, or a rotating charged mass, within a spatially flat, spatially homogeneous Robertson-Walker background whose scale factor is denoted the cosmological expansion function $\tau(t_0)$ (Yi, 2020, 2021). This paper concentrates specifically on the black-hole solutions of this framework: the cosmologically embedded Schwarzschild solution and the cosmologically embedded Kerr-Newman solution (Yi, 2021, 2025). We present the CGTR field equation and its perturbative metric ansatz, derive the Schwarzschild and Kerr-Newman metric functions and their horizon structure, work out the radial and circular photon and massive-particle geodesics, the ergosphere and extremality condition of the rotating solution, and the homogeneous (source-free) limit that recovers the background Robertson-Walker cosmology. We show that, in every case considered, the physically measured horizon radii, extremality condition, and photon-sphere radius reduce to their ordinary, epoch-independent general-relativistic values once expressed in the physically expanded radial coordinate, even though the corresponding local-coordinate expressions carry explicit $\tau(t_0)$-dependence. We close with a comparison to the mainstream McVittie and Schwarzschild-de Sitter embeddings of a point mass in an expanding or accelerating universe, and with the open questions raised by an independent critical review of the underlying CSTR framework (A Critical Review, 2025).

Keywords: cosmological general theory of relativity; Schwarzschild solution; Kerr-Newman solution; cosmological black holes; Robertson-Walker metric; McVittie solution

1. Introduction

A localized mass – whether static and uncharged, or rotating and charged – does not exist in isolation from the cosmological expansion that surrounds it. The Cosmological General Theory of Relativity (CGTR) addresses this by extending the Cosmological Special Theory of Relativity (CSTR) – in which a local, unexpanded spatial coordinate x'^i is related to the physically expanded coordinate x^i of a spatially flat Robertson-Walker universe through a cosmological expansion function $\tau(t_0)$ of the cosmological time t_0, $x^i = \tau(t_0)x'^i$ (Yi, 2020) – to curved spacetime, retaining the standard Einstein field equation $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ but solving it on a metric background that reduces to the Robertson-Walker line element away from any localized source (Yi, 2021). This construction yields cosmologically embedded generalizations of the two most fundamental exact solutions of stationary general relativity: the Schwarzschild solution for a static, spherically symmetric, uncharged mass, and the Kerr-Newman solution for a stationary, axisymmetric, rotating, charged mass. Both are consolidated, together with the rest of the CSTR program, in the 2025 SSRN paper (Yi, 2025).

The Cosmological General Theory of Relativity was introduced as an extension of the original CSTR construction (Yi, 2020), and its black-hole solutions – the Schwarzschild and Kerr-Newman metrics reviewed here – were derived together in a single paper the following year (Yi, 2021). Because the CGTR field equation of Section 2 is solved perturbatively on top of the same Robertson-Walker background that underlies the flat-space wave equations of the companion CSTR papers, the black-hole solutions reviewed in this paper depend directly on the cosmological inertial frame construction of the original 2020 paper, and inherit the same expansion function $\tau(t_0)$ that appears throughout the rest of the CSTR program.

The present paper concentrates specifically on this general-relativistic sector of the CSTR program and develops it in more depth than is possible in a broader survey: Section 2 presents the CGTR field equation, its perturbative metric ansatz, and the background Friedmann equations; Section 3 derives the cosmologically embedded Schwarzschild solution and its horizon, redshift, geodesic, and curvature structure; Section 4 derives the cosmologically embedded Kerr-Newman solution, its horizons, ergosphere, extremality condition, circular orbits, frame dragging, and Penrose process; Section 5 examines the homogeneous and weak-field, slow-rotation limits of both solutions, including their parametrized post-Newtonian content; Section 6 discusses the thermodynamic properties of the cosmologically embedded Schwarzschild horizon; and Section 7 compares this construction with the mainstream McVittie and Schwarzschild-de Sitter embeddings of a point mass in an expanding or accelerating universe, and discusses the open questions raised by an independent critical review of the underlying CSTR framework.

Throughout, primed quantities (r', x'^i) refer to the local, unexpanded radial or spatial coordinate of the cosmological inertial frame, while unprimed quantities (r_c, x^i) refer to the physically expanded coordinate related to the local one by $r_c = \tau(t_0)r'$. All metric functions and derived geometric quantities – horizon radii, photon-sphere radii, curvature invariants, and orbital parameters – are given directly in terms of the physical coordinate r_c wherever possible, since it is in this coordinate that the recurring epoch-independence pattern noted throughout this paper is most directly visible; the corresponding local-coordinate expression, when needed, is obtained simply by substituting $r_c = \tau(t_0)r'$.

2. The CGTR Field Equation and Metric Ansatz

CGTR retains the standard Einstein field equation,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (1)$$

and takes the total stress-energy tensor to be the sum of a spatially homogeneous perfect-fluid contribution $T^{\rm bg}_{\mu\nu}$, sourcing the Robertson-Walker background through the ordinary Friedmann equations for $\tau(t_0)$, and a localized contribution $T^{\rm loc}_{\mu\nu}$ concentrated at the spatial origin, representing the point mass, point charge, or rotating source under consideration (Yi, 2021). Because these two contributions are additive, CGTR solves Equation (1) perturbatively: the background expansion function $\tau(t_0)$ is fixed by $T^{\rm bg}_{\mu\nu}$ alone through the ordinary Friedmann equations, and the metric functions describing the localized source are then solved for as a correction superposed on this fixed, spatially flat Robertson-Walker background.

For a static, spherically symmetric localized source, this construction gives the metric ansatz (Yi, 2021)

$$ds^2 = -c^2 B(r_c)\, dt_0^2 + \tau(t_0)^2 \left[A(r_c)\, dr'^2 + r'^2 d\Omega^2\right], \quad (2)$$

where $r_c \equiv \tau(t_0)\, r'$ is the physical radial coordinate and $A(r_c)$, $B(r_c)$ are metric functions determined by Equation (1) sourced by $T^{\rm loc}_{\mu\nu}$, subject to the boundary condition $A, B \to 1$ as $r_c \to \infty$ so that Equation (2) reduces to the pure Robertson-Walker background far from the source.

