Spectral analysis of arithmetic functions

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Abstract

This paper explores the normalization of Fourier series kernels associated with elementary arithmetic functions. By analyzing key functions such as divisor related functions, the prime gap function and the inverse prime counting function. With these results, a framework for their spectral decomposition in terms of complex exponential functions was developed. Through contour integration techniques and normalization of the Dirac delta function, new methods can be derived from the analytic structure of these functions. These findings contribute to the broader understanding of arithmetic functions and their link to Fourier series.

Keywords: Number theory, arithmetic functions, Lambert series, prime gaps.

1 Introduction

Arithmetic functions such as the divisor function, the prime counting function, and the prime gap sequence occupy a central role in analytic number theory. Traditionally, these objects are studied through Dirichlet series, complex analytic methods, and asymptotic techniques rooted in the theory of the Riemann zeta function. At the same time, Fourier analysis provides a powerful language for decomposing functions into elementary oscillatory components. This paper aims to bridge these perspectives by developing a unified spectral framework for the study of arithmetic functions.

The starting point is an integral representation of the integer indicator function, which enables the expression of arithmetic constraints in terms of normalized Fourier kernels. Through contour integration and the residue theorem, the divisor indicator function is represented as infinite spectral sums of complex exponentials. This perspective naturally leads to analytic constructions of the classical divisor function and its generalizations.

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Building on this foundation, the same spectral methodology is applied to prime-related quantities. Generating functions for the prime gap sequence and the prime counting function are introduced and analyzed through inverse transforms and contour integrals. By employing Dirichlet series techniques and asymptotic analysis consistent with the Prime Number Theorem, explicit approximations are derived for the inverse prime counting function and for prime gaps. In particular, the appearance of the Lambert W function provides compact asymptotic expressions that reflect the expected logarithmic growth behavior of primes.

The overall objective of this work is to present arithmetic functions as analytically invertible spectral objects. By unifying Fourier-analytic normalization with classical tools from analytic number theory, the paper offers new integral representations and asymptotic expansions that illuminate structural connections between divisor functions, prime distributions, and their generating mechanisms.

2 Divisor functions

This section develops a spectral and analytic construction of divisor functions. By expressing arithmetic constraints through normalized Fourier kernels and contour integrals, the divisor indicator function is written as an infinite exponential sum. This formulation allows divisor functions to be derived via complex-analytic methods, including residue calculus and inverse transforms. Define the integer indicator function:

$$\mathbb{1}_{\mathbb{Z}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in \mathbb{R}$. It admits the Fourier-type representation:

$$\mathbb{1}_{\mathbb{Z}}(x) = \lim_{T\to\infty} \frac{1}{4\pi T} \int_{-2\pi T}^{2\pi T} \frac{e^{itx}}{e^{it}-1}\, dt, \tag{2.0.1}$$

which, under the conformal change of variables $it = 2Tlog(z)$, becomes the contour integral:

$$\mathbb{1}_{\mathbb{Z}}(x) = \lim_{T\to\infty} \frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{2Tx}}{z^{2T}-1} \frac{dz}{z}. \tag{2.0.2}$$

By the residue theorem, this yields the classical exponential sum:

$$\mathbb{1}_{\mathbb{Z}}(x) = \lim_{T\to\infty} \frac{1}{2T} \sum_{n=-T}^{T} e^{2\pi inx}. \tag{2.0.3}$$

Consequently, the divisor indicator function satisfies:

$$\mathbb{1}_{k|m} = \mathbb{1}_{\mathbb{Z}}(m/k) = \lim_{T\to\infty} \frac{1}{2T} \sum_{n=-T}^{T} e^{2\pi inm/k}\, m, k \in \mathbb{Z}\ , k \neq 0, \tag{2.0.4}$$

