Cosmological Quantum Physics: Relativistic Wave Equations, Field Quantization, and the Uncertainty Principle in the Cosmological Inertial Frame

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Abstract

The Cosmological Special Theory of Relativity (CSTR) re-derives the relativistic wave equations of quantum theory within an auxiliary cosmological inertial frame*, in which a local, unexpanded spatial coordinate is related to the physically expanded coordinate of an expanding Robertson-Walker universe through a cosmological expansion function $\tau(t_0)$ of the cosmological time t_0 (Yi, 2020,?). This paper focuses specifically on the quantum-theoretic content of that program: the cosmological Klein-Gordon equation, its plane-wave solutions, conserved current, and Yukawa-potential solution (Yi, 2020, 2021); the non-relativistic Schrödinger limit and the associated cosmological position-momentum and time-energy uncertainty relations (Yi, 2020); the Dirac equation in the cosmological inertial frame, its consistency with the Klein-Gordon equation, its non-relativistic (Pauli) reduction, and its negative-energy spectrum (Yi, 2021); the Klein-Gordon-Maxwell theory of an interacting charged scalar and electromagnetic field, together with its gauge invariance and static Coulomb limit (Yi, 2021); and the canonical quantization of the free Klein-Gordon field, including its Fock space, vacuum energy, and equal-time propagator (Yi, 2021, 2025). We conclude with a discussion of how this formalism compares with the treatment of quantum fields in curved and expanding spacetimes in the mainstream literature, and with the open questions raised by an independent critical review of CSTR (A Critical Review, 2025).*

Keywords: cosmological quantum physics; cosmological special theory of relativity; Klein-Gordon equation; Dirac equation; canonical quantization; cosmological uncertainty principle

1. Introduction

Quantum theory in its ordinary form – the Klein-Gordon and Dirac equations, the canonical quantization of free fields, and the position-momentum uncertainty principle – is formulated on a fixed Minkowski background. The Cosmological Special Theory of Relativity (CSTR) instead re-derives each of these results within an auxiliary *cosmological inertial frame*, defined by relating a local (unexpanded) spatial coordinate x'^i to the physically expanded coordinate x^i of a spatially flat Robertson-Walker universe through a cosmological expansion function $\tau(t_0)$ of the cosmological time t_0: $x^i = \tau(t_0)\, x'^i$ (Yi, 2020). Substituting this relation into the differential operators of the ordinary relativistic wave equations, so that $\partial/\partial x^i \to (1/\tau(t_0))\, \partial/\partial x'^i$, generates cosmological versions of the Klein-Gordon equation, the Yukawa potential, the Schrödinger equation, the Dirac equation, and the Klein-Gordon-Maxwell theory of an interacting charged scalar field, each of which has been developed in a short paper of its own (Yi, 2020, 2021,?,?,?), and all of which are drawn together under a single abstract in the 2025 consolidation (Yi, 2025).

The individual results reviewed below were developed over a two-year period: the cosmological inertial frame and its coordinate transformation were introduced first (Yi, 2020), followed within the same year by the Klein-Gordon equation and its free-particle wave function (Yi, 2020); the Yukawa potential, the Dirac equation, the interacting Klein-Gordon-Maxwell theory, and the canonical quantization of the scalar field were each developed in the following year (Yi, 2021,?,?,?). Because each of these later results depends on the substitution rule of Equation (3) introduced in the first two papers of the series, the quantum-theoretic program surveyed here is best understood as a single cumulative construction rather than as a set of unrelated results, and the present paper aims to make the logical dependence between its parts, and the specifically quantum-mechanical consequences of the cosmological substitution rule, more explicit than is possible in a broader survey of the full CSTR and CGTR program.

Concentrating on the quantum-theoretic subset of the program also makes it possible to pursue several derivations further than a single consolidated survey allows: the scattering theory implicit in the Yukawa potential can be developed into an explicit cross section (Section 4), the algebra of the uncertainty relations can be extended from the position-momentum case to the time-energy and angular-momentum cases (Section 5), the Dirac equation's non-relativistic and negative-energy content can each be worked out explicitly (Section 6), and the canonical quantization of the scalar field can be extended to its Fock-space and propagator structure

(Section 8) in a way that directly exposes the points of comparison, taken up in Section 9, with the mainstream theory of quantum fields in curved and expanding spacetimes.

Whereas a companion review has surveyed the full CSTR program, including its extension to a Cosmological General Theory of Relativity and cosmologically embedded black-hole solutions, the present paper concentrates specifically on the *quantum-theoretic* content of the program – the relativistic wave equations, their non-relativistic limits and associated uncertainty relations, and the canonical quantization of the free scalar field – and develops this material in greater depth than is possible in a single consolidated survey. Sections 2 and 3 establish the cosmological inertial frame and the Klein-Gordon equation; Section 4 treats the Yukawa potential and a cosmological Born approximation for scattering; Section 5 derives the Schrödinger equation and the cosmological uncertainty relations; Section 6 treats the Dirac equation, its non-relativistic limit, and its negative-energy spectrum; Section 7 treats the Klein-Gordon-Maxwell theory; and Section 8 treats the canonical quantization of the free scalar field. Section 9 compares this formalism with the mainstream treatment of quantum fields in curved and expanding spacetimes.

Throughout, primed quantities (x'^i, $\mathbf{p}'$, ∇') refer to the local, unexpanded coordinate of the cosmological inertial frame, while unprimed quantities (x^i, r_c) refer to the physically expanded coordinate related to the local one by $x^i = \tau(t_0)x'^i$. The cosmological time t_0 is left unprimed and unrescaled throughout, since Wave Function Type A of Equation (2) acts only on the spatial coordinates. Explicit factors of c and $\hbar$ are retained throughout, following the convention of the original papers, rather than being set to unity, so that the dimensional analysis noted at several points below (for instance, in Equation (14)) can be checked directly from the equations as written.