2.1. The Background Friedmann Equations

The background contribution $T^{\rm bg}_{\mu\nu}$, taken to be a perfect fluid of energy density $\rho(t_0)$ and pressure $p(t_0)$ comoving with the local coordinate x'^i, sources the expansion function $\tau(t_0)$ through the ordinary Friedmann equations,

$$\left(\frac{\dot\tau}{\tau}\right)^2 = \frac{8\pi G}{3}\rho, \qquad \frac{\ddot\tau}{\tau} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right), \quad (3)$$

exactly as in standard Robertson-Walker cosmology, since the spatially homogeneous part of Equation (1) is unaffected by the presence of the localized source at the origin. Equation (3) is what fixes $\tau(t_0)$ as an explicit function of cosmological time, given a specified cosmological fluid content (matter-, radiation-, or dark-energy-dominated), and is used, but not re-derived, throughout the remainder of this paper: the black-hole solutions of Sections 3 and 4 are compatible with any background expansion history satisfying Equation (3), and none of the horizon, redshift, or orbital results derived below depend on the specific functional form of $\tau(t_0)$.

2.2. Boundary and Junction Conditions

Beyond the asymptotic condition $A, B \to 1$ as $r_c \to \infty$, physical viability of the metric ansatz of Equation (2) requires that the induced metric and extrinsic curvature match continuously across any boundary between the vacuum exterior region and a possible interior matter distribution, in direct analogy with the Israel junction conditions used to join interior and exterior solutions in ordinary general relativity. For the idealized point-mass and point-charge sources considered in Sections 3 and 4 below, this reduces to the requirement that $A(r_c)$ and $B(r_c)$ be finite and match their asymptotic values smoothly outside the (unmodeled) interior of the source, which is assumed throughout rather than derived from a specific interior stress-energy model (Yi, 2021).

2.3. On a Possible Time-Dependent Mass Function

The metric ansatz of Equation (2) takes the source mass M to be a fixed constant, appropriate for an isolated black hole that neither accretes background cosmological fluid nor loses mass over the timescale of interest. A more general treatment, paralleling the McVittie construction discussed in Section 7, might instead allow a time-dependent mass function $M(t_0)$ to capture the effect of accretion from the surrounding cosmological fluid onto the central source as the universe expands. Because such a generalization would reintroduce explicit t_0-dependence into the metric functions A and B themselves, rather than confining all epoch dependence to the coordinate relation $r_c = \tau(t_0)r'$ as in the fixed-mass construction of Equation (2), it would substantially change the analysis of every subsequent section of this paper – the horizon would no longer sit at a strictly fixed physical radius, the Kretschmann scalar of Section 3 would acquire explicit time dependence, and the thermodynamic quantities of Section 6 would need to be reformulated for a dynamically evolving horizon. The papers reviewed here consider only the fixed-mass case, and extending the construction to a time-dependent mass function is accordingly listed among the open questions of Section 7.

3. The Cosmologically Embedded Schwarzschild Solution

3.1. The Metric Functions

For a single point mass M at the spatial origin and $T^{\rm loc}_{\mu\nu} = 0$ in the exterior vacuum region $r_c > 0$, solving Equation (1) with the ansatz of Equation (2) gives the cosmologically embedded Schwarzschild metric functions (Yi, 2021)

$$B(r_c) = A(r_c)^{-1} = 1 - \frac{2GM}{c^2\,\tau(t_0)\,r'} = 1 - \frac{2GM}{c^2 r_c}, \quad (4)$$

where the second equality follows from $r_c = \tau(t_0)r'$. Written in terms of the physical radial coordinate r_c, Equation (4) is identical in form to the ordinary Schwarzschild metric function, with no explicit residual dependence on t_0 or $\tau(t_0)$; the entire cosmological content of the solution resides in the relation between r_c and the local coordinate r' used in Equation (2).

This result can also be understood as a cosmological analogue of Birkhoff's theorem: because the vacuum field equation $G_{\mu\nu} = 0$ in the exterior region $r_c > 0$ is solved subject to spherical symmetry and the boundary condition that Equation (2) approach the Robertson-Walker background as $r_c \to \infty$, the solution for $A(r_c)$ and $B(r_c)$ is uniquely fixed, at each fixed cosmological time t_0, by the single parameter M, exactly as the ordinary Birkhoff theorem uniquely fixes the asymptotically flat Schwarzschild solution as the only spherically symmetric vacuum solution with a given mass. The cosmological embedding of Section 2 therefore does not introduce any additional freedom into the exterior vacuum solution beyond the single mass parameter M already present in the ordinary theory, consistent with the pattern, established throughout this paper, that the cosmological expansion function $\tau(t_0)$ enters only through the coordinate relation $r_c = \tau(t_0)r'$ and not through any new physical degree of freedom in the solution itself.

3.2. Horizon and Gravitational Redshift

The event horizon, located at $B(r_c) = 0$, occurs at the ordinary Schwarzschild radius $r_\mathrm{h} = 2GM/c^2$ in the physical coordinate, and correspondingly at $r'_\mathrm{h} = 2GM/(c^2\tau(t_0))$ in the local coordinate, so that the local-coordinate horizon contracts as the universe expands even though its physical radius is fixed (Yi, 2021). The gravitational redshift of a photon emitted at physical radius r_c within a fixed cosmological epoch t_0 and observed at spatial infinity within the same epoch is $1 + z_\mathrm{grav} = B(r_c)^{-1/2}$, identical to the ordinary Schwarzschild result; a photon that instead propagates over a cosmologically significant interval of t_0 between emission and observation acquires an additional cosmological redshift factor $\tau(t_0^\mathrm{obs})/\tau(t_0^\mathrm{em})$ on top of z_grav, so that the two redshift contributions combine multiplicatively, $1 + z_\mathrm{tot} = (1 + z_\mathrm{grav})(1 + z_\mathrm{cos})$, in the same way as in standard treatments of gravitationally redshifted light propagating through an expanding background.

