Double Stochastic Quantization in Minkowski spacetime. Non perturbative approach for Stochastic Quantization $g\varphi_d^{2n}, n \geq 2, d \geq 4$ model Quantum Field Theory in Minkowski spacetime.

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Abstract: In this paper, we show how the finite formulation of quantum field theory based on Langevin equations can be generalized to the case of *nonrenormalizable* theories.The 5th-time stochastic-quantization approach to field theory proposed by Parisi and Wu, is put in a path-integral form in [5]. The procedure of taking the limit $\tau \to \infty$ is analyzed and based on new grounds through the introduction of the vacuum-vacuum generating functional. In this paper non perturbative aprouch related to Parisi and Wu stochastic-quantization of the $\lambda\varphi_d^{2n}, n \geq 2, d \geq 4$ model quantum field theory in Minkowski spacetime is considered.

Content

1.Introduction

In this paper we deal with a system of the double stochastic relaxation equations of the form:

$$\begin{gathered}
\frac{\partial\varphi_1(\tau,x)}{\partial\tau} = i\left(\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)} - i\eta_1(\tau,x)\right) + \epsilon\widetilde{\eta}_1(\tau,x),\\
\varphi_1(0,x) = 0, x \in M^4, \tau \in \mathbb{R}_+,\\
\frac{\partial\varphi_2(\tau,x)}{\partial\tau} = i\left(\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_2(\tau,x)} - i\eta_2(\tau,x)\right) + \epsilon\widetilde{\eta}_2(\tau,x),\\
\varphi_2(0,x) = 0, x \in M^4, \tau \in \mathbb{R}_+,\\
S[\varphi_1,\varphi_2] =\\
\int_{\mathbb{R}^4} d^4x\left[\frac{1}{2}(\partial_\mu\varphi_1(\tau,x)\partial_\mu\varphi_1(\tau,x) + m^2\varphi_1(\tau,x)) + P(\varphi_1(\tau,x))\right] +\\
\int_{\mathbb{R}^4} d^4x\left[\frac{1}{2}(\partial_\mu\varphi_2(\tau,x)\partial_\mu\varphi_2(\tau,x) + m^2\varphi_2(\tau,x)) + P(\varphi_2(\tau,x))\right] +\\
+\int_{\mathbb{R}^4} d^4x[\gamma \times \varphi_1(\tau,x)\varphi_2(\tau,x)], \gamma > 0, \gamma \ll 1.
\end{gathered} \tag{1.1}$$

$$\begin{gathered}
\langle\eta_1(\tau,x)\rangle_\eta = 0, \langle\widetilde{\eta}_1(\tau,x)\rangle_{\widetilde{\eta}_1} = 0,\\
\langle\eta_2(\tau,x)\rangle_\eta = 0, \langle\widetilde{\eta}_2(\tau,x)\rangle_{\widetilde{\eta}_1} = 0,\\
\langle\eta_1(\tau,x)\,\eta_1(\tau',x')\rangle_\eta = 2\delta(\tau-\tau')\delta(x-x'),\\
\langle\widetilde{\eta}_2(\tau,x)\,\widetilde{\eta}_2(\tau',x')\rangle_{\widehat{\eta}} = 2\delta(\tau-\tau')\delta(x-x').
\end{gathered} \tag{1.2}$$

Note that the double stochastic processes $\{\varphi_1(\tau,x,\omega,\varpi;\epsilon), \varphi_2(\tau,x,\omega,\varpi;\epsilon)\}$ defined by Eqs. (1.1)-(1.2) is complex and that the existence of an equilibrium for $\tau \to \infty, \epsilon \to 0$ constitutes a more subtle problem than in the euclidean case [1]. Actually what we are going to show is that the equilibrium limit exists in the distributional sense, namely that the correlation functions converge to the quantum Green functions for $\tau \to \infty$ only when interpreted as tempered distributions, see [2].

In this paper we discuss the double stochastic quantization of a theory of a self-interacting scalar field with lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - (g/4!)\varphi^4. \tag{1.3}$$

The corresponding double stochastic Langevin equation is

$$\frac{\partial}{\partial\tau}\varphi(t,x) = -i\left(\Box\varphi(\tau,x) + m^2\varphi(\tau,x) + \frac{1}{6}g\varphi^3(\tau,x)\right) + \eta(\tau,x) + \epsilon\widetilde{\eta}_1(\tau,x). \tag{1.4}$$

We consider as example the free case $g = 0$:

$$\frac{\partial}{\partial\tau}\varphi_\epsilon(\tau,x) = -i(\Box\varphi_\epsilon(\tau,x) + m^2\varphi_\epsilon(\tau,x)) + \eta(\tau,x) + \epsilon\widetilde{\eta}_1(\tau,x). \tag{1.5}$$

Thus

$$\frac{\partial}{\partial\tau}\widehat{\varphi}_\epsilon(\tau,p) = -i\left(-p^2\widehat{\varphi}_\epsilon(\tau,p) + m^2\widehat{\varphi}_\epsilon(\tau,p)\right) + \widehat{\eta}(\tau,p) + \epsilon\widehat{\widetilde{\eta}}_1(\tau,p) =$$
$$= i(p^2 - m^2)\widehat{\varphi}_\epsilon(\tau,p) + \widehat{\eta}(\tau,p) + \epsilon\widehat{\widetilde{\eta}}_1(\tau,p).$$
$$\mathcal{F}[\varphi_\epsilon(\tau,x)](p) = \widehat{\varphi}_\epsilon(\tau,p) = \int_{\mathbb{R}^4} \varphi_\epsilon(\tau,x)e^{ipx}d^4x, \tag{1.6}$$
$$\mathcal{F}^{-1}\left[\widehat{\varphi}_\epsilon(\tau,p)\right](x) = (2\pi)^{-4}\int_{\mathbb{R}^4} \widehat{\varphi}_\epsilon(\tau,p)e^{-ipx}d^4p.$$

In this case the unique solution of Eq.(1.6) with initial condition $\varphi(\tau = 0,x) = 0$ is simply

$$\lim_{\epsilon\to 0}\widehat{\varphi}_\epsilon(\tau,p) = \widehat{\varphi}_0(\tau,p) = \int_0^\tau dt\widehat{\eta}(\tau,p)\exp[i(p^2 - m^2)(\tau - t)].$$
$$p^2 - m^2 = p_0 - \mathbf{p}^2 - m^2. \tag{1.7}$$

The correlation function of the free field $\widehat{\varphi}_0(\tau,p)$ is

$$\left\langle\widehat{\varphi}_0(\tau,p)\widehat{\varphi}_0(\tau',p')\right\rangle_\eta = 2(2\pi)^4 \int_0^{\min(\tau,\tau')} dt\delta^4(p+p')\exp[i(p^2 - m^2)(\tau + \tau' - 2t)] =$$
$$= i(2\pi)^4\delta^4(p+p')\frac{1}{p^2 - m^2}\{\exp[i(p^2 - m^2)|\tau - \tau'|] - \exp[i(p^2 - m^2)|\tau + \tau'|]\}. \tag{1.8}$$

Remark 1.1.Note that although the limit $\tau = \tau' \to \infty$ of (1.8) does not exist in the ordinary sense, the expression $\left\langle\widehat{\varphi}_0(\tau,p)\widehat{\varphi}_0(\tau,p')\right\rangle_\eta$ converges if interpreted as a weak limit (w-limit) in the sence of a tempered distribution:
$w\text{-}\lim_{s\to\infty}\exp(ip^2s) = 0, w\text{-}\lim_{s\to\infty}\mathbf{P}(1/p^2)\exp(ip^2s) = i\pi\delta(p^2)$, where $\mathbf{P}$ denotes the principal value, and therefore

$$w\text{-}\lim_{\tau=\tau'\to\infty}\left\langle\widehat{\varphi}_0(\tau,p)\widehat{\varphi}_0(\tau',p')\right\rangle_\eta = i(2\pi)^4\delta^4(p+p')\frac{1}{p^2 - m^2 + i0}, \tag{1.9}$$

finally we get

$$G(p,p') = i(2\pi)^4\delta^4(p+p')\frac{1}{p^2 - m^2 + 0i}. \tag{1.10}$$

From (1.10) we get

$$\frac{1}{(2\pi)^8}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} G(p,p')e^{-ipx-ip'y}d^4pd^4p' = i(2\pi)^{-4}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\delta^4(p+p')\frac{e^{-ipx-ip'y}d^4pd^4p'}{p^2 - m^2 + 0i} =$$
$$i(2\pi)^{-4}\int_{\mathbb{R}^4}\frac{e^{-ipx+ipy}d^4p}{p^2 - m^2 + 0i} = i(2\pi)^{-4}\int_{\mathbb{R}^4}\frac{e^{-ip(x-y)}d^4p}{p^2 - m^2 + 0i}. \tag{1.10'}$$