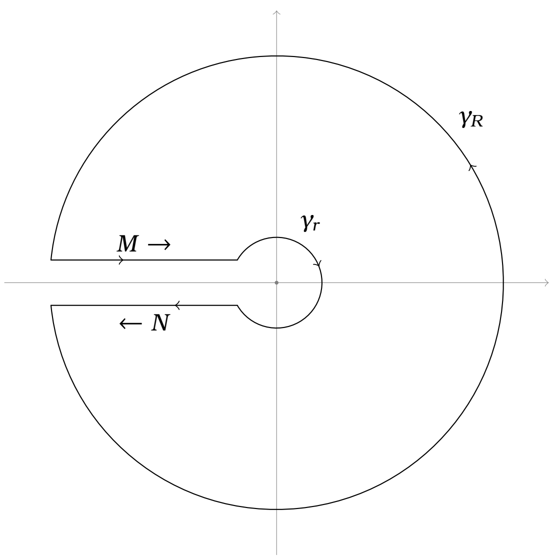
where:

$$\mathbb{1}_{m|n} = \begin{cases} 1, & \text{if } m|n, \\ 0, & \text{otherwise,} \end{cases}$$

is the divisor indicator function. For subsequent developments we use the contour identity:

$$\frac{1}{2\pi i}\oint_C z^n (log(z))^k\, dz = \frac{d^k}{ds^k}\frac{sin(\pi(s+n))}{\pi(s+n)}|_{s=0}, \tag{2.0.5}$$

where curve C is a keyhole contour centered at the origin with argument $-\pi \leq arg(z) \leq \pi$, the sketch is the following:



Where it is centered to zero with argument $-\pi \leq arg(z) \leq \pi$, with the small radius $\epsilon > 0$ really small and the big radius equal to one $R = 1$. The generating function of the divisor function $d(n)$ admits the Mellin-type representation:

$$\sum_{n=1}^{\infty} d(n) z^{-n} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(s)}{(\log(z))^s}\zeta^2(s)\, ds. \tag{2.0.6}$$

Evaluating the integral by residues and using $\zeta(-k) = \frac{B_{k+1}}{k+1}$ [3], we obtain:

$$\sum_{n=1}^{\infty} d(n) z^{-n} = \frac{-\gamma - log(log(z))}{log(z)} + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{B_{k+1}^2}{(k+1)^2}(log(z))^k, \tag{2.0.7}$$

where γ denotes the Euler–Mascheroni constant. Applying the inverse z-transform and the contour identity above 2.0.5 yields:

$$d(n) = log(n) + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{B_{k+1}^2}{(k+1)^2}\frac{d^k}{ds^k}\frac{sin(\pi(s+n))}{\pi(s+n)}|_{s=0}, \tag{2.0.8}$$

where B_k denotes the Bernoulli numbers. More generally,

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_{k-1}(n)}{n^s}, \tag{2.0.9}$$

where:

$$d_{k-1}(n) = \sum_{a_1 a_2 \ldots a_k = n} 1 \tag{2.0.10}$$

and $d_1(n) = d(n)$. Inversion of the zeta function gives:

$$\mathbb{1}_{n\geq 1} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} n^{\sigma+it} \zeta(\sigma+it)\, dt. \tag{2.0.11}$$

Using the expansions:

$$\zeta(s) = \sum_{k=0}^{\infty} \frac{(log(\zeta(s)))^k}{k!}, \tag{2.0.12}$$

and:

$$log(\zeta(s)) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{log(n)} \frac{1}{n^s}, \tag{2.0.13}$$

one obtains:

$$\mathbb{1}_{n\geq 1} = 1, \ \ n = 1, \tag{2.0.14}$$

$$\mathbb{1}_{n\geq 1} = \frac{\Lambda(n)}{log(n)} + \frac{1}{2!} \sum_{ab=n} \frac{\Lambda(a)\Lambda(b)}{log(a)log(b)} + \frac{1}{3!} \sum_{abc=n} \frac{\Lambda(a)\Lambda(b)\Lambda(c)}{log(a)log(b)log(c)} + \ldots, \ \ n \geq 2, \tag{2.0.15}$$

where $\Lambda(n)$ is the Von Mangoldt function defined by [1]:

$$\Lambda(n) = \lim_{T\to\infty} -\frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ -T\leq\Im(\rho)\leq T \\ 0<\Re(\rho)<1}} n^{\rho} \ , n \in \mathbb{R}, n > 1. \tag{2.0.16}$$