3.3. Radial Geodesics

For a massive test particle released from rest at physical radius r_c^0 within a fixed epoch t_0, the radial equation of motion following from the geodesic equation of Equation (2) is, to leading order in a fixed-epoch approximation ($\dot{\tau} \approx 0$ over the infall time), identical to the ordinary Schwarzschild radial infall equation,

$$\left(\frac{dr_c}{d\lambda}\right)^2 = \frac{2GM}{r_c} - \frac{2GM}{r_c^0}, \quad (5)$$

with λ an affine parameter along the geodesic, because the explicit $\tau(t_0)$-dependence of Equation (2) enters only through the relation between r_c and r' and cancels once the geodesic equation is expressed entirely in the physical coordinate r_c (Yi, 2021). Corrections to Equation (5) from the time-dependence of $\tau(t_0)$ over the course of the infall are suppressed by the ratio of the infall time to the Hubble time $1/H(t_0)$, and are therefore negligible for infall onto any solar-mass or stellar-mass black hole, becoming relevant only for infall times that are themselves a cosmologically significant fraction of $1/H(t_0)$.

3.4. Proper Time to the Singularity

Integrating Equation (5), the proper time elapsed, as measured by the infalling observer's own clock, between release at r_c^0 and arrival at the central singularity $r_c = 0$ is

$$\Delta\tau_\mathrm{proper} = \frac{\pi}{2}\sqrt{\frac{(r_c^0)^3}{2GM}}, \quad (6)$$

identical to the ordinary Schwarzschild result and, like the horizon and photon-sphere radii above, independent of the cosmological epoch t_0 once expressed in the physical coordinate r_c^0. This finiteness of the proper time to the singularity, exactly as in the ordinary Schwarzschild case, holds regardless of how much cosmological expansion has occurred since the infalling observer's release, reflecting the fact that the interior dynamics captured by Equation (5) depend on $\tau(t_0)$ only through the fixed relation $r_c = \tau(t_0)r'$ and not through any additional explicit time dependence within the fixed-epoch approximation adopted in this section.

3.5. Circular Photon Orbits and the Photon Sphere

Null circular orbits, obtained from the effective potential for photon motion in Equation (2) at fixed t_0, occur at the physical radius

$$r_\mathrm{ph} = \frac{3GM}{c^2}, \quad (7)$$

identical to the ordinary Schwarzschild photon-sphere radius, once again because the $\tau(t_0)$-dependence cancels between the metric functions of Equation (4) and the relation $r_c = \tau(t_0)r'$ when the orbit condition is expressed in the physical coordinate (Yi, 2021). The corresponding local-coordinate photon-sphere radius, $r'_\mathrm{ph} = 3GM/(c^2\tau(t_0))$, contracts with cosmic time in the same manner as the event horizon of Section 3.

3.6. Tidal Forces and the Kretschmann Scalar

The curvature invariant that measures tidal forces and the strength of the singularity, the Kretschmann scalar $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, computed from the metric of Equation (2) with the Schwarzschild functions of Equation (4), takes the value

$$K(r_c) = \frac{48\,G^2M^2}{c^4\,r_c^6}, \quad (8)$$

identical in form to the ordinary Schwarzschild Kretschmann scalar when expressed in the physical radial coordinate r_c, diverging as $r_c \to 0$ and vanishing as $r_c \to \infty$ exactly as in the asymptotically flat case, apart from the replacement of the asymptotic flat background by the asymptotic Robertson-Walker background implicit in Equation (2) (Yi, 2021). Because Equation (8) depends on t_0 only through r_c and not explicitly, the physical tidal force experienced by an infalling observer at fixed physical radius is, at any single epoch, indistinguishable from its ordinary Schwarzschild value, and the central singularity at $r_c = 0$ retains its usual curvature-singular character at every cosmological epoch.

3.7. Energy Conditions

The localized stress-energy contribution $T_{\mu\nu}^{\rm loc}$ sourcing the Schwarzschild vacuum solution vanishes identically outside the source, so that the weak, null, and dominant energy conditions are satisfied trivially in the exterior region $r_c > 0$ considered throughout Section 3, exactly as in the ordinary vacuum Schwarzschild solution. Whether the energy conditions are satisfied *inside* the (unmodeled) source itself depends on the interior stress-energy model, which is not specified in the papers reviewed here; this is the same limitation already noted for the boundary and junction conditions of Section 2, and affects the interior structure of the source rather than any of the exterior geometric results – the horizon, redshift, geodesics, photon sphere, and curvature invariant – derived in this section.

4. The Cosmologically Embedded Kerr-Newman Solution

4.1. The Metric Function and Horizons

For a rotating, charged source of mass M, angular momentum $J = Mac$, and charge Q, CGTR gives a cosmologically embedded Kerr-Newman metric whose equatorial radial structure is governed by (Yi, 2021)

$$\Delta(r') = \tau(t_0)^2 r'^2 - \frac{2GM}{c^2}\tau(t_0)\, r' + a^2 + \frac{GQ^2}{4\pi\epsilon_0 c^4}, \quad (9)$$

with the same identification $r_c = \tau(t_0)r'$ used above. Expressed in the physical radial coordinate, Equation (9) becomes $\Delta(r_c) = r_c^2 - (2GM/c^2)r_c + a^2 + GQ^2/(4\pi\epsilon_0 c^4)$, identical in form to the ordinary Kerr-Newman Δ-function. Solving $\Delta(r_c) = 0$ gives the inner and outer horizon radii

$$r_{c,\pm} = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2 - \frac{GQ^2}{4\pi\epsilon_0 c^4}}, \quad (10)$$

independent of t_0, with the corresponding local-coordinate radii $r'_\pm = r_{c,\pm}/\tau(t_0)$ carrying the entire epoch dependence (Yi, 2021).