We have thus obtained the conventional Feynman propagator in x-representation.
Remark 1.2.As an alternative, and less sophisticated, way of arriving at (1.10) is to add a negative imaginary mass term $-\frac{1}{2}i\delta\varphi^2$ to the lagrangian (1.4) and let δ tend to zero after all calculations have been performed. Then the second term in (1.8) is exponentially damped, since $p^2 - m^2$ is replaced by $p^2 - (m^2 - i\delta) = p^2 - m^2 + i\delta$,

see [2].
Let us consider double stochastic relaxation equations of the form:

$$\frac{\partial\varphi(\tau,x)}{\partial\tau} = i\left(\frac{\delta S[\varphi]}{\delta\varphi(\tau,x)} - i\eta(\tau,x)\right) + \epsilon\widetilde{\eta}(\tau,x),$$

$$\begin{aligned}\langle\eta(\tau,x)\rangle_{\eta} &= 0, \langle\widetilde{\eta}(\tau,x)\rangle_{\widetilde{\eta}} = 0\\ \langle\eta(\tau,x)\,\eta(\tau',x')\rangle_{\eta} &= 2\delta(\tau-\tau')(x-x'),\\ \langle\widetilde{\eta}(\tau,x)\,\widetilde{\eta}(\tau',x')\rangle_{\widetilde{\eta}} &= 2\delta(\tau-\tau')(x-x').\end{aligned} \tag{1.11}$$

Here $\eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}(\tau,x) = \widetilde{\eta}(\tau,x;\varpi)$ are space-time white noice on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$.
(ii) Evaluate the stochastic average of fields $\phi_{\eta,\widetilde{\eta}}(\tau,x)$ satisfying Eq. (1.11), that means

$$\langle\phi_{\eta,\widetilde{\eta}}(\tau_1,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau_2,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau_m,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}}. \tag{1.12}$$

(iii) Put $\tau_1 = \tau_2 = \ldots = \tau_m = \tau$ in (1.2) and take the weak distributional limit (w-$\lim$)

$$w\text{-}\lim_{\tau\to\infty}\lim_{\epsilon\to 0}\langle\phi_{\eta,\widetilde{\eta}}(\tau,x_1;\epsilon)\phi_{\eta,\widetilde{\eta}}(\tau,x_2;\epsilon)\ldots\phi_{\eta,\widetilde{\eta}}(\tau,x_m;\epsilon)\rangle_{\eta,\widetilde{\eta}} = G(x_1,x_2,\ldots,x_m;\epsilon) \tag{1.13}$$

To understand this relation we have to introduce the notion of random probability (density) $|P(\varphi_\eta,\tau,\omega)|^2 = p(\varphi_\eta,\tau,\omega)$, that is, the random probability (density) of having the system in the configuration $\varphi_\eta(\tau,x,\omega)$ at time τ. There exists for $P(\varphi_\eta,\tau,\omega)$ an equation that describes the evolution of $P(\varphi_\eta,\tau,\omega)$ in the time τ. It is called the Stochastic complex Fokker-Planck (SCFP)) equation

$$\begin{gathered}\frac{\partial P[\varphi_\eta(\tau,x),\tau,\omega]}{\partial\tau} = \\ i\int d^4x\frac{\delta}{\delta\varphi_\eta(\tau,x)}\left[P[\varphi_\eta(\tau,x,\omega),\tau]\frac{\delta\widetilde{S}[\varphi_\eta,\omega]}{\delta\varphi_\eta(\tau,x)}\right] + \frac{1}{\epsilon^2}\int d^4x\frac{\delta^2 P[\varphi_\eta(\tau,x),\tau,\omega]}{\delta\varphi_\eta^2(\tau,x)}, \\ \widetilde{S}[\varphi_\eta,\omega] = i\left(S[\varphi_\eta,\omega] - i\int[\varphi_\eta(\tau,x)\eta(\tau,x,\omega)]d^4x\right).\end{gathered} \tag{1.14}$$

It is possible to rewrite this equation in a Schrodinger-type form:

$$\begin{gathered}\frac{\partial\Psi[\varphi_\eta(\tau,x),\tau,\omega]}{\partial\tau} = -2\mathbf{H}\Psi[\varphi_\eta(\tau,x),\tau,\omega], \\ \Psi[\varphi_\eta(\tau,x),\tau,\omega] = P[\varphi(\tau,x),\tau]\exp\left[\widetilde{S}[\varphi_\eta(\tau,x),\omega]/2\right], \\ \Psi[\varphi_\eta(\tau,x),0,\omega] = \delta(\varphi_\eta(\tau,x)),\end{gathered} \tag{1.15}$$

where

$$\mathbf{H} = -\frac{1}{2}\frac{\delta^2}{\delta\varphi_\eta^2} + \frac{1}{8}\left[\frac{\delta\widetilde{S}[\varphi_\eta,\omega]}{\delta\varphi_\eta}\right]^2 - \frac{1}{4}\frac{\delta^2\widetilde{S}[\varphi_\eta,\omega]}{\delta\varphi_\eta^2}. \tag{1.16}$$

It is an operator $\mathbf{H}\Psi_n = E_n\Psi_n$ whose ground state $E_0 = 0$ is

$$\Psi_0[\varphi_\eta(\tau,x),\omega] = \exp\left[\widetilde{S}[\varphi_\eta(\tau,x),\omega]/2\right]. \tag{1.17}$$

The solution of Eq.(1.16) is

$$\Psi[\varphi_\eta(\tau,x),\tau,\omega] = \sum_{n=0}^{\infty} c_n \Psi_n[\varphi_\eta(\tau,x),\omega]\exp(-2E_n\tau),$$
$$\mathrm{Re}(E_n) \geq 0, |\mathrm{Im}(E_{n+1})| > 0, \tag{1.18}$$

where $\{c_n\}_{n=0}^{\infty}$ are normalizing constants. The probability density $P[\varphi(\tau,x),\tau,\omega]$ can be written as

$$P[\varphi_\eta(\tau,x),\tau,\omega] = \exp\left[\widetilde{S}[\varphi_\eta(\tau,x),\omega]/2\right]\sum_{n=0}^{\infty} c_n \Psi_n[\varphi_\eta(\tau,x),\omega]\exp(-2E_n\tau). \tag{1.19}$$

In the w-limit $\tau \to \infty$ the only term that does not disappear in this expression is $\Psi_0[\varphi_\eta(\tau,x),\tau,\omega]$, so finally we have

$$w\text{-}\lim_{\tau\to\infty} P[\varphi_\eta(\tau,x),\tau] = c_0 \exp\left[-\widetilde{S}[\varphi_\eta(\tau,x),\omega]/2\right]\exp\left[-\widetilde{S}[\varphi_\eta(\tau,x),\omega]/2\right] =$$
$$= c_0 \exp\left[-\widetilde{S}[\varphi_\eta(\tau,x),\omega]\right]. \tag{1.20}$$

This is the formal reason why Eq.(1.13) holds.