2. The Cosmological Inertial Frame

The spatially flat Robertson-Walker line element,

$$ds^2 = -c^2dt_0^2 + \tau(t_0)^2\left[dx'^2 + dy'^2 + dz'^2\right], \qquad (1)$$

underlies CSTR's cosmological inertial frame, in which the local coordinate x'^i is related to the physically expanded coordinate x^i by *Wave Function Type A*,

$$x^i = \tau(t_0)\,x'^i, \qquad t = t_0, \qquad (2)$$

and every ordinary spatial derivative is replaced according to

$$\frac{\partial}{\partial x^i} \longrightarrow \frac{1}{\tau(t_0)}\frac{\partial}{\partial x'^i} \qquad (3)$$

(Yi, 2020,?). The expansion function $\tau(t_0)$ plays the role of the Robertson-Walker scale factor and is identified with the cosmological redshift z of light emitted at t_0^{em} and observed at t_0^{obs} through

$$1+z = \frac{\tau(t_0^{\mathrm{obs}})}{\tau(t_0^{\mathrm{em}})}. \qquad (4)$$

In the limit that $\tau(t_0)$ is treated as constant over the time and distance scales of interest, Equations (2)–(3) reduce to the identity map and every equation derived below reduces to its ordinary flat-space form, so that CSTR's quantum theory is a strict generalization of, rather than a replacement for, ordinary relativistic quantum mechanics.

3. The Cosmological Klein-Gordon Equation

Applying Equation (3) to the free-particle Klein-Gordon equation gives its cosmological form (Yi, 2020),

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t_0^2} + \frac{1}{\tau(t_0)^2}\nabla'^2 - \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0. \qquad (5)$$

The free-particle plane-wave solution is

$$\psi_{\mathbf{p}'}(x', t_0) = \exp\left[\frac{i}{\hbar}\left(\tau(t_0)\,\mathbf{p}'\cdot\mathbf{x}' - E\,t_0\right)\right], \qquad (6)$$

with dispersion relation $E^2 = c^2\mathbf{p}'^2 + m^2c^4$ (Yi, 2020). At fixed cosmological time t_0, the solutions of Equation (6) form a complete, delta-normalized basis,

$$\int d^3x'\,\psi^*_{\mathbf{p}''}\psi_{\mathbf{p}'} = (2\pi\hbar)^3\,\delta^3(\mathbf{p}'-\mathbf{p}''), \qquad (7)$$

so that a general solution can be written as a time-independent superposition of the basis states, which underlies the canonical quantization of Section 8.

3.1. Box Normalization and Mode Counting

For the purposes of the field quantization carried out in Section 8, it is convenient to confine the system to a large but finite comoving cubical volume $V' = L'^3$ in the local coordinate, with periodic boundary conditions on ψ. The continuous local momentum $\mathbf{p}'$ is then replaced by the discrete set $\mathbf{p}'_{\mathbf{n}} = (2\pi\hbar/L')\,\mathbf{n}$ for integer triples $\mathbf{n}$, and the delta-normalization of Equation (7) is replaced by the discrete orthonormality condition $\int_{V'} d^3x'\,\psi^*_{\mathbf{n}'}\psi_{\mathbf{n}} = V'\,\delta_{\mathbf{n}\mathbf{n}'}$. Because the local coordinate volume V' is held fixed while the physical volume $V = \tau(t_0)^3V'$ grows with the expansion, the density of momentum-space modes per unit local-coordinate volume, $V'/(2\pi\hbar)^3$, is independent of t_0, and the entire epoch dependence of the mode spectrum is carried by the dispersion relation $E(p', t_0)$ of Section 3 rather than by the mode density itself. This observation is used without further comment in the construction of the Fock space in Section 8.

Taking the thermodynamic limit $L' \to \infty$ recovers the continuum normalization of Equation (7) in the usual way, $\sum_{\mathbf{n}} \to V'\int d^3p'/(2\pi\hbar)^3$, and this limit is taken without further comment throughout the remainder of this paper whenever a continuum mode sum is written as an integral. It is worth noting explicitly that this construction fixes the box in the *local* coordinate rather than in the physical coordinate: an observer who instead insisted on a fixed physical box $V = L^3$ would find that the corresponding local-coordinate box $V' = V/\tau(t_0)^3$ shrinks as the universe expands, so that the discrete mode spacing $2\pi\hbar/L'$ in that alternative convention would itself become explicitly time-dependent. CSTR's convention of fixing the box in the local coordinate, adopted throughout this paper, avoids this complication and is the reason the mode density above is manifestly t_0-independent.

3.2. The Cosmological Klein-Gordon Current

Multiplying Equation (5) by ψ^*, subtracting the complex conjugate equation multiplied by ψ, and applying Equation (3) gives the continuity equation

$$\frac{\partial \rho}{\partial t_0} + \frac{1}{\tau(t_0)} \nabla' \cdot \mathbf{j}' = 0, \tag{8}$$

with $\rho = (i\hbar/2mc^2)(\psi^* \partial_{t_0} \psi - \psi \partial_{t_0} \psi^*)$ and $\mathbf{j}' = -(i\hbar/2m\tau(t_0))(\psi^* \nabla' \psi - \psi \nabla' \psi^*)$ (Yi, 2020). As in ordinary Klein-Gordon theory, ρ is not positive definite for a general superposition of positive- and negative-energy solutions, which motivates the first-order Dirac equation of Section 6.

3.3. Negative-Energy Solutions and Charge Conjugation

Equation (5) admits both $E = +\sqrt{c^2 \mathbf{p}'^2 + m^2 c^4}$ and $E = -\sqrt{c^2 \mathbf{p}'^2 + m^2 c^4}$ plane-wave solutions at every fixed $\mathbf{p}'$, since the equation is second order in $\partial/\partial t_0$. As in the ordinary flat-space theory, the negative-energy solutions are reinterpreted, for a charged scalar field φ of the type introduced in Section 7, as positive-energy solutions of the opposite charge propagating forward in cosmological time, via the charge-conjugation map $\varphi \to \varphi_c \equiv \varphi^*$, which leaves Equation (20) invariant provided the charge e is simultaneously reversed. Because this reinterpretation is carried out at fixed cosmological time and does not involve the spatial substitution of Equation (3) in any new way beyond its appearance in Equation (5) itself, charge conjugation in the cosmological inertial frame is structurally identical to its ordinary flat-space counterpart at every epoch t_0.

4. The Yukawa Potential and Cosmological Scattering

4.1. The Static Solution

Introducing a static point source into Equation (5) and solving the resulting inhomogeneous, time-independent, spherically symmetric equation gives the cosmologically modified Yukawa potential (Yi, 2021),

$$V(r') = -\frac{g^2}{r'} \exp\left[-\frac{mc}{\hbar}\, \tau(t_0)\, r' \right], \tag{9}$$

where g is the source coupling constant. Expressed in the physical radial coordinate $r_c = \tau(t_0) r'$, the exponential screening length is fixed at the ordinary Compton wavelength $\hbar/(mc)$ of the mediating particle at every cosmological epoch, so that only the local-coordinate expression of the potential carries explicit $\tau(t_0)$ dependence (Yi, 2021).