The full cosmologically embedded Kerr-Newman line element, of which Equation (9) gives only the equatorial radial structure, takes the Boyer-Lindquist-type form

$$\begin{aligned} ds^2 = &-c^2\left(1 - \frac{2GMr_c}{c^2\Sigma}\right)dt_0^2 - \frac{4GMar_c\sin^2\theta}{c\,\Sigma}\,dt_0\,d\phi \\ &+ \frac{\Sigma}{\Delta}\,\tau(t_0)^2 dr'^2 + \Sigma\,d\theta^2 \\ &+ \left(r_c^2 + a^2 + \frac{2GMa^2 r_c\sin^2\theta}{c^2\Sigma}\right)\sin^2\theta\,d\phi^2, \quad (11) \end{aligned}$$

with $\Sigma \equiv r_c^2 + a^2\cos^2\theta$ and Δ given by Equation (9), generalizing the metric ansatz of Equation (2) beyond spherical symmetry in the same way that the ordinary Kerr-Newman metric generalizes the ordinary Schwarzschild metric (Yi, 2021). Every term in Equation (11) that depends on the radial coordinate does so only through $r_c = \tau(t_0)r'$ and Σ, Δ evaluated at that same r_c, so that the horizon, ergosphere, and orbital results of this section, all derived from Equation (11), follow the same epoch-independence pattern already established for the simpler spherically symmetric Schwarzschild case.

4.2. Extremality and the No-Horizon Condition

The extremal condition $r_{c,+} = r_{c,-}$, reached when $(GM/c^2)^2 = a^2 + GQ^2/(4\pi\epsilon_0 c^4)$, is likewise independent of the cosmological epoch, since it depends only on the physical source parameters M, a, Q. When $(GM/c^2)^2 < a^2 + GQ^2/(4\pi\epsilon_0 c^4)$, Equation (10) yields no real horizon, exposing a naked ring singularity in the physical geometry, exactly as in the ordinary Kerr-Newman case; because this condition is stated entirely in terms of epoch-independent physical parameters, CGTR predicts that whether a given cosmologically embedded rotating charged mass possesses a horizon at all is a fixed property of the source, unaffected by how much the universe has expanded since the source's formation.

This has a direct bearing on the cosmological status of the weak cosmic censorship conjecture, which holds, in the ordinary theory, that gravitational collapse from generic, physically reasonable initial data does not produce a naked singularity. Because CGTR's extremality condition above depends only on the physical parameters M, a, Q of the source and not on $\tau(t_0)$, any process that would violate cosmic censorship by driving a Kerr-Newman source past extremality – for instance, by adding angular momentum or charge to a near-extremal source – is neither helped nor hindered by the cosmological expansion within this framework: the physical threshold for censorship violation is fixed once and for all by the source parameters, and the passage of cosmic time affects only the local-coordinate description of the resulting geometry, not whether censorship is respected or violated. This is a further instance of the general pattern, established throughout this paper, that the cosmological embedding of Section 2 leaves the physical content of stationary black-hole solutions unchanged at every epoch.

4.3. The Ergosphere

The static limit surface, at which the timelike Killing vector ∂_{t_0} becomes null, is located, in the physical coordinate and at polar angle θ, at

$$r_{c,\rm ergo}(\theta) = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2\cos^2\theta - \frac{GQ^2}{4\pi\epsilon_0 c^4}}, \quad (12)$$

identical in form to the ordinary Kerr-Newman ergosphere and, as with the horizons of Section 4, independent of the cosmological epoch when expressed in the physical radial coordinate. The ergosphere lies outside the outer horizon of

Equation (10) for all $\theta \neq 0, \pi$, exactly as in the ordinary theory, so that the region in which negative-energy orbits permit a cosmological analogue of the Penrose process remains geometrically well defined at every cosmological epoch (Yi, 2021).

At the poles ($\theta = 0, \pi$), the ergosphere radius of Equation (12) coincides with the outer horizon radius $r_{c,+}$ of Equation (10), while at the equator ($\theta = \pi/2$) it attains its maximum separation from the horizon, exactly as in the ordinary Kerr-Newman geometry; the ergosphere therefore retains its familiar oblate, horizon-hugging shape at every cosmological epoch when both surfaces are plotted in the physical radial coordinate r_c. In the local coordinate $r' = r_c/\tau(t_0)$, both the ergosphere and horizon surfaces contract together by the same overall factor $1/\tau(t_0)$, so that their relative shape – as opposed to their absolute local-coordinate size – is likewise unaffected by the cosmological epoch.

4.4. Equatorial Circular Orbits

For equatorial circular geodesic orbits around the cosmologically embedded Kerr-Newman solution, the specific energy and angular momentum per unit rest mass, evaluated at fixed cosmological time t_0 and expressed in the physical radial coordinate r_c, take the same functional form as in the ordinary Kerr-Newman spacetime,

$$\frac{E}{mc^2} = \frac{r_c^{3/2} - 2\frac{GM}{c^2} r_c^{1/2} \pm a\frac{GM}{c^2} r_c^{-1/2} \cdot \frac{c}{\sqrt{GM}}}{r_c^{3/4}\left(r_c^{3/2} - 3\frac{GM}{c^2} r_c^{1/2} \pm 2a\sqrt{\frac{GM}{c^2}}\right)^{1/2}}, \quad (13)$$

with the upper (lower) sign for co-rotating (counter-rotating) orbits, because the same cancellation of $\tau(t_0)$ between the metric functions and the coordinate relation $r_c = \tau(t_0)r'$ that occurred for the Schwarzschild geodesics of Section 3 occurs here as well (Yi, 2021). The innermost stable circular orbit (ISCO), defined by the condition that the effective radial potential built from Equation (13) develops a vanishing second derivative, is correspondingly located at a fixed physical radius depending only on M, a, and Q, and not on the cosmological epoch t_0.

4.5. Frame Dragging

The Lense-Thirring frame-dragging angular velocity of a zero-angular- momentum observer, $\omega(r_c, \theta) = -g_{t_0\phi}/g_{\phi\phi}$, computed from the cosmologically embedded Kerr-Newman metric, takes the same functional form as in the ordinary Kerr-Newman spacetime when expressed in the physical coordinate r_c, and in particular retains the ordinary asymptotic falloff $\omega \to 2GJ/(c^2 r_c^3)$ at large physical radius. Because this falloff depends only on the physical angular momentum $J = Mac$ and radius r_c, the frame-dragging effect of a cosmologically embedded rotating mass – for instance, its contribution to geodetic and Lense-Thirring precession of a nearby gyroscope – is, at any single epoch, indistinguishable from the ordinary, non-cosmological prediction, with the local-coordinate expression of ω carrying the usual explicit factors of $\tau(t_0)$ inherited from $r_c = \tau(t_0)r'$.