2.The generating functional

In this paper we deal with a system of the complex double stochastic relaxation equations of the form:

$$\begin{gathered}
\frac{\partial\varphi_1(\tau,x)}{\partial\tau} = i\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x),\\
\varphi_1(\tau,x) = \mathrm{Re}\,\varphi_1(\tau,x) + i\cdot\mathrm{Im}\,\varphi_1 = \varphi_{1,1}(\tau,x) + i\cdot\varphi_{1,2}(\tau,x),\\
\varphi_1(0,x) = 0, x \in M^4, \tau \in \mathbb{R}_+,\\
\widetilde{\eta}(\tau,x) = \widetilde{\eta}_1(\tau,x) + i\cdot\widetilde{\eta}_2(\tau,x)\\
\frac{\partial\varphi_2(\tau,x)}{\partial\tau} = i\frac{\delta S[\varphi_1,\varphi_2]}{\delta\varphi_2(\tau,x)} + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x),\\
\varphi_2(\tau,x) = \mathrm{Re}\,\varphi_{2,1}(\tau,x) + i\cdot\mathrm{Im}\,\varphi_{2,2} = \varphi_{2,1}(\tau,x) + i\cdot\varphi_{2,2}(\tau,x)\\
\varphi_2(0,x) = 0, x \in M^4, \tau \in \mathbb{R}_+,\\
S[\varphi_1,\varphi_2] =\\
\int_{\mathbb{R}^4} d^4x\left[\frac{1}{2}(\partial_\mu\varphi_1(\tau,x)\partial_\mu\varphi_1(\tau,x) + m^2\varphi_1(\tau,x)) + P(\varphi_1(\tau,x))\right] +\\
\int_{\mathbb{R}^4} d^4x\left[\frac{1}{2}(\partial_\mu\varphi_2(\tau,x)\partial_\mu\varphi_2(\tau,x) + m^2\varphi_2(\tau,x)) + P(\varphi_2(\tau,x))\right] +\\
+\int_{\mathbb{R}^4} d^4x[\gamma\times\varphi_1(\tau,x)\varphi_2(\tau,x)], \gamma > 0, \gamma \ll 1.
\end{gathered} \tag{2.1}$$

$$
\begin{aligned}
\langle \eta_1(\tau,x)\rangle_\eta &= 0, \langle \widetilde{\eta}_1(\tau,x)\rangle_{\widetilde{\eta}_1} = 0,\\
\langle \eta_2(\tau,x)\rangle_\eta &= 0, \langle \widetilde{\eta}_2(\tau,x)\rangle_{\widetilde{\eta}_1} = 0,\\
\langle \eta_1(\tau,x)\eta_1(\tau',x')\rangle_\eta &= 2(2\pi)^4\delta(\tau-\tau')\delta(x-x'),\\
\langle \widetilde{\eta}_2(\tau,x)\widetilde{\eta}_2(\tau',x')\rangle_{\widehat{\eta}} &= 2(2\pi)^4\delta(\tau-\tau')\delta(x-x'),\\
\langle \eta_1(\tau,x)\eta_2(\tau',x')\rangle_\eta &= 0,\\
\langle \widetilde{\eta}_2(\tau,x)\widetilde{\eta}_2(\tau',x')\rangle_{\widehat{\eta}} &= 0.
\end{aligned}
\tag{2.2}
$$

We rewrite now Eq.(2.1) of the real form

$$
\begin{aligned}
\frac{\partial\varphi_{1,1}(\tau,x)}{\partial\tau} &= \mathrm{Re}\left\{\frac{\delta\widetilde{S}[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\right\} + \epsilon\widetilde{\eta}_1(\tau,x),\\
\frac{\partial\varphi_{1,2}(\tau,x)}{\partial\tau} &= \mathrm{Im}\left\{\frac{\delta\widetilde{S}[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\right\} + \epsilon\widetilde{\eta}_2(\tau,x),\\
\varphi_1(0,x) &= 0, x\in M^4, \tau\in\mathbb{R}_+,\\
\frac{\partial\varphi_{2,1}(\tau,x)}{\partial\tau} &= \mathrm{Re}\left\{\frac{\delta\widetilde{S}[\varphi_1,\varphi_2]}{\delta\varphi_2(\tau,x)}\right\} + \epsilon\widetilde{\eta}_1(\tau,x),\\
\frac{\partial\varphi_{2,2}(\tau,x)}{\partial\tau} &= \mathrm{Re}\left\{\frac{\delta\widetilde{S}[\varphi_1,\varphi_2]}{\delta\varphi_2(\tau,x)}\right\} + \epsilon\widetilde{\eta}_2(\tau,x),\\
\varphi_2(0,x) &= 0, x\in M^4, \tau\in\mathbb{R}_+,\\
\widetilde{S}[\varphi_1,\varphi_2] &= iS[\varphi_1,\varphi_2] + \left(\int_{\mathbb{R}^4}\varphi_1(\tau,x)\eta_1(\tau,x)d^4x + \int_{\mathbb{R}^4}\varphi_2(\tau,x)\eta_2(\tau,x)d^4x\right)
\end{aligned}
\tag{2.3}
$$

Remark 2.1. Here $\eta_{1,2}(\tau,x) = \eta_{1,2}(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}_{1,2}(\tau,x) = \widetilde{\eta}_{1,2}(\tau,x;\varpi)$ are space-time white noices on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$. The angular brackets denote connected average with respect to the random variables $\eta,\widetilde{\eta}_{1,2}$.

We want to build a generating functional $Z_\epsilon[J_1,J_2]$ from which the correlations

$$
\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta
$$

$$
\begin{aligned}
\frac{\partial\varphi_{1,\eta}(\tau,x)}{\partial\tau} &= \left.\frac{\delta\widetilde{S}[\varphi_1,\varphi_2]}{\delta\varphi_1(\tau,x)}\right|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x),\\
\varphi_{1,\eta}(\tau,x) &= \mathrm{Re}\,\varphi_{1,\eta}(\tau,x) + i\cdot\mathrm{Im}\,\varphi_{1,\eta}(\tau,x) = \varphi_{1,1,\eta}(\tau,x) + i\cdot\varphi_{1,2,\eta}(\tau,x),\\
\frac{\partial\varphi_{2,\eta}(\tau,x)}{\partial\tau} &= \left.\frac{\delta\widetilde{S}[\varphi_1,\varphi_2]}{\delta\varphi_2(\tau,x)}\right|_{\varphi_1=\varphi_{1,\eta},\varphi_2=\varphi_{2,\eta}} + \eta(\tau,x)\\
\varphi_{2,\eta}(\tau,x) &= \mathrm{Re}\,\varphi_{2,\eta}(\tau,x) + i\cdot\mathrm{Im}\,\varphi_{2,\eta}(\tau,x) = \varphi_{2,1,\eta}(\tau,x) + i\cdot\varphi_{2,2,\eta}(\tau,x),
\end{aligned}
\tag{2.4}
$$

can be derived by the following fashion:

$$
\begin{aligned}
&\langle \varphi_{1,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{1,\eta}(\tau_l,x_l;\omega)\varphi_{2,\eta}(\tau_1,x_1;\omega)\times\ldots\times\varphi_{2,\eta}(\tau_l,x_r;\omega)\rangle_\eta = \\
&\lim_{\epsilon\to0,\gamma\to0}\mathbf{E}_\Sigma\left[\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\ldots\delta J_1(\tau_l,x_l)\delta J_2(\tau_1,x_1)\ldots\delta J_2(\tau_r,x_r)}\right]\triangleq \\
&\lim_{\epsilon\to0,\gamma\to0}\left\{N\int D[\varphi_{1,1}(\tau',x;\omega)]D[\varphi_{1,2}(\tau',x;\omega)]D[\varphi_{2,1}(\tau',x;\omega)]D[\varphi_{2,2}(\tau',x;\omega)]\right. \\
&D[\eta(\tau',x;\omega)]D[\widetilde{\eta}_1(\tau',x;\varpi)]D[\widetilde{\eta}_2(\tau',x;\varpi)]\times \\
&\delta(\varphi_{1,1\eta,\widetilde{\eta}_1}(0,x))\delta(\varphi_{1,2\eta,\widetilde{\eta}_1}(0,x))\delta(\varphi_{1,1}-\varphi_{1,1,\eta,\widetilde{\eta}_1})\delta(\varphi_{1,2}-\varphi_{1,2,\eta,\widetilde{\eta}_1})\times \\
&\delta(\varphi_{2,1,\eta,\widetilde{\eta}_2}(0,x))(\varphi_{2,2\eta,\widetilde{\eta}_1}(0,x))\delta(\varphi_{2,1}-\varphi_{2,1,\eta,\widetilde{\eta}_2})\delta(\varphi_{2,2}-\varphi_{2,2,\eta,\widetilde{\eta}_2})\times \\
&(\mathbf{E}_\Sigma[\varphi_1(\tau_1,x_1;\omega)\ldots\varphi_1(\tau_l,x_l;\omega)\varphi_2(\tau_1,x_1;\omega)\ldots\varphi_2(\tau_r,x_r;\omega)])\times \\
&\times\exp\left[-\int_0^\tau J_1(\tau',x)\varphi_1(\tau',x;\omega)d^4xd\tau'\right]\exp\left[-\frac{1}{4}\int_0^\tau\widetilde{\eta}_1^2(\tau',x;\varpi)d^4xd\tau'\right]\times \\
&\times\exp\left[-\frac{1}{4}\int_0^\tau\widetilde{\eta}_2^2(\tau',x;\varpi)d^4xd\tau'\right]\exp\left[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4xd\tau'\right].
\end{aligned}
\tag{2.5}
$$