4.2. The Cosmological Born Approximation

The scattering of a particle described by Equation (6) off the potential of Equation (9) can be treated, to leading order, by the same Born-approximation method used in ordinary non-relativistic and relativistic scattering theory. Writing the scattering amplitude as the Fourier transform of the potential with respect to the momentum transfer $\mathbf{q}' = \mathbf{p}'_{\text{out}} - \mathbf{p}'_{\text{in}}$, and using the substitution of Equation (3) in the matrix element, gives

$$f(\mathbf{q}') \propto \frac{g^2}{\mathbf{q}'^2 + [(mc/\hbar)\tau(t_0)]^2}, \tag{10}$$

which reduces to the ordinary Yukawa scattering amplitude when $\tau(t_0)$ is held fixed. Equation (10) shows that the effective range of momentum transfer over which the scattering cross section is appreciable contracts, in the local-momentum variable $\mathbf{q}'$, as the universe expands, in parallel with the contraction of the local-coordinate screening length noted above.

Squaring the amplitude of Equation (10) gives the cosmological differential cross section in the local-momentum variable,

$$\frac{d\sigma}{d\Omega'} = \frac{4m^2 g^4}{\hbar^4} \frac{1}{\left(\mathbf{q}'^2 + [(mc/\hbar)\tau(t_0)]^2 \right)^2}, \tag{11}$$

in direct analogy with the ordinary Yukawa (screened Coulomb) cross section, but with the mediating particle's inverse range parameter $mc/\hbar$ rescaled by $\tau(t_0)$ in the local-momentum variable $\mathbf{q}'$. As with the potential itself, converting Equation (11) to a cross section differential in the physical momentum transfer $\mathbf{q} = \tau(t_0)\mathbf{q}'$ removes the explicit $\tau(t_0)$-dependence and recovers the ordinary, epoch-independent Yukawa cross section, consistent with the general pattern – already seen in the Schwarzschild horizon and Coulomb potential of the companion review – that physically measured quantities in CSTR are epoch-independent even where their local-coordinate expressions are not (Yi, 2021).

4.3. Scale of the Local-Coordinate Effect

It is useful to have a rough sense of the size of the local-coordinate rescaling entering Equations (9)–(11). Normalizing $\tau(t_0^{\text{obs}}) = 1$ at the present epoch, a source at redshift z corresponds, via Equation (4), to $\tau(t_0^{\text{em}}) = 1/(1+z)$, so that the local-coordinate screening length $\hbar/(mc\,\tau(t_0^{\text{em}}))$ evaluated at the source is larger than its present-day value by a factor of $(1+z)$. For a source at $z \sim 1$, typical of a large fraction of observed extragalactic sources, this is an $\mathcal{O}(1)$ effect in the local coordinate, whereas for sources at $z \ll 1$, such as those within the Local Group, the local-coordinate rescaling is negligible and CSTR's predictions for locally measured scattering and bound-state observables reduce to their ordinary, flat-space values to high accuracy. This is the sense in which CSTR's departures from ordinary quantum theory are, by construction, concentrated at cosmological rather than laboratory or solar-system scales.

5. Non-Relativistic Limit and the Uncertainty Principle

5.1. The Cosmological Schrödinger Equation

Writing $\psi = \exp(-imc^2 t_0/\hbar)\phi(x', t_0)$ and substituting into Equation (5), retaining only the leading time derivative of ϕ, gives the cosmological Schrödinger equation (Yi, 2020),

$$i\hbar \frac{\partial \phi}{\partial t_0} = -\frac{\hbar^2}{2m\,\tau(t_0)^2} \nabla'^2 \phi. \tag{12}$$

The corresponding Ehrenfest relation for the expectation value of the local coordinate is

$$\frac{d}{dt_0}\langle \mathbf{x}' \rangle = \frac{1}{m\,\tau(t_0)^2}\langle \mathbf{p}' \rangle, \tag{13}$$

so that a wave packet's local-coordinate group velocity is suppressed by $\tau(t_0)^{-2}$ at fixed local momentum (Yi, 2020).

5.2. The Cosmological Position-Momentum Uncertainty Relation

Because the momentum conjugate to $\mathbf{x}'$ in Equation (6) is $\tau(t_0)$ times the physical momentum, the ordinary uncertainty relation, referred to the physically expanded coordinate $\mathbf{x} = \tau(t_0)\mathbf{x}'$, is rescaled to (Yi, 2020)

$$\Delta x\,\Delta p \;\geq\; \frac{\hbar}{2}\,\tau(t_0), \tag{14}$$

so that the minimum attainable uncertainty product, in physically expanded coordinates, grows in direct proportion to the expansion function.

5.3. The Time-Energy and Angular-Momentum Relations

Because the energy E appearing in Equation (6) is conjugate to the cosmological time t_0 itself, rather than to a rescaled time coordinate, the time-energy uncertainty relation is left in its ordinary form, $\Delta E\,\Delta t_0 \,\geq\, \hbar/2$ (Yi, 2020). The orbital angular-momentum operator $\mathbf{L}' = \mathbf{x}' \times \mathbf{p}'$, constructed entirely from local-coordinate operators, retains the ordinary commutation algebra $[L'_i, L'_j] = i\hbar\epsilon_{ijk}L'_k$ unchanged by $\tau(t_0)$, since the expansion function cancels between the two factors of the cross product; the cosmological rescaling of Equation (14) therefore affects the linear position-momentum uncertainty relation without altering the algebra, or hence the quantization, of angular momentum.

5.4. Time-Independent Perturbation Theory

For a particle governed by Equation (12) in an additional static potential $V'(\mathbf{x}')$, ordinary time-independent perturbation theory carries over with the free Hamiltonian $H_0 = -\hbar^2\nabla'^2/(2m\tau(t_0)^2)$ evaluated at the (approximately fixed) cosmological time t_0 of interest. To first order, the energy shift of a non-degenerate eigenstate $|n\rangle$ of H_0 is $\Delta E_n = \langle n|V'|n\rangle$, unchanged in form from the ordinary theory, while the second-order shift,

$$\Delta E_n^{(2)} = \sum_{k\neq n} \frac{|\langle k|V'|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}, \tag{15}$$

inherits an implicit $\tau(t_0)$-dependence through the unperturbed energy denominators $E_n^{(0)}, E_k^{(0)}$, which scale as $\tau(t_0)^{-2}$ through H_0. Equation (15) is used without further modification whenever perturbation theory is applied to a bound-state problem, such as a cosmological hydrogen-like system bound by the potential of Equation (21), within a time interval short enough that $\tau(t_0)$ may be treated as constant.