This expansion generalizes to:

$$d_k(n) = 1, \ \ n = 1, \tag{2.0.17}$$

$$d_{k-1}(n) = \frac{\Lambda(n)}{log(\sqrt[k]{n})} + \frac{1}{2!} \sum_{ab=n} \frac{\Lambda(a)\Lambda(b)}{log(\sqrt[k]{a})log(\sqrt[k]{b})} + \frac{1}{3!} \sum_{abc=n} \frac{\Lambda(a)\Lambda(b)\Lambda(c)}{log(\sqrt[k]{a})log(\sqrt[k]{b})log(\sqrt[k]{c})} + \ldots, \ \ n \geq 2. \tag{2.0.18}$$

A generalized version of divisor functions is defined by the sum:

$$\sigma_a(n) = \sum_{d|n} d^a. \tag{2.0.19}$$

The Dirichlet series associated with $\sigma_a(n)$ is given by:

$$D(\sigma_a, s) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \zeta(s)\zeta(s-a), \ \ \Re(s) > 1 + max\{\Re(a), 0\}. \tag{2.0.20}$$

Consider now the contour integral representation:

$$\sigma_a(n) = n^{a/2} \oint_C z^{n-1} Q(z,a) \frac{dz}{2\pi i}, \tag{2.0.21}$$

where the function $Q(z,a)$ is defined by the generating series:

$$Q(z,a) = \sum_{n=1}^{\infty} \sum_{d|n} d^{a/2} \frac{d^{a/2}}{n^{a/2}} z^{-n}. \tag{2.0.22}$$

This function admits the Mellin integral representation:

$$Q(z,a) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(s)}{(\log(z))^s} \zeta(s - \frac{a}{2}) \zeta(s + \frac{a}{2})\, ds. \tag{2.0.23}$$

Evaluating the integral using the residue theorem yields:

$$Q(z,a) = \zeta(a+1) \frac{\Gamma(\frac{a}{2}+1)}{(log(z))^{a/2+1}} + \zeta(1-a) \frac{\Gamma(-\frac{a}{2}+1)}{(log(z))^{-a/2+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(-k-\frac{a}{2}) \zeta(-k+\frac{a}{2}) log^k(z). \tag{2.0.24}$$

Substituting this result into 2.0.21 (together with 2.0.5) gives the following expression for the divisor power function:

$$\sigma_a(n) = \zeta(a+1) n^a + \zeta(1-a) + n^{a/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \zeta(-k-\frac{a}{2}) \zeta(-k+\frac{a}{2}) \frac{d^k}{ds^k} \frac{sin(\pi(s+n))}{\pi(s+n)}|_{s=0}. \tag{2.0.25}$$

An alternative construction uses the Lambert series:

$$L(q) = \sum_{n=1}^{\infty} b_n q^{-n} = \sum_{n=1}^{\infty} a_n \frac{1}{1-q^n}, \tag{2.0.26}$$

with:

$$b_n = \sum_{d|n} a_d. \tag{2.0.27}$$

The coefficients satisfy the inversion formula:

$$b_x = \lim_{T\to\infty} \frac{1}{2iT} \int_{\sigma - iT}^{\sigma + iT} e^{sx} L(e^s)\, ds, \tag{2.0.28}$$

with convergence estimate:

$$|\frac{1}{2iT} \int_{\sigma - iT}^{\sigma + iT} e^{sx} L(e^s)\, ds| \le \frac{1}{2T} \int_{-T}^{T} e^{\sigma x} |L(e^{\sigma + it})|\, dt \le e^{\sigma x} \sum_{n=1}^{\infty} |b_n| e^{-\sigma n}, \sigma > 0. \tag{2.0.29}$$

Equivalently, by the residue theorem under the mapping $s = \frac{T}{\pi} log(z)$:

$$b_x = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{|L(p_i)| = +\infty \\ -T \le \arg(p_i) \le T}} Res(z^{x-1} L(z)\, ; p_i), \tag{2.0.30}$$

where p_i are the poles of the Lambert series.