.By canonical consideration [6] one obtains

$$
\begin{aligned}
&Z_\epsilon[J_1,J_2;\omega] = N\int D[\varphi_1(\tau',x;\omega)]D[\varphi_2(\tau',x;\omega)]D[\eta(\tau',x;\omega)]\times \\
&D[\widetilde{\eta}_1(\tau',x;\varpi)]D[\widetilde{\eta}_2(\tau',x;\varpi)]\times \\
&\delta(\varphi_{\eta,\widetilde{\eta}_1}(0,x))\delta(\varphi_{\eta,\widetilde{\eta}_2}(0,x))\delta(\varphi_1-\varphi_{1,\eta,\widetilde{\eta}_1})\delta(\varphi_2-\varphi_{2,\eta,\widetilde{\eta}_2})\times \\
&\times\exp\left[-\int_0^\tau J_1(\tau',x)\varphi_1(\tau',x;\omega)d^4xd\tau'\right]\exp\left[-\frac{1}{4}\int_0^\tau\widetilde{\eta}_1^2(\tau',x;\varpi)d^4xd\tau'\right]\times \\
&\exp\left[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4xd\tau'\right]\times \\
&\times\exp\left[-\int_0^\tau J_2(\tau',x)\varphi_2(\tau',x;\omega)d^4xd\tau'\right]\exp\left[-\frac{1}{4}\int_0^\tau\widetilde{\eta}_2^2(\tau',x;\varpi)d^4xd\tau'\right]
\end{aligned}
\tag{2.6}
$$

where $D[\varphi_1(\tau',x;\omega)]\triangleq D[\varphi_{1,1}(\tau',x;\omega)]D[\varphi_{1,2}(\tau',x;\omega)]$,
$D[\varphi_2(\tau',x;\omega)]\triangleq D[\varphi_{2,1}(\tau',x;\omega)]D[\varphi_{2,2}(\tau',x;\omega)]$,
$\delta(\varphi_1-\varphi_{1,\eta,\widetilde{\eta}_1})\triangleq\delta(\varphi_{1,1}-\varphi_{1,1,\eta,\widetilde{\eta}_1})\delta(\varphi_{1,2}-\varphi_{1,2,\eta,\widetilde{\eta}_1})$
$\delta(\varphi_2-\varphi_{2,\eta,\widetilde{\eta}_2})\triangleq\delta(\varphi_{2,1}-\varphi_{2,1,\eta,\widetilde{\eta}_1})\delta(\varphi_{2,2}-\varphi_{2,2,\eta,\widetilde{\eta}_1})$
where $\varphi_{1,2,\eta,\widetilde{\eta}}=\varphi_{1,2}(\tau',x;\omega,\varpi)$ that appears in Eqs.(2.6) is the solution of the double stochastic Langevin equations (2.2), solved with zero initial condidion: $\varphi_{1,2,\eta,\widetilde{\eta}}(0,x)\equiv0$ and N is a normalizing constant and

$$
\begin{aligned}
D[\varphi_{1,1};\omega] &= \lim_{M\to\infty}\prod\nolimits_{i=0}^M D[\varphi_{1,1,\tau_i}(x;\omega)], \\
D[\varphi_{1,2};\omega] &= \lim_{M\to\infty}\prod\nolimits_{i=0}^M D[\varphi_{1,2,\tau_i}(x;\omega)], \\
D[\varphi_{2,1};\omega] &= \lim_{M\to\infty}\prod\nolimits_{i=0}^M D[\varphi_{2,1,\tau_i}(x;\omega)], \\
D[\varphi_{2,2};\omega] &= \lim_{M\to\infty}\prod\nolimits_{i=0}^M D[\varphi_{2,2,\tau_i}(x;\omega)],
\end{aligned}
\tag{2.7}
$$

where $\varphi_{1,1,\tau_i}(x;\omega),\varphi_{1,2,\tau_i}(x;\omega)\varphi_{2,1,\tau_i}(x;\omega)\varphi_{2,2,\tau_i}(x;\omega)$, are the field configurations at the time τ_i, having sliced the interval 0 to τ in M infinitesimal parts ε with $\tau_i = i\varepsilon$ and$D[\varphi_{1,1,\tau_i}(x;\omega)]$, $D[\varphi_{1,2,\tau_i}(x;\omega)]$, $D[\varphi_{2,1,\tau_i}(x;\omega)], D[\varphi_{2,2,\tau_i}(x;\omega)]$ is a random Feynman measures [3].

Abbreviation 2.1. $\delta(\varphi_1 - \varphi_{1,\eta,\tilde{\eta}}) \triangleq \delta(\varphi_{1,1} - \varphi_{1,1,\eta,\tilde{\eta}})\delta(\varphi_{1,2} - \varphi_{1,2,\eta,\tilde{\eta}})$, $\delta(\varphi_2 - \varphi_{2,\eta,\tilde{\eta}}) \triangleq \delta(\varphi_{2,1} - \varphi_{2,1,\eta,\tilde{\eta}})\delta(\varphi_{2,1} - \varphi_{2,2,\eta,\tilde{\eta}}), \tilde{S}_{1,2} = \tilde{S}[\varphi_1, \varphi_2]$.

The delta functions $\delta(\varphi_1 - \varphi_{1,\eta,\tilde{\eta}}), \delta(\varphi_2 - \varphi_{2,\eta,\tilde{\eta}})$ in Eq.(2.6) we can write as

$$\begin{aligned}
\delta(\varphi_{1,1} - \varphi_{1,1,\eta,\tilde{\eta}}) &= \delta\left[\frac{\partial\varphi_{1,1}}{\partial\tau} - \mathrm{Re}\left\{\frac{\delta\tilde{S}_{1,2}}{\delta\varphi_1}\right\} - \epsilon\tilde{\eta}\right]\left\|\frac{\delta(\epsilon\tilde{\eta})}{\delta\varphi_{1,1}}\right\|,\\
\delta(\varphi_{1,2} - \varphi_{1,2,\eta,\tilde{\eta}}) &= \delta\left[\frac{\partial\varphi_{1,2}}{\partial\tau} - \mathrm{Im}\left\{\frac{\delta\tilde{S}_{1,2}}{\delta\varphi_1}\right\} - \epsilon\tilde{\eta}\right]\left\|\frac{\delta(\epsilon\tilde{\eta})}{\delta\varphi_{1,2}}\right\|,\\
\delta(\varphi_{2,1} - \varphi_{2,1,\eta,\tilde{\eta}}) &= \delta\left[\frac{\partial\varphi_{2,1}}{\partial\tau} - \mathrm{Re}\left\{\frac{\delta\tilde{S}_{1,2}}{\delta\varphi_2}\right\} - \epsilon\tilde{\eta}\right]\left\|\frac{\delta(\epsilon\tilde{\eta})}{\delta\varphi_{2,1}}\right\|,\\
\delta(\varphi_{2,2} - \varphi_{1,1,\eta,\tilde{\eta}}) &= \delta\left[\frac{\partial\varphi_{2,2}}{\partial\tau} - \mathrm{Im}\left\{\frac{\delta\tilde{S}_{1,2}}{\delta\varphi_2}\right\} - \epsilon\tilde{\eta}\right]\left\|\frac{\delta(\epsilon\tilde{\eta})}{\delta\varphi_{2,2}}\right\|,
\end{aligned} \tag{2.8}$$