6. The Dirac Equation in the Cosmological Inertial Frame

With $\gamma^0 = \mathrm{diag}(I', -I')$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, the Dirac equation in the cosmological inertial frame is (Yi, 2021)

$$\left[i\hbar\gamma^0 \frac{\partial}{\partial(ct_0)} + \frac{i\hbar}{\tau(t_0)}\gamma^i \frac{\partial}{\partial x'^i} - mc\,I \right]\psi = 0. \tag{16}$$

Left-multiplying by the conjugate operator and using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}I$ reduces Equation (16) to the matrix form of Equation (5), demonstrating its consistency with the cosmological Klein-Gordon equation (Yi, 2021). The probability density $j^0 = \bar{\psi}\gamma^0\psi = \sum_{n=1}^{4}|\psi_n|^2$ built from the four-component spinor $\psi = (\psi_1, \ldots, \psi_4)$ and its adjoint $\bar{\psi} = \psi^\dagger\gamma^0$ is positive definite, resolving the indefinite-density problem of Equation (8) (Yi, 2021).

6.1. Plane-Wave Spinor Solutions and Spin

The positive-energy plane-wave solutions of Equation (16) take the form

$$\psi^{(+)}_{\mathbf{p}',s}(x', t_0) = u_s(\mathbf{p}', t_0)\,\exp\left[\frac{i}{\hbar}\left(\tau(t_0)\,\mathbf{p}'\cdot\mathbf{x}' - E\,t_0\right)\right], \tag{17}$$

where the four-component spinor amplitude $u_s(\mathbf{p}', t_0)$, labeled by spin projection $s = \pm\frac{1}{2}$, satisfies the same algebraic relation as in ordinary Dirac theory between its upper and lower two-component blocks, $\chi_{\mathrm{lower}} = \dfrac{c\,\boldsymbol{\sigma}\cdot\mathbf{p}'}{\tau(t_0)^{-1}(E + mc^2)}\,\chi_{\mathrm{upper}}$, with the local momentum $\mathbf{p}'$ appearing exactly as it does in the exponent of Equation (17). Because the spin operator $\mathbf{S} = (\hbar/2)\mathrm{diag}(\boldsymbol{\sigma}, \boldsymbol{\sigma})$ is built entirely from the local-coordinate Pauli matrices and does not involve $\tau(t_0)$ explicitly, the helicity operator $\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}'$ commutes with the free Dirac Hamiltonian in Equation (16) at every fixed cosmological time, exactly as in the ordinary theory, so that helicity remains a good quantum number for a free cosmological Dirac particle (Yi, 2021).

6.2. Zitterbewegung in the Cosmological Frame

As in ordinary Dirac theory, the local-coordinate velocity operator $\dot{x}'^i = (i/\hbar)[H_{\mathrm{Dirac}}, x'^i]$, computed from Equation (16), is proportional to the matrix $\gamma^0\gamma^i/\tau(t_0)$ rather than to the classical velocity $c^2p'^i/E$, so that a wave packet built from a superposition of positive- and negative-energy solutions exhibits the same rapidly oscillating *Zitterbewegung* motion familiar from the flat-space theory, here superposed on the overall local-coordinate contraction induced by $\tau(t_0)^{-1}$. Because the Zitterbewegung frequency is set by $2mc^2/\hbar$, the same combination that fixes the ordinary Compton frequency, and does not otherwise involve $\tau(t_0)$, this effect is predicted by CSTR to retain its ordinary flat-space frequency at every cosmological epoch, while its local-coordinate amplitude is modulated by the same $\tau(t_0)^{-1}$ factor that suppresses the ordinary velocity operator (Yi, 2021).

6.3. Non-Relativistic Limit: The Cosmological Pauli Equation

Splitting $\psi = (\chi, \eta)^T$ into upper and lower two-component blocks and eliminating η in the non-relativistic limit gives the cosmological Pauli equation,

$$i\hbar \frac{\partial \chi}{\partial t_0} = \frac{1}{2m\,\tau(t_0)^2}\left(-i\hbar\nabla' - \frac{e}{c}\mathbf{A}'\right)^2 \chi - \frac{e\hbar}{2mc\,\tau(t_0)}\,\boldsymbol{\sigma}\cdot\mathbf{B}'\,\chi, \tag{18}$$

in which the spin-magnetic coupling to $\mathbf{B}' = \nabla' \times \mathbf{A}'$ is suppressed by one additional power of $\tau(t_0)^{-1}$ relative to the orbital kinetic term (Yi, 2021), so that spin-dependent effects are predicted to be more sensitive to the cosmological epoch than orbital effects at the same order.

6.4. The Negative-Energy Spectrum

As in ordinary Dirac theory, Equation (16) admits both positive- and negative-energy plane-wave solutions, with $E = \pm\sqrt{c^2\mathbf{p}'^2 + m^2c^4}$ at every cosmological time t_0, since the dispersion relation inherited from Equation (5) is unaffected in form by $\tau(t_0)$. The Dirac-sea and hole-theoretic interpretation of the negative-energy states, and the corresponding prediction of an antiparticle of charge $-e$, therefore carry over unchanged from the flat-space theory into the cosmological inertial frame at any fixed epoch t_0; whether pair-creation rates computed within Equation (16) acquire an explicit $\tau(t_0)$-dependence once genuinely time-dependent processes (rather than fixed-t_0 eigenstates) are considered is addressed further in Section 9.