3 Prime generating and prime gap function

This section extends the spectral framework to prime-related quantities by introducing generating functions for the prime gap sequence and the prime counting function. Using contour integration and inverse transform techniques, the prime gap function is represented through complex-analytic constructions analogous to those developed earlier for divisor functions. The section also derives formulas for the inverse prime counting function via residue calculus, establishing a structural link between generating functions, Dirichlet series, and prime distribution. These results prepare the ground for the asymptotic approximations developed in the following section. The generating function of the prime gap sequence g_n is defined by:

$$G(z) = \sum_{k=0}^{\infty} g_k z^{-k} = \sum_{k=0}^{\infty} z^{-\pi(k)} \ , |z| > 1 \ , \tag{3.0.1}$$

where $\pi(k)$ denotes the prime counting function and $g_0 = p_1 = 2$. Equivalently, $G(z)$ can be expressed by the integral:

$$G(z) = \int_0^{\infty} z^{-\pi(t)} \, dt \ , |z| > 1. \tag{3.0.2}$$

The prime gap function admits the inversion formula:

$$g_x = \lim_{T\to\infty} \frac{1}{2iT} \int_{\sigma-iT}^{\sigma+iT} e^{sx} G(e^s) \, ds, \tag{3.0.3}$$

with convergence estimate is given by the inequalities:

$$|\frac{1}{2iT} \int_{\sigma-iT}^{\sigma+iT} e^{sx} G(e^s) \, ds| \le \frac{1}{2T} \int_{-T}^{T} e^{\sigma x} |G(e^{\sigma+it})| \, dt \le e^{\sigma x} G(e^{\sigma}), \sigma > 0. \tag{3.0.4}$$

By the residue theorem under the conformal mapping $s = \frac{T}{\pi} \log z$, this becomes:

$$g_x = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{|G(p_i)|=+\infty \\ -T \le \arg(p_i) \le T}} Res(z^{x-1} G(z) \ ; p_i) \ , \tag{3.0.5}$$

where p_i are the poles of the prime gap generating function $G(z)$. Using the inverse Z-transform and identity 2.0.5, we obtain:

$$g_n = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\pi(m))^k \frac{d^k}{ds^k} \frac{\sin(\pi(s+n))}{\pi(s+n)}|_{s=0}. \tag{3.0.6}$$

The inverse prime counting function is given by the contour integral:

$$\pi^{-1}(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma-i\infty} \frac{e^{s(x-1)}}{s} G(e^s) \, ds, \tag{3.0.7}$$

and, by residue calculus:

$$\pi^{-1}(x) = Res(\frac{e^{s(x-1)}}{s} G(e^s) \ ; 0) + \sum_{\substack{|G(p_i)|=+\infty \\ p_i \ne 1}} Res(\frac{e^{s(x-1)}}{s} G(e^s) \ ; log(p_i)). \tag{3.0.8}$$

4 Asymptotic approximations

This section develops asymptotic formulas for the prime gap function and the inverse prime counting function using Dirichlet series techniques and the Prime Number Theorem. By approximating the associated Dirichlet series and evaluating the resulting contour integrals, explicit asymptotic expansions are obtained. In particular, closed-form approximations involving the Lambert W function are derived, leading to logarithmic-type growth estimates for prime gaps. These results provide analytic insight into the large-scale behavior of prime-related functions within the spectral framework established earlier. The asymptotic approximations of the prime gap function and the inverse prime counting function, are derived by considering the Dirichlet series:

$$D(g,s) = \sum_{n=1}^{\infty} \frac{g_n}{n^s} = \sum_{n=2}^{\infty} \frac{1}{\pi(n)^s}, \tag{4.0.1}$$

using the asymptotic approximation of the prime counting function $\pi(x) \sim \frac{x}{logx}$, according to the prime number theorem [3], the Dirichlet series for the prime gap function gets the form:

$$D(g,s) \sim \sum_{n=1}^{\infty} \frac{(\log n)^s}{n^s} \ , \Re(s) > 1, \tag{4.0.2}$$

the Dirichlet series for the prime gap function can be expressed by the integral:

$$D(g,s) = \int_2^{\infty} \frac{1}{(\pi(x))^s}\, dx, \tag{4.0.3}$$

equivalently, using the asymptotic approximation of the prime counting function:

$$D(g,s) \sim \int_1^{\infty} \frac{(\log x)^s}{x^s}\, dx, \tag{4.0.4}$$

by evaluating the integral gives the result:

$$D(g,s) \sim \frac{\Gamma(s+1)}{(s-1)^{s+1}}. \tag{4.0.5}$$

The inverse prime counting function admits the Mellin inversion formula:

$$\pi^{-1}(x) = 2 + \int_{2-i\infty}^{2+i\infty} \frac{(x-1)^s}{s} D(g,s) \frac{ds}{2\pi i} \ , x > 1. \tag{4.0.6}$$