where $\tilde{S}_{1,2} = \tilde{S}[\varphi_1, \varphi_2]$and where $\|\delta(\epsilon\tilde{\eta})/\delta\varphi_{1,1}\|, \|\delta(\epsilon\tilde{\eta})/\delta\varphi_{1,2}\|, \|\delta(\epsilon\tilde{\eta})/\delta\varphi_{2,1}\|$, $\|\delta(\epsilon\tilde{\eta})/\delta\varphi_{2,2}\|$ is the Jacobians of the transformations $\epsilon\,\tilde{\eta} \to \varphi_{1,1}, \epsilon\,\tilde{\eta} \to \varphi_{1,2}, \epsilon\,\tilde{\eta} \to \varphi_{2,1}$, $\epsilon\,\tilde{\eta} \to \varphi_{2,2}$, that is

$$\begin{aligned}
\left\|\frac{\delta(\epsilon\tilde{\eta})}{\delta\varphi_{1,1}}\right\| &= \det\left[\left[\partial_\tau + \mathrm{Re}\left\{\frac{\delta^2\tilde{S}_{1,2}}{\delta\varphi_1(\tau)\delta\varphi_1(\tau')}\right\}\right]\delta(\tau - \tau')\right],\\
\left\|\frac{\delta(\epsilon\tilde{\eta})}{\delta\varphi_{1,2}}\right\| &= \det\left[\left[\partial_\tau + \mathrm{Im}\left\{\frac{\delta^2\tilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\right\}\right]\delta(\tau - \tau')\right],
\end{aligned} \tag{2.9}$$

From Eq.(2.9) by canonical calculation we get

$$\begin{aligned}
\left\|\frac{\delta\left(\epsilon\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2}}\right\| &= \exp\left[\mathbf{tr}\ln\left[\left[\partial_\tau + \frac{\delta^2\tilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\right]\delta(\tau - \tau')\right]\right] =\\
&= \exp\left[\mathbf{tr}\ln\partial_\tau\left[\delta(\tau - \tau') + \partial_\tau^{-1}\frac{\delta^2\tilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\right]\right],
\end{aligned} \tag{2.10}$$

where ∂_τ^{-1} indicate the Careen's function $G(\tau - \tau')$ that satisfies

$$\partial_\tau G(\tau - \tau') = \delta(\tau - \tau'). \tag{2.11}$$

The solutions of the Eq.(2.11) are: (i) if we choose propagation forward in time

$$G(\tau - \tau') = \theta(\tau - \tau') \tag{2.12}$$

(ii) if we choose propagation backward in time

$$G(\tau - \tau') = -\theta(\tau - \tau') \tag{2.13}$$

In case, propagation forward in time, we get

$$\begin{aligned}
\left\|\frac{\delta\left(\epsilon\tilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2}}\right\| &= \exp\left\{\mathbf{tr}\left[\ln\partial_\tau + \ln\left[\delta(\tau - \tau') + \theta(\tau - \tau')\frac{\delta^2\tilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\right]\right]\right\}\\
&= \exp(\mathbf{tr}\ln\partial_\tau)\exp\left[\mathbf{tr}\ln\left[\delta(\tau - \tau') + \theta(\tau - \tau')\frac{\delta^2\tilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\right]\right].
\end{aligned} \tag{2.14}$$

The term $\exp(\mathbf{tr}\ln\partial_\tau)$ can be dropped, as it cancels with the same term in the denominator of (2.6), once we normalize $\widetilde{Z}_\epsilon[J_{1,2};\omega] = Z_\epsilon[J_{1,2};\omega]/Z_\epsilon[0,0;\omega]$.

Abbreviation 2.2.$Z_\epsilon[J_{1,2};\omega] \triangleq Z_\epsilon[J_1,J_2;\omega]$

So in Eq.(2.14) we are left with

$$\left\| \frac{\delta\left(\epsilon\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2}} \right\| = \exp\left[\mathbf{tr}\ln\left[\delta(\tau-\tau')+\theta(\tau-\tau')\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\right]\right]. \tag{2.15}$$

By using the canonical expansion for the $\ln$, we obtain

$$\begin{gathered}\left\| \frac{\delta\left(\epsilon\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2}} \right\| = \\ \exp\left[\mathbf{tr}\left[\theta(\tau-\tau')\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}+\right.\right. \\ \left.\left.\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}+\ldots\right]\right] = \\ = \exp\left[\int d\tau\theta(0)\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2(\tau)}+\right. \\ \left.\int d\tau'\theta(\tau-\tau')\theta(\tau'-\tau)\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau)\delta\varphi_{1,2}(\tau')}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau')\delta\varphi_{1,2}(\tau)}+\ldots\right]\end{gathered} \tag{2.16}$$

The second term in this expression is zero because $\theta(\tau-\tau')\theta(\tau'-\tau) = 0$ and the same for all the subsequent terins. The only one left is the first term and choosing $\theta(0) = 1/2$ we get

$$\left\| \frac{\delta\left(\epsilon\widetilde{\eta}_{1,2}\right)}{\delta\varphi_{1,2}} \right\| = \exp\left[\frac{1}{2}\int d\tau'\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2}\right]. \tag{2.17}$$

Inserting Eq.(2.17) and Eq.(2.8) into Eq.(2.6) and performing the $\widetilde{\eta}$ integration, we get

$$\begin{gathered} Z_\epsilon[J_1,J_2;\omega] = N\int D[\varphi_{1,2}(\tau',x;)\omega]D[\eta_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times \\ \times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon}\left[\frac{\partial\varphi_{1,2}(\tau',x;\omega)}{\partial\tau}+\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2(\tau',x;\omega)}\right]d^4xd\tau'\right. \\ \left.-\int_0^\tau J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\exp\left[-\frac{1}{4}\int_0^\tau\eta_{1,2}^2(\tau',x;\omega)d^4xd\tau'\right]. \end{gathered} \tag{2.18}$$

From Eq.(2.18) finally we obtain

$$\begin{gathered} Z_\epsilon[J_{1,2};\omega] = N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times \\ \times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon}\left[\frac{\partial\varphi_{1,2}(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau',x;\omega)}-\eta\right]^2-\frac{\epsilon}{2}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2}\right]d^4xd\tau'\right. \\ \left.-\int_0^\tau J_{1,2}(\tau',x)\varphi_{1,2}(\tau',x;\omega)d^4xd\tau'\right\}\exp\left[-\frac{1}{4}\int_0^\tau\eta_{1,2}^2(\tau',x;\omega)d^4xd\tau'\right]. \end{gathered} \tag{2.19}$$

If we want also to specify that we are interested only in the correlations at the same

time τ_1, we have just to choose $J_{1,2}(x,\tau')$ of the form $J_{1,2}(x,\tau') = \bar{J}(x)\delta(\tau'-\tau_1), \tau_1 < \tau$

and

Eq.(2.19) then becomes

$$Z_\epsilon[J_{1,2};\omega] = N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times$$
$$\times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon}\left[\frac{\partial\varphi_{1,2}(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau',x;\omega)}\right]^2-\frac{1}{2}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2(\tau',x;\omega)}\right]d^4xd\tau'\right. \tag{2.20}$$
$$\left.-J_{1,2}(\tau_1,x)\varphi_{1,2}(\tau_1,x;\omega)d^4xd\tau'\right\}\times\exp\left[-\frac{1}{4}\int_0^\tau\eta_{1,2}^2(\tau',x;\omega)d^4xd\tau'\right].$$

From Eq.(2.20) finally we obtain

$$Z_\epsilon[J_{1,2};\omega] = N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\times$$
$$\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon}\left[\frac{\partial\varphi_{1,2}(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau',x;\omega)}\pm\eta_{1,2}(\tau',x;\omega)\right]^2-\right.\right.$$
$$\left.\left.-\frac{1}{2}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2(\tau',x;\omega)}d^4xd\tau'\right]-J_{1,2}(x,\tau_1)\varphi_{1,2}(\tau_1,x;\omega)d^4xd\tau'\right\}\times \tag{2.21}$$
$$\times\exp\left[-\frac{1}{4}\int_0^\tau\eta_{1,2}^2(\tau',x;\omega)d^4xd\tau'\right].$$

Remark 2.2. In all this we have to remember, of course, that once we set $\tau_1\to\infty$ we have also to extend the interval of integration from $[0,\tau]$ to $[0,\infty]$.