6.5. The Cosmological Klein Paradox

The ordinary Klein paradox – the counterintuitive result that a relativistic electron incident on a sufficiently strong, sharp potential step exhibits a reflection coefficient exceeding unity, with the excess carried by a transmitted current of negative-energy particles – carries over into the cosmological inertial frame with the potential step's height rescaled by $\tau(t_0)$ in the local coordinate, in the same way already seen for the Yukawa and Coulomb potentials above. Writing a static step potential as $V'(x') = V_0'\theta(x')$ in the local coordinate, the critical condition for the paradox to occur, $V_0 > E + mc^2$ in the ordinary theory, becomes $V_0'\,\tau(t_0) > E + mc^2$ when V_0' is measured in the local-coordinate potential and E retains its ordinary, epoch-independent meaning as the conserved energy conjugate to t_0. This shows that a fixed local-coordinate potential step becomes increasingly likely to exhibit the Klein paradox as the universe expands, since the critical threshold on V_0' required for over-the-barrier reflection to occur decreases as $1/\tau(t_0)$; equivalently, a step of fixed physical height $V_0 = V_0'\tau(t_0)$ exhibits the paradox under exactly the ordinary, epoch-independent condition $V_0 > E + mc^2$, consistent with the pattern established elsewhere in this paper that physical quantities recover their ordinary flat-space behavior once properly referred to the physically expanded coordinate.

7. Klein-Gordon-Maxwell Theory

A complex scalar field φ of charge e, minimally coupled to the electromagnetic potential A_μ through $D_\mu = \partial_\mu + ieA_\mu/\hbar$, is described by the cosmological Klein-Gordon-Maxwell Lagrangian (Yi, 2021)

$$\mathcal{L} = (D_\mu\varphi)^*(D^\mu\varphi) - \left(\frac{mc}{\hbar}\right)^2 \varphi^*\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{19}$$

whose Euler-Lagrange equations, with the spatial part of D_μ rescaled according to Equation (3), give the field equation

$$\left[D_0D^0 - \frac{1}{\tau(t_0)^2}D_iD^i + \left(\frac{mc}{\hbar}\right)^2\right]\varphi = 0 \tag{20}$$

together with the inhomogeneous Maxwell equation sourced by the scalar field's electromagnetic four-current (Yi, 2021).

7.1. Gauge Invariance

Equation (19) is invariant under $\varphi \to e^{-ie\Lambda(x)/\hbar}\varphi$, $A_\mu \to A_\mu + \partial_\mu\Lambda$, since Equation (3) acts identically on every spatial derivative in D_μ and therefore commutes with the gauge transformation; the resulting conserved current satisfies $\partial_\mu j^\mu = 0$, with the spatial divergence carrying the same $1/\tau(t_0)$ factor as in Equation (8) (Yi, 2021). As in ordinary quantum electrodynamics, this current conservation implies a corresponding Ward-Takahashi identity relating the divergence of the photon self-energy or vertex functions computed from the perturbative expansion of Section 8 to lower-order Green's functions of the theory; because the substitution of Equation (3) enters identically in every diagram contributing to a given Green's function at fixed cosmological time, the same cancellations that enforce the Ward identity in the ordinary theory are expected to persist at fixed t_0 in the cosmological theory, though, as with the propagator itself, a genuinely cosmological (multi-time) version of the identity has not been derived in the papers reviewed here.

7.2. The Cosmological Coulomb Limit

For $m = 0$ and a static point charge q, the time component of the field equation reduces to $(1/\tau(t_0)^2)\nabla'^2 A_0(r') = -(q/\epsilon_0)\delta^3(\mathbf{x}')$, with solution

$$A_0(r') = \frac{q\,\tau(t_0)}{4\pi\epsilon_0\,r'}, \tag{21}$$

which is consistent, in its qualitative $\tau(t_0)$-dependence, with the pattern already noted for the Yukawa potential of Section 4: the local-coordinate expression carries an explicit factor of the expansion function, while the corresponding physically measured potential at fixed physical separation $r_c = \tau(t_0)r'$ is expected, by analogy with the Schwarzschild and Yukawa cases, to be far less sensitive to the cosmological epoch. A careful, explicit verification that Equation (21) reduces *exactly* to the ordinary, epoch-independent Coulomb potential once re-expressed in the physical coordinate r_c – rather than only approximately or up to an additional power of $\tau(t_0)$ – has not been carried out in the papers reviewed here, and is noted as a specific, well-posed consistency check in Section 9 (Yi, 2021).

7.3. The Cosmological Hydrogen-Like Bound State

Combining the Coulomb potential of Equation (21) with the non-relativistic reduction of Section 5, a particle of charge $-e$ bound to a fixed charge $+Ze$ obeys the cosmological Schrödinger equation of Equation (12) with $V'(r') = -Ze^2\tau(t_0)/(4\pi\epsilon_0 r')$. Over an interval of cosmological time short enough that $\tau(t_0)$ may be treated as constant, this is identical in form to the ordinary hydrogen-atom problem with an effectively rescaled reduced mass $m\,\tau(t_0)^2$, so that the bound-state energy levels take the familiar form

$$E_n = -\frac{m\,\tau(t_0)^2}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2}, \tag{22}$$

for principal quantum number $n = 1, 2, 3, \ldots$, in direct analogy with the ordinary Bohr energy levels. Equation (22) shows that the cosmological hydrogen-like binding energy grows as $\tau(t_0)^2$ at fixed local-coordinate quantum numbers, which – through the redshift identification of Section 2 – amounts to a prediction that atomic transition energies computed within this non-relativistic approximation would appear to change with cosmic time when referred to the local coordinate. Whether the corresponding physically measured transition energy, referred to the physical coordinate r_c, is exactly independent of t_0, as the pattern established for the Schwarzschild and Yukawa cases would suggest, or retains some residual epoch dependence inherited from the power-counting noted in Section 7, is a concrete quantitative question that a full treatment of this bound-state problem would need to resolve.

The angular part of the cosmological hydrogen-like problem separates from the radial part in exactly the same way as in the ordinary theory, since the potential $V'(r')$ depends only on the radial local coordinate and the angular-momentum algebra of Section 5 is unaffected by $\tau(t_0)$; the eigenfunctions are therefore products of the ordinary spherical harmonics $Y_{\ell m}(\theta,\phi)$ with a radial function governed by an effective, $\tau(t_0)$-rescaled Bohr radius $a_0(t_0) = \hbar^2/(m\,\tau(t_0)^2\,Ze^2/(4\pi\epsilon_0))$. Because the angular selection rules for electric-dipole transitions, $\Delta\ell = \pm1$, $\Delta m = 0, \pm1$, follow purely from the angular integral of the dipole operator against the spherical harmonics and do not involve the radial rescaling at all, CSTR predicts that the ordinary dipole selection rules of atomic spectroscopy are preserved unchanged at every cosmological epoch, even though the transition energies themselves are subject to the open question noted above.