4.6. The Penrose Process

Within the ergosphere of Equation (12), a particle can in principle split into two fragments such that one fragment carries negative energy (as measured at infinity within the fixed epoch t_0) and falls into the horizon while the other escapes with more energy than the original particle possessed, extracting rotational energy from the source. Because the maximum extractable energy in the ordinary Penrose process is a fixed fraction of the irreducible mass, determined entirely by the physical parameters M and a, and the ergosphere and horizon locations of Sections 4 are themselves epoch-independent in the physical coordinate, CGTR predicts that the maximum energy extractable via the Penrose process from a cosmologically embedded rotating source is the same, at every cosmological epoch, as the ordinary Kerr-Newman result, with no additional cosmological enhancement or suppression of the extractable energy fraction.

4.7. Gyroscope Precession: A Worked Comparison

As a concrete illustration of the frame-dragging result of Section 4, consider a gyroscope in circular orbit at physical radius r_c in the equatorial plane of a cosmologically embedded Kerr source. The geodetic (de Sitter) precession rate and the Lense-Thirring precession rate, both computed from the metric of Equation (2) with the Kerr-Newman functions of Section 4, take the ordinary forms $\Omega_{\text{geodetic}} \approx \frac{3}{2}\sqrt{GM/r_c^3} \times (GM/c^2 r_c)$ and $\Omega_{\text{LT}} \approx 2GJ/(c^2 r_c^3)$ once expressed in the physical radial coordinate, with no additional $\tau(t_0)$-dependence beyond what is already implicit in r_c itself. Comparing the two, the ratio $\Omega_{\text{LT}}/\Omega_{\text{geodetic}} \sim (a/r_c)\sqrt{r_c c^2/(GM)}$ depends only on the physical spin parameter a, mass M, and orbital radius r_c, so that – exactly as in ordinary general relativity, and as tested by gyroscope experiments such as Gravity Probe B in the Earth's field – the relative size of the two precession effects predicted by CGTR for a cosmologically embedded rotating source is fixed by the source's physical parameters alone and does not depend on the cosmological epoch at which the measurement is performed.

5. Limiting Cases

5.1. The Homogeneous Limit

Taking $M \to 0$, $Q \to 0$, $a \to 0$ in Equations (4) and (9) gives $B(r_c) = A(r_c)^{-1} \to 1$, so that Equation (2) reduces identically to the spatially flat Robertson-Walker line element underlying CSTR. This recovery of the homogeneous background in the source-free limit is a basic consistency requirement on any embedding of a localized source within an expanding universe, and is satisfied by construction here, since the metric ansatz of Equation (2) was built as a perturbation of the Robertson-Walker background rather than as an independent ansatz (Yi, 2021).

It is worth noting that this limit is taken at fixed r' (equivalently, fixed $r_c/\tau(t_0)$) as the source parameters are sent to zero, rather than at fixed r_c as $r_c \to \infty$; the two limits coincide here only because the metric functions of Equations (4) and (9) depend on the source parameters and on r_c only through the combination $GM/(c^2 r_c)$ (and its charge and spin counterparts), which vanishes in either limit. This is a mild but non-trivial consistency feature of the specific perturbative construction adopted in Section 2: a metric ansatz in which the source and background contributions to $T_{\mu\nu}$ were combined non-additively would not, in general, guarantee that the small-mass and large-radius limits coincide in this way.

5.2. The Weak-Field, Slow-Rotation Limit

Expanding Equation (4) to leading order in $GM/(c^2 r_c)$ gives $B(r_c) \approx 1 - 2GM/(c^2 r_c)$, identical to the Newtonian-limit metric function of ordinary general relativity, with Newtonian potential $\Phi = -GM/r_c$; expanding Equation (9) to leading order in a/r_c and $Q^2/(r_c^2)$ in addition recovers the ordinary gravitomagnetic frame-dragging term at order a/r_c^3 familiar from the weak-field Kerr metric. Both limits confirm that the cosmological embedding of Section 2 modifies the relation between the physical and local radial coordinates without altering the weak-field content of the localized solutions themselves.

Carrying the expansion of $B(r_c)$ and $A(r_c)$ to the next order in $GM/(c^2 r_c)$ reproduces the standard parametrized post-Newtonian (PPN) form $B(r_c) \approx 1 - 2GM/(c^2 r_c) + 2\beta_{\rm PPN}(GM/c^2 r_c)^2$, $A(r_c)^{-1} \approx 1 - 2GM/(c^2 r_c)$, with the CGTR Schwarzschild solution of Equation (4) giving the general-relativistic values $\beta_{\rm PPN} = \gamma_{\rm PPN} = 1$ at every cosmological epoch, since the exact metric function of Equation (4) reduces identically to the ordinary Schwarzschild form once expressed in r_c. This shows that CGTR predicts no cosmological-epoch-dependent deviation from the standard solar-system tests of general relativity (light bending, perihelion precession, Shapiro delay), all of which are governed by $\beta_{\rm PPN}$ and $\gamma_{\rm PPN}$ evaluated at the present epoch.

As concrete instances of this general statement, the perihelion precession per orbit of a test body in the weak field of Equation (4), $\Delta\phi = 6\pi GM/(c^2 r_c(1-e^2))$ for an orbit of semi-latus rectum set by eccentricity e, and the light-bending angle for a photon with impact parameter b, $\delta\phi = 4GM/(c^2 b)$, both depend on t_0 only insofar as r_c and b themselves might be defined with reference to a specific epoch, and neither formula contains an explicit residual factor of $\tau(t_0)$ beyond this. This is consistent with the more general PPN statement above and confirms, at the level of two of the three classical solar-system tests of general relativity, that the cosmological embedding of Section 2 leaves the weak-field phenomenology of the Schwarzschild solution completely unchanged at fixed physical radius.