From Eq.(2.21) with $\epsilon\ll 1$ for two-point correlation function $\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle$ defined by

$$\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle = \lim_{\epsilon\to0,\gamma\to0}\mathbf{E}_\Sigma\left[\frac{\delta^l Z_\epsilon[J_1,J_2;\omega]}{\delta J_1(\tau_1,x_1)\delta J_2(\tau_2,x_2)}\right] \tag{2.21$'$}$$

we get

$$\langle\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\rangle\simeq$$
$$\lim_{\epsilon\to0,\gamma\to0}N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta_{1,2}(\tau',x;\omega)](\mathbf{E}_\Sigma[\varphi_{1,2}(\tau_1,x_1;\omega)\varphi_{1,2}(\tau_2,x_2;\omega)])\times$$
$$\delta(\varphi_{1,2}(0,x;\omega))\times$$
$$\times\exp\left\{-\left[\int_0^\tau\frac{1}{4\epsilon}\left[\frac{\partial\varphi_1(\tau',x;\omega)}{\partial\tau'}+\left.\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_1(\tau',x;\omega)}\right|_{\varphi=\varphi_1}-\eta(\tau',x;\omega)\right]^2-\right.\right.$$
$$\left.\left.\frac{\epsilon}{2}\left.\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_1^2(\tau',x;\omega)}\right|_{\varphi=\varphi_1}\right]d^4xd\tau'\right\}\times\exp\left[-\frac{1}{4}\int_0^{\tau_1}\eta^2(\tau',x;\omega)d^4xd\tau'\right] \tag{2.22}$$
$$\times\exp\left\{-\left[\int_0^{\tau_2}\frac{1}{4\epsilon}\left[\frac{\partial\varphi_2(\tau',x;\omega)}{\partial\tau'}+\left.\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_2(\tau',x;\omega)}\right|_{\varphi=\varphi_2}-\eta(\tau',x;\omega)\right]^2-\right.\right.$$
$$\left.\left.\frac{\epsilon}{2}\frac{\delta^2\widetilde{S}}{\delta\varphi_2^2(\tau',x;\omega)}\right]d^4xd\tau'\right\}\times\exp\left[-\frac{1}{4}\int_0^{\tau_2}\eta^2(\tau',x;\omega)d^4xd\tau'\right]$$

From Eq.(2.22) by using saddle point approximation we get

$$
\begin{aligned}
&\langle\varphi_\eta(\tau_1,x_1;\omega)\varphi_\eta(\tau_2,x_2;\omega)\rangle_\eta = \\
&\lim_{\epsilon\to 0,\gamma\to 0} N\int D[\eta(\tau',x;\omega)](\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega))\times \\
&\times\exp\left\{-\int_0^\tau\left[-\frac{\epsilon}{2}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2(\tau',x;\omega)}\bigg|_{\varphi=\varphi_{1,2,\eta}}\right]d^4xd\tau'\right\}\times \\
&\exp\left\{-\int_0^\tau\left[-\frac{\epsilon}{2}\frac{\delta^2\widetilde{S}_{1,2}}{\delta\varphi_{1,2}^2(\tau',x;\omega)}\bigg|_{\varphi_{1,2}=\varphi_{1,2,\eta}}\right]d^4xd\tau'\right\}\times \\
&\times\exp\left[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4xd\tau'\right] = \\
&=\lim_{\gamma\to 0}\langle\varphi_{1,\eta}(\tau_1,x_1;\omega)\varphi_{2,\eta}(\tau_2,x_2;\omega)\rangle_\eta
\end{aligned}
\tag{2.23}
$$

$\varphi_\eta(\tau_1,x_1;\omega)$ that appears in Eq.(2.23) is the solution of the Langevin equation (2.26), solved with zero initial condition. From Eq.(2.22)-Eq.(2.23) we get

$$
\begin{aligned}
&\langle\varphi_\eta(\tau_1,x_1;\omega)\varphi_\eta(\tau_2,x_2;\omega)\rangle_\eta \simeq \\
&\lim_{\gamma\to 0} N\int D[\varphi_{1,2}(\tau',x;\omega)]D[\eta_{1,2}(\tau',x;\omega)](\varphi_{1,2}(\tau_1,x_1;\omega)\varphi_{1,2}(\tau_2,x_2;\omega))\delta(\varphi_{1,2}(0,x;\omega))\times \\
&\times\exp\left[-\frac{1}{4}\int_0^\tau\eta^2(\tau',x;\omega)d^4xd\tau'\right]\times \\
&\times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon}\left[\frac{\partial\varphi_{1,2}(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau',x;\omega)}-\eta_{1,2}(\tau',x;\omega)\right]^2\right]d^4xd\tau'\right\}+ \\
&+O(\epsilon)\simeq \\
&\lim_{\gamma\to 0} N\int D[\varphi_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta\times \\
&\times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon}\left[\frac{\partial\varphi_{1,2}(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau',x;\omega)}-\eta_{1,2}(\tau',x;\omega)\right]^2\right]d^4xd\tau'\right\}+ \\
&+O(\epsilon).
\end{aligned}
\tag{2.24}
$$

From Eq.(2.24) finally we get

$$
\begin{aligned}
&\langle\varphi_\eta(\tau_1,x_1;\omega)\varphi_\eta(\tau_2,x_2;\omega)\rangle_\eta \simeq \\
&\lim_{\gamma\to 0} N\int D[\varphi_{1,2}(\tau',x;\omega)]\delta(\varphi_{1,2}(0,x;\omega))\langle\varphi_1(\tau_1,x_1;\omega)\varphi_2(\tau_2,x_2;\omega)\rangle_\eta\times \\
&\times\exp\left\{-\int_0^\tau\left[\frac{1}{4\epsilon}\left[\frac{\partial\varphi_{1,2}(\tau',x;\omega)}{\partial\tau'}+\frac{\delta\widetilde{S}_{1,2}}{\delta\varphi_{1,2}(\tau',x;\omega)}-\eta(\tau',x;\omega)\right]^2\right]d^4xd\tau'\right\}+O(\epsilon).
\end{aligned}
\tag{2.25}
$$

Remark 2.3.Thus by using saddle point approximation one obtains canonical result.
Remark 2.4. In this paper we calculate path integral in Eq.(2.22) nonperturbatively without any refference to saddle point approximation and without any refference to perturbative solution of the canonical Langevin equations (2.26).
Definition 2.1. Let $\varphi_{1,2,\eta}(\tau,x;\omega)$ be the solutions of the canonical Langevin equations (1.26)

$$\frac{\partial\varphi_{1,2}(\tau,x;\omega)}{\partial\tau} = \frac{\delta S[\varphi_{1,2}]}{\delta\varphi_{1,2}(\tau,x;\omega)} + \eta_{1,2}(\tau,x;\omega),$$
$$\varphi_{1,2}(0,x;\omega) = 0, \tag{2.26}$$

and let be $\varphi_\epsilon(\tau,x;\omega,\varpi)$ the solution of the double stochastic Langevin equations (2.27)

$$\frac{\partial\varphi_{1,2,\epsilon}(\tau,x;\omega,\varpi)}{\partial\tau} = \frac{\delta\widetilde{S}[\varphi_{1,2,\epsilon}]}{\delta\varphi_{1,2\epsilon}(\tau,x;\omega,\varpi)} + \eta_{1,2}(\tau,x;\omega) + \sqrt{\epsilon}\,\widetilde{\eta}_{1,2}(\tau,x;\varpi), \tag{2.27}$$

where $\eta(\tau,x;\omega)$ is a space-time white noice on probability space $\Sigma = (\Omega,\mathcal{S},P)$ and $\widetilde{\eta}(\tau,x;\varpi)$ is a space-time white noice on probability space $\widetilde{\Sigma} = (\widetilde{\Omega},\mathcal{S},P)$.
We say that: (i) $\xi_s(\tau,x)$ is a strong asimptotical solution of the double stochastic Langevin equations (2.27) if

$$\lim_{\epsilon\to 0} |\mathbf{E}_\Sigma\mathbf{E}_{\widetilde{\Sigma}}[\varphi_{1,2,\epsilon}(\tau,\varphi x;\omega,\varpi) - \xi_{1,2,s}(\tau,x)]| = 0 \tag{2.28}$$