8. Canonical Quantization of the Klein-Gordon Field

Treating the free, real Klein-Gordon field $\phi(x', t_0)$ of Equation (5) as an operator with conjugate momentum density $\pi = \partial_{t_0}\phi/c^2$, CSTR imposes the equal-time canonical commutation relation (Yi, 2021)

$$[\phi(\mathbf{x}', t_0), \pi(\mathbf{x}'', t_0)] = i\hbar\,\delta^3(\mathbf{x}' - \mathbf{x}''). \tag{23}$$

The Lagrangian and Hamiltonian corresponding to Equation (5) are

$$\mathcal{L} = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t_0}\right)^2 - \frac{1}{2\tau(t_0)^2}(\nabla'\phi)^2 - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2, \tag{24}$$

$$H = \int d^3x'\left[\frac{c^2}{2}\pi^2 + \frac{1}{2\tau(t_0)^2}(\nabla'\phi)^2 + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2\right]. \tag{25}$$

8.1. Fock Space and the Particle-Number Operator

Expanding ϕ in the plane-wave basis of Equation (6), with time-dependent ladder operators $a_{\mathbf{p}'}(t_0)$, $a^\dagger_{\mathbf{p}'}(t_0)$ satisfying $[a_{\mathbf{p}'}, a^\dagger_{\mathbf{p}''}] = \delta^3(\mathbf{p}' - \mathbf{p}'')$, the Hamiltonian of Equation (25) becomes, after normal ordering to remove the zero-point contribution,

$$:H: = \int d^3p'\,E(p', t_0)\,a^\dagger_{\mathbf{p}'}a_{\mathbf{p}'}, \tag{26}$$

with $E(p', t_0)$ fixed by the dispersion relation of Section 3. The particle-number operator $N = \int d^3p'\,a^\dagger_{\mathbf{p}'}a_{\mathbf{p}'}$ commutes with $: H :$ at fixed t_0, so particle number is conserved instantaneously, with any change in the energy of a fixed-particle-number state attributable entirely to the explicit $\tau(t_0)$-dependence of $E(p', t_0)$ rather than to particle creation (Yi, 2021).

The full Fock space at fixed cosmological time t_0 is built in the usual way from the vacuum $|0\rangle$ by repeated action of creation operators, $|\mathbf{p}'_1, \ldots, \mathbf{p}'_n; t_0\rangle = a^\dagger_{\mathbf{p}'_1}(t_0)\cdots a^\dagger_{\mathbf{p}'_n}(t_0)|0\rangle$, with total energy $\sum_{i=1}^n E(p'_i, t_0)$ and total momentum $\sum_{i=1}^n \mathbf{p}'_i$ as eigenvalues of $:H:$ and the local-coordinate momentum operator, respectively. Because the creation operators at fixed t_0 satisfy the ordinary bosonic commutation algebra, these states are symmetric under exchange of any two particles, exactly as in flat-space Klein-Gordon theory, and the construction places no cosmological-epoch-dependent constraint on the occupation number of any given mode. The explicit t_0-dependence of the ladder operators means, however, that the Fock space itself should be understood as a family of Hilbert spaces parametrized by t_0, connected to one another only through the (as yet unspecified, see Section 9) relation between $a_{\mathbf{p}'}(t'_0)$ and $a_{\mathbf{p}'}(t''_0)$ at different epochs, rather than as a single, fixed Hilbert space carrying a manifestly unitary time evolution in t_0.

8.2. Wick's Theorem and Higher-Point Functions

Because the equal-time field $\phi(x', t_0)$ is a free field at any fixed t_0, expectation values of products of field operators between vacuum states obey the same combinatorial structure as in ordinary free-field theory: Wick's theorem applies unchanged, so that any equal-time $2n$-point function decomposes into a sum over all pairwise contractions (equal-time two-point functions) of the $2n$ field operators, and every equal-time odd-point function of the free field vanishes identically. This follows because the mode expansion underlying Equation (26) is, at fixed t_0, formally identical to the ordinary flat-space mode expansion, with only the single-particle energy $E(p', t_0)$ carrying epoch dependence; Wick's theorem is a purely combinatorial consequence of the bilinear mode expansion and is insensitive to the specific form of $E(p', t_0)$ (Yi, 2021).

8.3. The Equal-Time Propagator

The two-point function $G_F(x' - x''; t_0) = \langle 0|T\{\phi(x', t_0)\phi(x'', t_0)\}|0\rangle$, evaluated between field insertions at a common cosmological time t_0, takes the same functional form as the ordinary flat-space Feynman propagator, with $E(p', t_0)$ evaluated at that common epoch (Yi, 2021). Explicitly, carrying out the momentum integral in the usual way gives

$$G_F(x'-x''; t_0) = \int \frac{d^3p'}{(2\pi\hbar)^3} \frac{\hbar c^2}{2E(p', t_0)} e^{i\mathbf{p}'\cdot(\mathbf{x}'-\mathbf{x}'')\tau(t_0)/\hbar}, \tag{27}$$

for field insertions taken at equal cosmological time. Extending this construction to field insertions at two different cosmological times, which would be required for a genuine scattering-amplitude calculation in an expanding background, has not been carried out in the papers reviewed here and is identified in Section 9 as an open problem.

8.4. Perturbative Expansion of the Interacting Theory

For the Klein-Gordon-Maxwell interaction of Section 7, an interaction-picture treatment analogous to the ordinary Dyson series can be set up at fixed cosmological time by writing the interacting Hamiltonian as $H = H_0 + H_{\text{int}}(t_0)$, with H_0 the free Hamiltonian of Equation (25) for the scalar field and its electromagnetic counterpart, and H_{int} the minimal-coupling interaction term obtained from Equation (19). The time-evolution operator in the interaction picture is then given, exactly as in ordinary field theory, by the time-ordered exponential

$$U(t_0, t_0^i) = T \exp\left[-\frac{i}{\hbar}\int_{t_0^i}^{t_0} dt_0'\, H_{\text{int}}(t_0')\right], \tag{28}$$

whose perturbative expansion generates the same diagrammatic structure as the ordinary Dyson series, with each internal scalar-field line evaluated using the equal-time propagator of Equation (27) whenever the two ends of the line fall at the same cosmological time, and with the extension to internal lines connecting different cosmological times subject to the same open question already noted for Equation (27). Because $H_{\text{int}}(t_0')$ itself carries explicit $\tau(t_0')$-dependence through the covariant derivative of Equation (19), the interaction-picture perturbation series for a process spanning a cosmologically significant interval of t_0 would need to integrate over the time-dependence of $\tau(t_0')$ inside Equation (28) explicitly, rather than treating it as a fixed external parameter, which is a further respect in which the formalism reviewed here remains to be completed for genuinely cosmological (as opposed to locally approximately flat) scattering processes.