6. Black Hole Thermodynamics

The surface gravity of the cosmologically embedded Schwarzschild horizon, $\kappa = c^4/(4GM)$, computed from the metric function of Equation (4) at $r_c = r_{\rm h} = 2GM/c^2$, is identical to the ordinary Schwarzschild surface gravity and, as with the horizon radius itself, independent of the cosmological epoch t_0. Via the ordinary identification of the Hawking temperature with the surface gravity, $T_H = \hbar\kappa/(2\pi k_B c)$, this suggests that a cosmologically embedded Schwarzschild black hole radiates at the ordinary, epoch-independent Hawking temperature $T_H = \hbar c^3/(8\pi GMk_B)$ at any fixed cosmological time, and that its horizon area $\mathcal{A} = 4\pi r_{\rm h}^2$ and corresponding Bekenstein-Hawking entropy $S = k_B c^3 \mathcal{A}/(4G\hbar)$ likewise take their ordinary, epoch-independent values when expressed in the physical radial coordinate. A full treatment of black-hole thermodynamics in CGTR would, however, need to account for the fact that the horizon is embedded in a time-dependent Robertson-Walker background rather than in asymptotically flat space, so that the usual assumption of a well-defined asymptotic region at future null infinity – used to derive the Hawking radiation spectrum in the ordinary case – does not directly apply, and the radiated flux would need to be defined instead relative to the cosmological horizon of the background expansion, a calculation not carried out in the papers reviewed here.

A related question concerns the generalized second law of thermodynamics, which in the ordinary asymptotically flat case states that the sum of black-hole entropy and ordinary matter entropy outside the horizon is non-decreasing. In a cosmological setting, this statement is usually generalized to include the entropy associated with the cosmological (apparent) horizon of the background expansion itself. Because the CGTR construction of Section 2 treats the background expansion and the localized black-hole solution as additive but otherwise separate contributions to the metric, formulating a generalized second law that consistently combines the black-hole entropy of this section with a cosmological horizon entropy for the background $\tau(t_0)$ has not been attempted in the papers reviewed here, and is a further, thermodynamic counterpart to the mass-function question raised in Section 2 above.

A further complication specific to the cosmological setting is that the Robertson-Walker background itself possesses an associated Gibbons-Hawking temperature when its expansion is exponential (de-Sitter- like), analogous to the Hawking temperature of a black-hole horizon; for a general expansion function $\tau(t_0)$ satisfying the Friedmann equations of Section 2 rather than pure exponential expansion, no single, constant background temperature of this kind is generally defined, and the two temperature scales – the black hole's T_H derived above and any instantaneous cosmological analogue defined from $H(t_0)$ – would need to be compared at each epoch separately rather than treated as two fixed, comparable constants. This is a further respect in which the thermodynamic treatment of the cosmologically embedded Schwarzschild solution remains incomplete relative to the purely static, asymptotically flat case.

7. Discussion

7.1. Comparison with the McVittie Solution

The problem addressed by CGTR – embedding a point mass in an expanding Robertson-Walker background – was first addressed in the mainstream general-relativity literature by McVittie, whose solution similarly reduces to the Schwarzschild metric near the source and to the Robertson-Walker metric far from it. In its original form, the McVittie line element is written as

$$ds^2 = -c^2 \left(\frac{1-\mu(t_0)}{1+\mu(t_0)}\right)^2 dt_0^2 + [1+\mu(t_0)]^4 a(t_0)^2 \left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right], \qquad (14)$$

with $\mu(t_0) = GM/(2c^2 a(t_0)\bar{r})$ and $\bar{r}$ a comoving radial coordinate, so that the single mass parameter M enters the metric functions divided by the scale factor $a(t_0)$ itself, in contrast with the CGTR ansatz of Equation (2), in which the mass enters only through the fixed combination $GM/(c^2 r_c)$ with $r_c = \tau(t_0)r'$ the physical coordinate. This structural difference is the origin of the differing horizon behavior noted below: in Equation (14), the location at which the metric degenerates depends on t_0 both through $\bar{r}$ and through the explicit $a(t_0)$ appearing in $\mu(t_0)$, whereas in Equation (4) the analogous degeneration condition $B(r_c) = 0$ depends on t_0 only through the fixed relation between r_c and r'.

The McVittie metric is commonly written directly in terms of a physical (areal) radial coordinate and a single mass function, and its interpretation – in particular, whether it describes a black hole with a genuine event horizon or only an apparent horizon that evolves with the cosmological expansion, and whether the central singularity is spacelike or null – has been the subject of extensive discussion in the general-relativity literature. The CGTR construction reviewed here, by contrast, is formulated directly in the local coordinate r' of the cosmological inertial frame, with the physical radius $r_c = \tau(t_0)r'$ introduced only after the metric functions A and B have been solved for. Because the Schwarzschild metric function of Equation (4), expressed in r_c, carries no residual t_0-dependence, CGTR's horizon at $r_c = 2GM/c^2$ behaves as a fixed, non-evolving surface in the physical coordinate at any single epoch, in contrast with the explicitly time-dependent apparent horizon of the McVittie solution; reconciling this difference – in particular, determining whether CGTR's horizon structure is an artifact of the perturbative, fixed-background construction of Section 2, or reflects a genuine physical distinction from the McVittie embedding – is identified as an open question for further work.

7.2. Apparent Horizons and the Choice of Horizon Definition

The tension identified above reflects a broader point familiar from the mainstream literature on black holes in a dynamical or cosmological setting: in a time-dependent spacetime, the teleological, globally defined event horizon (the boundary of the causal past of future null infinity) need not coincide with the locally defined apparent or trapping horizon (the outermost surface at which outgoing null geodesics instantaneously fail to diverge). In the McVittie spacetime, it is the apparent horizon that is usually discussed, and it is generically time-dependent, tracking the interplay between the fixed mass parameter and the evolving background density. The CGTR construction of Section 2, by fixing the metric functions $A(r_c)$, $B(r_c)$ once and for all as functions of the physical coordinate alone, implicitly defines its horizon in a manner closer to the event-horizon concept of asymptotically flat spacetimes than to the locally defined apparent horizon of the McVittie solution. Determining the location of the apparent (rather than event) horizon within the CGTR metric of Equation (2) – by directly computing the expansion of outgoing null geodesic congruences at fixed t_0, rather than relying on the condition $B(r_c) = 0$ – and comparing it with the McVittie apparent horizon directly, would clarify whether the distinction noted in Section 7 above is a genuine physical difference between the two constructions or an artifact of the choice of horizon definition.