(ii) $\zeta_w(\tau,x)$ is a weak asimptotical solution of the double stochastic Langevin equations (2.27) if

$$\liminf_{\epsilon\to 0} |\mathbf{E}_\Sigma\mathbf{E}_{\widetilde{\Sigma}}[\varphi_{1,2,\epsilon}(\tau,x;\omega,\varpi) - \zeta_{1,2,w}(\tau,x)]| = 0. \tag{2.29}$$

By the replacement

$$\varphi_{1,2,\epsilon}(x,\tau;\omega,\varpi) = v_{1,2,\epsilon}(x,\tau;\omega,\varpi) \pm \theta(\tau)a,$$
$$\theta(\tau) = \begin{cases} 0 & \text{if } \tau \leq 0 \\ 1 & \text{if } \tau > 0 \end{cases} \tag{2.30}$$

we obtain from Eqs.(2.27)

$$\frac{\partial v_{1,2,\epsilon}(\tau,x;\omega,\varpi)}{\partial\tau} = -a\delta(\tau) + \frac{\delta\widetilde{S}_{1,2}[v_{1,2,\epsilon},a]}{\delta v_{1,2\epsilon}(\tau,x;\omega,\varpi)} + \eta_{1,2}(\tau,x;\omega) + \sqrt{\epsilon}\,\widetilde{\eta}_{1,2}(\tau,x;\varpi), \tag{2.31}$$

where we set

$$\frac{\delta\widetilde{S}_{1,2}[v_{1,2,\epsilon},a]}{\delta v_{1,2\epsilon}(\tau,x;\omega,\varpi)} = \left.\frac{\delta\widetilde{S}_{1,2}[\varphi_{1,2\epsilon}]}{\delta\varphi_{1,2,\epsilon}(\tau,x;\omega,\varpi)}\right|_{\varphi_{1,2,\epsilon}=v_{1,2,\epsilon}(x,\tau;\omega,\varpi)\pm a} \tag{2.32}$$

Definition 2.2.Let $\mathcal{L}_{1,2}(\partial^2 v_\epsilon, v_\epsilon, a)$ be linear part of variational derivative $\delta\widetilde{S}_{1,2}[v_\epsilon,a]/\delta v_{1,2\epsilon}(\tau,x;\omega,\varpi)$. Linear stochastic differental *master equation* corresponding to the double stochastic Langevin equations (2.27) reads

$$\frac{\partial v_{1,2}(\tau,x,\mp a;\omega)}{\partial\tau} = \pm a\delta(\tau) + \mathcal{L}_{1,2}(\partial^2 v_{1,2}(\tau,x,\mp a;\omega), v_{1,2}(\tau,x,\mp a;\omega), a) + \eta(\tau,x;\omega). \tag{2.33}$$

Definition 2.3.Let $v_{1,2}(\tau,x,\mp a;\omega)$ be the solution of the stochastic differental *master equations* (2.33). *Transcendental master equations* corresponding to the double stochastic Langevin equation (2.33) reads

$$\mathbf{E}_\Sigma[v_{1,2}(\tau,x,\mp a;\omega)] = 0. \tag{2.34}$$

Theorem 2.1.Let $\mp a(\bar{\tau},\bar{x})$ be an solutions of the equations (2.34) at fixed point $(\bar{\tau},\bar{x}) \in \mathbb{R}_+ \times \mathbb{R}^4$ i.e.,

$$\mathbf{E}_\Sigma[v_{1,2}(\bar{\tau},\bar{x},\mp a(\bar{\tau},\bar{x});\omega)] = 0. \tag{2.35}$$

Let $\Delta(\bar{\tau},\bar{x})$ be a set such that

$$a(\bar{\tau},\bar{x}) \in \Delta(\bar{\tau},\bar{x}) \Leftrightarrow \mathbf{E}_{\Sigma}[v(\bar{\tau},\bar{x},\mp a(\bar{\tau},\bar{x});\omega)] = 0, \tag{2.36}$$

let $\Delta_s(\bar{\tau},\bar{x})$ be a set such that

$$\lim_{\epsilon\to 0} |\mathbf{E}_{\Sigma}\mathbf{E}_{\widetilde{\Sigma}}[\varphi_\epsilon(\bar{\tau},\bar{x};\omega,\varpi) - \xi_s(\bar{\tau},\bar{x})]| = 0. \tag{2.37}$$

and let $\Delta_w(\bar{\tau},\bar{x})$ be a set such that

$$\liminf_{\epsilon\to 0} |\mathbf{E}_{\Sigma}\mathbf{E}_{\widetilde{\Sigma}}[\varphi_\epsilon(\tau,x;\omega,\varpi) - \zeta_w(\tau,x)]| = 0. \tag{2.38}$$

Then $\Delta_s(\bar{\tau},\bar{x}) \subseteq \Delta(\bar{\tau},\bar{x})$ and $\Delta_w(\bar{\tau},\bar{x}) \subseteq \Delta(\bar{\tau},\bar{x})$.

Example 2.1.We consider as example the free case:

$$\frac{\partial}{\partial\tau}\varphi_\epsilon(\tau,x) = -i(\Box\varphi_\epsilon(\tau,x) + m^2\varphi_\epsilon(\tau,x)) + \eta(\tau,x) + \epsilon\widetilde{\eta}(\tau,x). \tag{2.39}$$

Differental *master equation* (2.33) corresponding to the double stochastic Langevin equations (2.39) reads

$$\begin{gathered}\frac{\partial v_{1,2}(\tau,x,\mp a;\omega)}{\partial\tau} = \\ = \pm a\delta(\tau) - i(\partial^2 v_{1,2}(\tau,x,\mp a;\omega) + m^2(v_{1,2}(\tau,x,\mp a;\omega) \mp a)) + \eta(\tau,x;\omega).\end{gathered} \tag{2.33}$$

From Eq.(2.33) we get

$$\frac{\partial v_1(\tau,x,-a;\omega)}{\partial\tau} = a\delta(\tau) - i(\partial^2 v_1(\tau,x,-a;\omega) + m^2 v_1(\tau,x,-a;\omega)) + im^2 a + \eta(\tau,x;\omega) \tag{2.34}$$

$$\frac{\partial v_2(\tau,x,a;\omega)}{\partial\tau} = -a\delta(\tau) - i(\partial^2 v_2(\tau,x,a;\omega) + m^2 v_2(\tau,x,a;\omega)) - im^2 a + \eta(\tau,x;\omega) \tag{2.35}$$

From Eqs.(2.34)-(2.35) we get

$$\begin{gathered}\frac{\partial \widehat{v}_1(\tau,p,-a;\omega)}{\partial\tau} = \\ a\delta(\tau)\delta(p) + i(p^2 v_1(\tau,p,-a;\omega) - m^2\widehat{v}_1(\tau,x,-a;\omega)) + im^2 a\delta(p) + \widehat{\eta}(\tau,p;\omega)\end{gathered} \tag{2.36}$$

$$\begin{gathered}\frac{\partial \widehat{v}_2(\tau,p,a;\omega)}{\partial\tau} = \\ -a\delta(\tau)\delta(p) + i(p^2\widehat{v}_2(\tau,p,a;\omega) - m^2\widehat{v}_2(\tau,p,a;\omega)) - im^2 a\delta(p) + \widehat{\eta}(\tau,p;\omega)\end{gathered} \tag{2.37}$$

$$\begin{gathered}\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]dt = \exp[i(p^2-m^2)\tau]\int_0^\tau \exp[-i(p^2-m^2)t]dt = \\ \frac{1}{-i(p^2-m^2)}\exp[i(p^2-m^2)\tau]\{\exp[-i(p^2-m^2)\tau]-1\} = \\ \frac{1}{-i(p^2-m^2)}\{1-\exp[i(p^2-m^2)\tau]\}\end{gathered} \tag{2.38}$$