Substituting the asymptotic expression of $D(g,s)$ gives:

$$\pi^{-1}(x) \sim 2 + (x-1)log(x-1) - (x-1)\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \frac{1}{(x-1)^k} \ , x > 1 + e. \tag{4.0.7}$$

The contour integral can be evaluated in closed form using the Lambert W function [2]:

$$\int_{2-i\infty}^{2+i\infty} \frac{(x-1)^s}{s} \frac{\Gamma(s+1)}{(s-1)^{s+1}} \frac{ds}{2\pi i} = -(x-1)W_{-1}(\frac{1}{1-x}), \tag{4.0.8}$$

leading to the approximation:

$$\pi^{-1}(x) \sim 2 - (x-1)W_{-1}(\frac{1}{1-x}) \ , x > 1 + e, \tag{4.0.9}$$

the corresponding graph of the approximation above, is given below:

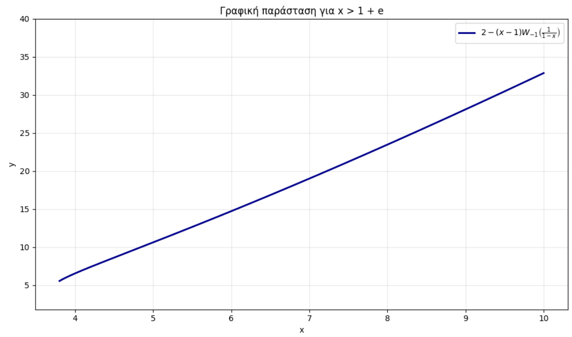


Figure 1: Inverse prime counting function $\pi^{-1}(x)$ approximation graph.

Using the average order theorem [4], $g_n \sim \frac{d\pi^{-1}(x+1)}{dx}|_{x=n}$:

$$g_n \sim log(n) + 1 - \sum_{k=1}^{\infty} \frac{k^{k-1}(1-k)}{k!} \frac{1}{n^k} \ , n > e, \tag{4.0.10}$$

equivalently, using the Lambert W function and its derivative:

$$g_n \sim -\frac{(W_{-1}(-\frac{1}{n}))^2}{1 + W_{-1}(-\frac{1}{n})} \ , n > e, \tag{4.0.11}$$

the corresponding graph of the approximation above, is given by:

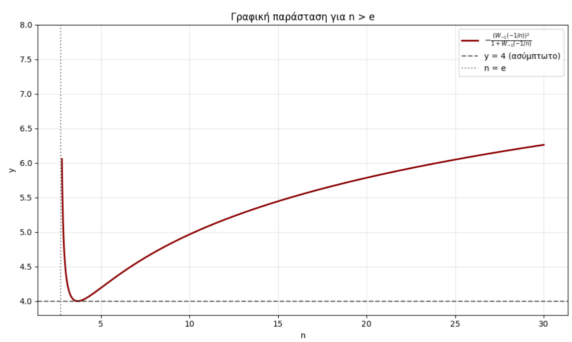


Figure 2: Prime gap function g_n approximation graph.

5 Conclusions and Final Remarks

In this paper, a unified spectral framework for the study of arithmetic functions is developed by combining Fourier kernel normalization, contour integration, and Dirichlet series methods. These constructions linked divisor functions to the Lambert series and illustrating the flexibility of the spectral approach.

The framework was then extended to prime-related quantities. By introducing generating functions for the prime gap sequence and the prime counting function, and applying inverse transform and residue techniques, explicit integral representations and summation formulas were obtained. Through asymptotic analysis grounded in the Prime Number Theorem, approximations for the inverse prime counting function and prime gaps was derived, including compact expressions involving the Lambert W function. These results are consistent with the expected logarithmic growth behavior of primes and their gaps.

Overall, the paper demonstrates that arithmetic functions can be treated as analytically invertible spectral objects. The unification of Fourier-analytic normalization with classical tools of analytic number theory provides new integral representations and asymptotic expansions, clarifying structural relationships between divisor functions, prime distributions, and their generating mechanisms.

Future work may further investigate the analytic properties of the associated generating functions, explore refinements of the asymptotic expansions, and examine potential applications of the spectral framework to other arithmetic functions and related problems in analytic number theory.

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