9. Discussion

9.1. Comparison with Quantum Field Theory in Curved Spacetime

The mainstream treatment of quantum fields on an expanding Robertson-Walker background – developed originally by Parker and collaborators in the late 1960s, and standard in the textbook literature on quantum field theory in curved spacetime – differs from the CSTR construction reviewed here in its point of departure: rather than rescaling the spatial coordinate of a flat-space wave equation by $\tau(t_0)$ and substituting the result into the differential operators as in Equation (3), the mainstream approach quantizes the Klein-Gordon field directly on the curved Robertson-Walker background using the covariant Klein-Gordon equation $(\Box - m^2c^2/\hbar^2)\phi = 0$, and finds that the notion of a "particle" becomes observer- and time-dependent: mode functions that define a vacuum state at one cosmological time do not, in general, remain purely positive-frequency at a later time, leading to cosmological particle creation even for a free field. This effect has no direct counterpart in the equal-time quantization of Section 8, in which the vacuum $|0\rangle$, defined by $a_{\mathbf{p}'}(t_0)|0\rangle = 0$, is implicitly constructed independently at each cosmological time. Reconciling the CSTR construction with this mainstream phenomenon of cosmological particle creation – in particular, determining whether the time-dependent ladder operators $a_{\mathbf{p}'}(t_0)$ of Section 8 are related to one another at different epochs by a nontrivial Bogoliubov transformation, as they would be in the mainstream treatment – is a natural and, at present, unresolved point of contact between CSTR and the standard theory of quantum fields in an expanding universe.

In the standard treatment, the two sets of mode functions appropriate to two different cosmological times are related by a Bogoliubov transformation $b_{\mathbf{k}} = \alpha_{\mathbf{k}} a_{\mathbf{k}} + \beta_{\mathbf{k}}^* a_{-\mathbf{k}}^\dagger$, and a nonzero Bogoliubov coefficient $\beta_{\mathbf{k}}$ signals that the vacuum defined at the earlier time contains particles when examined using the mode functions appropriate to the later time, with the number of created particles per mode given by $|\beta_{\mathbf{k}}|^2$. The construction of Section 8, by contrast, defines annihilation and creation operators $a_{\mathbf{p}'}(t_0)$ separately at each cosmological time using the instantaneous plane-wave basis of Equation (6), without specifying a transformation analogous to the Bogoliubov relation connecting $a_{\mathbf{p}'}(t_0')$ to $a_{\mathbf{p}'}(t_0'')$ for $t_0' \neq t_0''$. Two logical possibilities follow: either such a transformation exists and is trivial (that is, $\alpha_{\mathbf{p}'} = 1$, $\beta_{\mathbf{p}'} = 0$ at every pair of epochs, so that CSTR predicts no cosmological particle creation for the free field, in contrast with the mainstream result), or the transformation is nontrivial but has simply not yet been worked out within the CSTR literature reviewed here. Distinguishing between these two possibilities – which amounts to computing the overlap between the mode functions of Equation (6) at two different cosmological times directly – would settle whether CSTR agrees with or departs from one of the most well-established predictions of quantum field theory in an expanding universe, and is accordingly listed as the first open question of Section 9.4 below.

9.2. Relation to the Adiabatic Vacuum Construction

A related concept in the mainstream literature is the *adiabatic vacuum*: when the expansion rate $\dot{\tau}(t_0)/\tau(t_0)$ is small compared with the particle's characteristic frequency $mc^2/\hbar$, the mode functions of the covariant Klein-Gordon equation can be expanded in powers of this ratio, and a vacuum state defined order-by-order in this adiabatic expansion minimizes the particle production of Section 9 at each order. The equal-time construction of Section 8, which defines $a_{\mathbf{p}'}(t_0)$ using the instantaneous plane-wave basis of Equation (6) without further reference to the rate of change of $\tau(t_0)$, is structurally similar to the leading (zeroth) order of the adiabatic expansion, in which the mode functions are simply those of the instantaneously flat problem. This suggests that the CSTR quantization of Section 8 may be best understood as the leading adiabatic approximation to a more complete treatment, valid when $\hbar\dot{\tau}(t_0)/(\tau(t_0)mc^2) \ll 1$, rather than as an exact construction valid at every rate of cosmological expansion; making this correspondence precise, and identifying the higher adiabatic corrections in the language of Section 8, is a further concrete task suggested by the comparison of this section.

9.3. Implications for Cosmological Spectroscopy

The hydrogen-like bound-state analysis of Section 7, together with the adiabatic regime just identified, suggests a concrete observational context in which the open questions of this paper could be tested: the spectroscopy of resonance lines from sufficiently distant astrophysical sources. Since $\dot{\tau}(t_0)/\tau(t_0)$ is the ordinary Hubble parameter $H(t_0)$, and atomic binding energies are set by $mc^2/\hbar$ times the fine-structure constant squared, the adiabaticity condition $\hbar H(t_0)/(mc^2) \ll 1$ is satisfied to extraordinary precision for ordinary atomic transitions at any epoch accessible to observation, so that CSTR's predictions for atomic spectra should, if the adiabatic identification of the preceding subsection is correct, be indistinguishable from the ordinary, epoch-independent predictions of flat-space quantum mechanics to a precision far beyond current spectroscopic sensitivity. This is a useful consistency check in its own right: any observed deviation of high-redshift atomic or molecular transition energies from their laboratory values – of the kind searched for in tests of possible time-variation of the fine-structure constant – would, within the framework reviewed here, need to be attributed either to the open question concerning the exact power-counting of Equation (22) noted in Section 7, or to physics beyond the adiabatic regime assumed throughout this paper, rather than to the leading-order cosmological corrections computed explicitly above.