7.3. Comparison with the Schwarzschild-de Sitter (Kottler) Solution

A related mainstream construction, the Schwarzschild-de Sitter or Kottler solution, embeds a point mass in a background with a nonzero cosmological constant rather than a general, matter-dominated Robertson-Walker expansion, and produces both a black-hole event horizon and a cosmological horizon at a fixed physical radius set by the cosmological constant. Because CGTR's background is instead sourced by the ordinary Friedmann equations for a general $\tau(t_0)$ rather than by a cosmological constant specifically, the Kottler solution is not directly recovered as a special case of the construction in Section 2 without an additional assumption relating $\tau(t_0)$ to a constant vacuum energy density; identifying the precise conditions on $\tau(t_0)$ under which the CGTR Schwarzschild solution of Section 3 reduces to the Kottler metric is a further concrete comparison left for future work.

In the Kottler solution, the cosmological horizon at physical radius $r_\Lambda = \sqrt{3/\Lambda}$, set by the cosmological constant Λ, plays a role analogous to a second, outer horizon bounding the region in which the black-hole horizon of Section 3 can be meaningfully defined; a static observer can exist only between the black-hole horizon and this cosmological horizon. The CGTR construction of Section 2, formulated for a general expansion function $\tau(t_0)$ rather than for exponential, de-Sitter-like expansion specifically, does not by itself single out an analogous fixed cosmological horizon radius; whether a time-dependent analogue of r_Λ, expressed through the instantaneous Hubble radius $c/H(t_0)$ of the background at each epoch, plays a comparable geometric role in bounding the region of validity of the CGTR Schwarzschild solution of Equation (4) is a further question raised, but not resolved, by this comparison.

7.4. Scale of the Cosmological Correction

It is useful to have a rough sense of the physical scale at which the local-coordinate corrections discussed throughout this paper become significant. Normalizing $\tau(t_0^{\mathrm{obs}}) = 1$ at the present epoch, the local-coordinate horizon radius $r'_{\mathrm{h}} = 2GM/(c^2\tau(t_0))$ of a source that formed at redshift z is larger, in the local coordinate, by a factor of $(1+z)$ relative to the same physical horizon evaluated today, following directly from the redshift identification $1+z = \tau(t_0^{\mathrm{obs}})/\tau(t_0^{\mathrm{em}})$ used throughout the companion CSTR papers (Yi, 2020). For a stellar-mass black hole with Schwarzschild radius of order kilometers, or a supermassive black hole with Schwarzschild radius of order the size of the solar system, this local-coordinate rescaling has no effect whatsoever on locally measured observables – orbital dynamics, photon-sphere imaging, or tidal disruption radii – since, as emphasized throughout Sections 3 and 4, every physically measured quantity considered in this paper is independent of $\tau(t_0)$ once expressed in the physical coordinate r_c. The local-coordinate rescaling is therefore best understood as a bookkeeping device internal to the cosmological inertial frame, rather than as a predicted observable effect on black-hole physics itself.

7.5. Observational Context: Event Horizon Imaging

Very-long-baseline interferometric imaging of the shadows of the supermassive black holes at the centers of M87 and the Milky Way has provided a direct observational measurement of a black-hole photon-sphere- related angular scale, offering, in principle, an empirical anchor for the photon-sphere prediction of Equation (7). Because both imaged sources lie at negligible cosmological redshift ($z \ll 1$), the discussion of Section 3 above indicates that CGTR predicts no measurable departure from the ordinary, non-cosmological photon-sphere radius for either source, so that existing shadow measurements cannot, by themselves, distinguish between CGTR and ordinary general relativity. A more discriminating test would require imaging a black-hole shadow at cosmologically significant redshift, at which the local-coordinate rescaling emphasized throughout this paper becomes an order-unity effect (Section 5 above); no such measurement is presently feasible with existing very-long-baseline interferometric techniques, since the angular resolution required to resolve a photon-sphere-scale structure falls rapidly with increasing source distance. This is consistent with the broader pattern noted in Section 7: the distinguishing predictions of CGTR relative to ordinary general relativity are concentrated at cosmological distance and redshift scales that lie, for the time being, beyond the reach of direct black-hole imaging.

7.6. Open Questions

Five questions stand out from the discussion above as concrete directions for further developing the general-relativistic sector of CSTR: (i) determining whether the CGTR metric ansatz of Equation (2) solves the *exact*, rather than only a perturbative or linearized, Einstein field equation sourced by a point mass on an expanding background, which would place it on the same footing as the exact McVittie solution; (ii) clarifying the relationship between CGTR's fixed-radius horizon of Section 3 and the time-dependent apparent horizon of the McVittie solution; (iii) identifying the conditions under which the CGTR Schwarzschild solution reduces to the Kottler metric in the special case of cosmological-constant-dominated expansion; (iv) extending the fixed-mass construction of Equation (2) to a time-dependent mass function capable of describing accretion from the background cosmological fluid, as noted in Section 2; and (v) formulating a generalized second law of thermodynamics that consistently combines the black-hole entropy of Section 6 with a cosmological horizon entropy for the background expansion. An independent critical review of the underlying CSTR framework has additionally raised broader questions about the framework's consistency with the diffeomorphism invariance of general relativity and about its limited independent citation history (A Critical Review, 2025); addressing these, alongside the five technical questions above, would substantially strengthen the general-relativistic foundations of the program.

Quantity	Physical value	Local value
Schwarzschild horizon	$2GM/c^2$	$2GM/(c^2\tau(t_0))$
Photon-sphere radius	$3GM/c^2$	$3GM/(c^2\tau(t_0))$
Kerr-Newman horizons $r_{c,\pm}$	Eq. (10)	$r_{c,\pm}/\tau(t_0)$
Ergosphere $r_{c,\mathrm{ergo}}(\theta)$	Eq. (12)	$r_{c,\mathrm{ergo}}/\tau(t_0)$
Extremality condition	epoch-independent	epoch-independent
ISCO radius	epoch-independent	scales as $1/\tau(t_0)$

Table 1: Physical versus local-coordinate values of the principal geometric quantities derived in this paper.

Quantity	Epoch dependence (physical)
Kretschmann scalar $K(r_c)$	Independent of t_0
Proper time to singularity	Independent of t_0
Frame-dragging rate $\omega(r_c,\theta)$	Independent of t_0
Penrose-process energy yield	Independent of t_0
Geodetic/Lense-Thirring ratio	Independent of t_0
PPN parameters $\beta_{\mathrm{PPN}}, \gamma_{\mathrm{PPN}}$	Independent of t_0 (both equal 1)
Surface gravity, Hawking temperature	Independent of t_0 (fixed background)
Hawking flux in dynamical background	Open question (Sec. 6)

Table 2: Epoch dependence of the dynamical and thermodynamic quantities derived in Sections 4–6.

Table 1 summarizes the physical (epoch-independent) and local-coordinate (epoch-dependent) values of the principal geometric quantities derived in this paper.

Table 2 extends this comparison to the dynamical and thermodynamic quantities derived in Sections 4 through 6.

8. Conclusion

This paper has developed the general-relativistic sector of the Cosmological Special Theory of Relativity in more depth than a single consolidated survey allows, deriving the cosmologically embedded Schwarzschild solution and its horizon, redshift, and geodesic structure (Section 3), the cosmologically embedded Kerr-Newman solution and its horizons, ergosphere, extremality condition, and circular orbits (Section 4), and the homogeneous and weak-field limits that connect both solutions back to the underlying Robertson-Walker background and to ordinary general relativity (Section 5). In every case, a consistent pattern emerges, summarized in Table 1: the physically measured geometric properties of the cosmologically embedded black hole – its horizon radii, photon-sphere radius, ergosphere, extremality condition, and ISCO – recover their ordinary, epoch-independent general-relativistic values once expressed in the physical radial coordinate r_c, even though the corresponding local-coordinate expressions carry an explicit signature of the cosmological expansion function $\tau(t_0)$. Comparison with the mainstream McVittie and Schwarzschild-de Sitter embeddings of a point mass in an expanding or accelerating universe (Section 7) identifies the exact-versus-perturbative status of the CGTR field equation, and the relation between CGTR's fixed-radius horizon and the McVittie apparent horizon, as the most direct points of comparison – and of open tension – between this framework and the standard general-relativistic treatment of localized sources in cosmology, and these, together with the Kottler-limit question, the possible extension to a time-dependent mass function (Section 2), and the generalized second law of thermodynamics (Section 6), are identified as the five central open problems for the continued development of the Cosmological General Theory of Relativity.

Beyond these open questions, the results of Sections 3 through 6 establish a single, recurring pattern across every geometric and dynamical property considered: tidal forces (Section 3), frame dragging and the Penrose process (Section 4), the parametrized post-Newtonian coefficients and surface gravity (Sections 5 and 6) all take their ordinary, epoch-independent general-relativistic values once expressed in the physical radial coordinate r_c. This suggests, tentatively, that CGTR's cosmological embedding of a stationary black hole solution modifies only the bookkeeping relation $r_c = \tau(t_0)r'$ between physical and local coordinates, and leaves the physically measurable content of the Schwarzschild and Kerr-Newman solutions themselves unchanged at any fixed cosmological epoch – a conclusion that would need to be checked against an exact, rather than perturbative, solution of the CGTR field equation before it could be regarded as established, per the first open question identified in Section 7.

Taken as a whole, the results of this paper indicate that the mathematical content added by the cosmological embedding of Section 2 is, within the fixed-mass, perturbative construction reviewed here, concentrated entirely in the relation between the physical and local radial coordinates, $r_c = \tau(t_0)r'$, rather than in any modification of the physically measurable predictions of the Schwarzschild and Kerr-Newman solutions themselves. Whether this conclusion survives the transition to an exact solution of the field equation, to a time-dependent mass function, and to a fully consistent cosmological thermodynamics – the three principal extensions identified in Sections 2, 6, and 7 – is the central question left open by the present analysis, and resolving it is the most direct route toward establishing the Cosmological General Theory of Relativity as a complete and quantitatively predictive extension of general relativity to localized sources in an expanding universe.

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A. Glossary of Key Symbols

Table 3 collects the principal symbols used throughout this paper.

Symbol	Meaning
t_0	Cosmological time
$\tau(t_0)$	Cosmological expansion function (scale factor)
r_c, x^i	Physically expanded radial/spatial coordinate
r', x'^i	Local (unexpanded) radial/spatial coordinate
M, a, Q	Mass, spin parameter, charge of the localized source
$A(r_c)$, $B(r_c)$	CGTR metric functions
r_{h}, $r_{c,\pm}$	Schwarzschild / Kerr-Newman horizon radii
r_{ph}	Photon-sphere radius
$r_{c,\mathrm{ergo}}$	Ergosphere radius
$G_{\mu\nu}$, $T_{\mu\nu}$	Einstein tensor, stress-energy tensor
CGTR	Cosmological General Theory of Relativity
CSTR	Cosmological Special Theory of Relativity
ISCO	Innermost stable circular orbit

Table 3: Glossary of symbols used in this paper.

B. Index of Principal Results

Table 4 indexes the principal results reviewed in this paper against the original source in which each first appeared.

Result	Original source
Cosmological inertial frame	Yi (2020)
CGTR field equation, metric ansatz	Yi (2021)
Cosmologically embedded Schwarzschild solution	Yi (2021)
Cosmologically embedded Kerr-Newman solution	Yi (2021)
Full consolidation	Yi (2025)

Table 4: Index of the principal results reviewed in this paper and their original sources. Derived quantities not appearing explicitly in the original papers (e.g., the Kretschmann scalar, proper time to the singularity, PPN parameters, and thermodynamic quantities of Sections 3–6) are standard general-relativistic constructions applied here to the metric functions of Yi (2021).

References

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