9.4. Open Questions

Three questions raised above stand out as concrete, well-posed calculations for further developing the quantum-theoretic content of CSTR: (i) extending the equal-time propagator of Section 8 to field insertions at two different cosmological times, which is a prerequisite for computing scattering amplitudes such as the Born-approximation cross section of Section 4 from the quantized theory rather than from the first-quantized wave equation alone; (ii) determining whether the cosmological ladder operators of Section 8 undergo a Bogoliubov transformation between different cosmological times, in analogy with the particle creation found in the mainstream treatment of quantum fields in curved spacetime; and (iii) extending the canonical quantization carried out here for the Klein-Gordon field to the Dirac field of Section 6, which has not been performed in the papers reviewed here. An independent critical review of CSTR has additionally raised broader questions about the framework's consistency with the diffeomorphism invariance of general relativity and about its limited independent citation history (A Critical Review, 2025); addressing these, alongside the three technical questions above, would substantially strengthen the quantum-theoretic foundations of the program. A fourth, more elementary question – verifying the precise power of $\tau(t_0)$ with which the Coulomb potential and hydrogen-like binding energy of Section 7 reduce to their ordinary, epoch-independent values once expressed in the physical coordinate r_c – is comparatively straightforward and would be a natural first step, since it involves only the static field equations of Section 7 rather than the full quantized or dynamical theory.

Quantity	Epoch dependence (physical)
Yukawa range	Independent of t_0
Yukawa cross section	Independent of t_0
Coulomb potential	Independent of t_0 (tentative; Sec. 7)
Hydrogen-like binding energy	Open question (Sec. 7)
Position-momentum uncertainty	Grows with $\tau(t_0)$
Time-energy uncertainty	Independent of t_0
Spin-magnetic coupling (Pauli eq.)	Suppressed by extra $\tau(t_0)^{-1}$

Table 1: Epoch dependence of the principal physically measured quantities derived in this review.

Table 1 summarizes, for each major quantum-theoretic result reviewed above, whether the corresponding physically measured quantity is expected to be epoch-independent (as established explicitly for the Yukawa range and, tentatively, for the Coulomb potential) or carries an intrinsic $\tau(t_0)$-dependence even in physical variables (as for the position-momentum uncertainty relation).

Table 2 additionally indexes each major result reviewed in this paper against the original source in which it first appeared, providing a compact guide back to the primary literature.

10. Conclusion

This paper has developed the quantum-theoretic content of the Cosmological Special Theory of Relativity in greater depth than a single consolidated survey allows: the Klein-Gordon equation, its conserved current, and its Yukawa-potential and Born-approximation scattering behavior (Sections 3–4); the non-relativistic Schrödinger limit and the cosmological position-momentum, time-energy, and angular-

Result	Original source
Cosmological inertial frame, Wave Function Type A	Yi (2020)
Klein-Gordon equation, current, normalization	Yi (2020)
Schrödinger equation, uncertainty relations	Yi (2020)
Yukawa potential	Yi (2021)
Dirac equation, Pauli limit, spinor solutions	Yi (2021)
Klein-Gordon-Maxwell theory, Coulomb limit	Yi (2021)
Canonical quantization, propagator	Yi (2021)
Full consolidation	Yi (2025)

Table 2: Index of the principal results reviewed in this paper and their original sources.

momentum relations (Section 5); the Dirac equation, its non-relativistic Pauli limit, and its negative-energy spectrum (Section 6); the Klein-Gordon-Maxwell theory of an interacting charged scalar field, its gauge invariance, and its Coulomb limit (Section 7); and the canonical quantization of the free scalar field, its Fock space, and its equal-time propagator (Section 8). Comparison with the mainstream treatment of quantum fields in curved and expanding spacetimes (Section 9) identifies cosmological particle creation and the two-time propagator as the most direct points of contact – and of open tension – between CSTR and the standard theory, and these, together with the quantization of the Dirac field itself, are identified as the central open problems for the continued development of cosmological quantum physics within this framework. Throughout, a consistent pattern has emerged: physically measured quantities – the Yukawa interaction range and cross section, the time-energy uncertainty product, and, tentatively, the Coulomb potential – tend to recover their ordinary, epoch-independent values once properly referred to the physically expanded coordinate, while the position-momentum uncertainty product and the spin-magnetic coupling of the Pauli equation retain an explicit dependence on the cosmological expansion function even in physical variables. Table 1 collects this pattern across the results reviewed here, and Table 2 indexes each result against its original source. Establishing which of these two categories a given physical quantity belongs to – and, in the cases flagged above as open, resolving the ambiguity outright – is the most direct path toward a fully self-consistent cosmological quantum theory within the framework proposed here.

More broadly, the results collected here show that the cosmological substitution rule of Equation (3) – simple as it is to state – has nontrivial and, in places, still incompletely worked out consequences once it is carried systematically through the full apparatus of relativistic quantum mechanics and field theory: scattering cross sections, bound-state spectra and selection rules, spin-dependent couplings, and the vacuum structure of the quantized field each acquire a specific, calculable dependence on the cosmological expansion function, and in several cases (the Coulomb limit, the hydrogen-like spectrum, the two-time propagator, and the relation to cosmological particle creation) the calculation required to settle that dependence definitively remains to be carried out. Completing these calculations is the most direct route toward establishing cosmological quantum physics, as formulated here, as a fully quantitative and internally consistent extension of relativistic quantum theory to an expanding universe.

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A. Glossary of Key Symbols

Table 3 collects the principal symbols used throughout this paper.

Symbol	Meaning
t_0	Cosmological time
$\tau(t_0)$	Cosmological expansion function (scale factor)
x^i, r_c	Physically expanded coordinate
x'^i, r'	Local (unexpanded) coordinate
ψ	Klein-Gordon or Dirac wave function
m	Particle rest mass
E, $\mathbf{p}'$	Energy and momentum
g	Yukawa coupling constant
ϕ	Quantized Klein-Gordon field operator
$a_{\mathbf{p}'}, a^\dagger_{\mathbf{p}'}$	Annihilation/creation operators
φ	Complex (charged) Klein-Gordon field
e	Scalar field electric charge
A_μ	Electromagnetic four-potential
$\boldsymbol{\sigma}$, γ^μ	Pauli and Dirac matrices
$u_s(\mathbf{p}', t_0)$	Dirac plane-wave spinor amplitude
G_F	Equal-time Feynman propagator
CSTR	Cosmological Special Theory of Relativity

Table 3: Glossary of symbols used in this paper.

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