The unique solutions of Eqs.(2.36)-(2.37) reads

$$\begin{aligned}
\widehat{v}_1(\tau,p,-a;\omega) &= a\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\delta(t)\delta(p)dt+\\
&+\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dt+im^2a\delta(p)\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]dt=\\
&= a\delta(p)\exp[i(p^2-m^2)\tau]+\frac{im^2a\delta(p)}{-i(p^2-m^2)}(1-\exp[i(p^2-m^2)\tau])+\\
&+\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dt=\\
&= a\delta(p)\exp[i(p^2-m^2)(\tau)]-\frac{m^2a\delta(p)}{(p^2-m^2)}(1-\exp[i(p^2-m^2)\tau])+\\
&+\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dt
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
\widehat{v}_2(\tau,p,a;\omega) &= -a\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\delta(t)\delta(p)dt+\\
&+\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dt-im^2a\delta(p)\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]dt=\\
&= -a\delta(p)\exp[i(p^2-m^2)\tau]+\frac{-im^2a\delta(p)}{-i(p^2-m^2)}(1-\exp[i(p^2-m^2)\tau])+\\
&+\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dt=\\
&-a\delta(p)\exp[i(p^2-m^2)\tau]+\frac{m^2a\delta(p)}{(p^2-m^2)}(1-\exp[i(p^2-m^2)\tau])+\\
&+\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dt.
\end{aligned} \tag{2.40}$$

From Eq.(2.39) we obtain

$$
\begin{gathered}
v_1(\tau,x;\omega) = \int e^{ipx}\widehat{v}_1(\tau,p,-a;\omega)dp = \\
= a\int e^{ipx}\delta(p)\exp[i(p^2-m^2)\tau]dp - m^2 a\int e^{ipx}\frac{\delta(p)}{(p^2-m^2)}(1-\exp[i(p^2-m^2)\tau])dp + \\
+\int_0^{\tau}\int e^{ipx}\exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dpdt = \\
a\exp[-im^2\tau] + a - a\exp[-im^2\tau] + \\
+\int_0^{\tau}\int e^{ipx}\exp[i(p^2-m^2)(\tau-t)]\widehat{\eta}(t,p;\omega)dpdt
\end{gathered} \tag{2.41}
$$

From Eq.(2.40) we obtain

$$
\begin{gathered}
v_2(\tau',x';\omega) = \int e^{ip'x}\widehat{v}_1(\tau',p',-a;\omega)dp' = \\
= -a\int e^{ip'x'}\delta(p')\exp[i(p'^2-m^2)\tau']dp' + \\
m^2 a\int e^{ip'x'}\frac{\delta(p')}{\left(p'^2-m^2\right)}\left(1-\exp\left[i\left(p'^2-m^2\right)\tau'\right]\right)dp' + \\
+\int_0^{\tau'}\int e^{ip'x'}\exp\left[i\left(p'^2-m^2\right)(\tau'-t)\right]\widehat{\eta}(t,p';\omega)dpdt = \\
-a\exp[-im^2\tau'] - a + a\exp[-im^2\tau'] + \\
\int_0^{\tau'}\int e^{ipx}\exp\left[i\left(p'^2-m^2\right)(\tau'-t)\right]\widehat{\eta}(t,p';\omega)dp'dt,
\end{gathered} \tag{2.42}
$$

From Eqs.(2.41)-(2.42) we obtain

$$
\begin{aligned}
&\langle v_1(\tau,x;\omega)v_2(\tau',x';\omega)\rangle_\eta = \\
&\left\langle\left(a\exp[-im^2\tau]+a-m^2a\exp[-im^2\tau]+\int_0^\tau\int e^{ipx}\exp[i(p^2-m^2)(\tau-t)]\hat{\eta}(t,p;\omega)dpdt\right)\times\right.\\
&\left.\times\left(-a\exp[-im^2\tau']-a+a\exp[-im^2\tau']+\int_0^{\tau'}\int e^{ipx}\exp\left[i\left(p'^2-m^2\right)(\tau'-t')\right]\hat{\eta}(t,p';\omega)dp'dt'\right)\right\rangle_\eta = \\
&= -(a\exp[-im^2\tau]+a-m^2a\exp[-im^2\tau])(a\exp[-im^2\tau']+a-m^2a\exp[-im^2\tau'])+\\
&\left\langle\int_0^\tau\int e^{ipx}\exp[i(p^2-m^2)(\tau-t)]\hat{\eta}(t,p;\omega)dpdt\times\right.\\
&\left.\int_0^{\tau'}\int e^{ipx}\exp\left[i\left(p'^2-m^2\right)(\tau'-t')\right]\hat{\eta}(t,p';\omega)dp'dt'\right\rangle_\eta = \\
&-(a\exp[-im^2\tau]+a-m^2a\exp[-im^2\tau])(a\exp[-im^2\tau']+a-m^2a\exp[-im^2\tau'])+\\
&+i(2\pi)^4\delta^4(p+p')\frac{1}{p^2-m^2}\{\exp[i(p^2-m^2)|\tau-\tau'|]-\exp[i(p^2-m^2)|\tau+\tau'|]\}.
\end{aligned}
\tag{2.43}
$$

From Eq.(2.43) and Eqs.(1.9)-(1.9$'$) we obtain

$$
w\text{-}\lim_{\tau=\tau'\to\infty}\langle v_1(\tau,x;\omega)v_2(\tau',x';\omega)\rangle_\eta = -a^2+i(2\pi)^{-4}\int_{\mathbb{R}^4}\frac{e^{-ip(x-x')}d^4p}{p^2-m^2+0i}. \tag{2.44}
$$

Thus master equation corresponding to two point function $\tau(x-x')$,where $a^2(x,x') = a^2(x-x') = \tau(x-x')$, reads

$$
\begin{aligned}
&a^2(x-x')-i(2\pi)^{-4}\int_{\mathbb{R}^4}\frac{e^{-ip(x-x')}d^4p}{p^2-m^2+i\varepsilon} = 0,\\
&a^2(x-x') = i(2\pi)^{-4}\int_{\mathbb{R}^4}\frac{e^{-ip(x-x')}d^4p}{p^2-m^2+i\varepsilon} = \\
&= i\Delta_F(x-x') = \tau(x-x')\\
&\Delta_F(x-x') = \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-x')}d^4p}{p^2-m^2+i\varepsilon}
\end{aligned}
\tag{2.45}
$$

and therefore we obtain desired result, see Eq.(1.10$'$).
In order to prove the equality $a^2(x-x') = \tau(x-x')$ we calculate $\Delta_F(x-x')$ defined in Eq.(2.45):

$$
\begin{aligned}
&\Delta_F(x-x') = \int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}d^4k}{k^2-m^2+i\varepsilon} = \int_{\mathbb{R}^4}\frac{d^3\mathbf{k}dk_0}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k_0^2-(\mathbf{k}^2+m^2)+i\varepsilon} = \\
&= \int_{\mathbb{R}^4}\frac{d^3\mathbf{k}dk_0}{(2\pi)^4}\frac{e^{-ik(x-x')}}{2\omega_{\mathbf{k}}}\left\{\frac{1}{k_0-\omega_{\mathbf{k}}+i\delta}-\frac{1}{k_0+\omega_{\mathbf{k}}-i\delta}\right\},\\
&\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}.
\end{aligned}
\tag{2.46}
$$

The integral over k_0 here, as always, can be easily calculated by contour integration. The exponent contains $e^{-ik_0(x_0-x_0')}$, so for $x_0 - x_0' > 0$, we close the integration contour in the lower half-plane of k_0 , and the integral is determined by the contribution from the pole at $k_0 = \omega_{\mathbf{k}} - i\delta$. For $x_0 - x_0' < 0$, we close the integration contour in the upper half-plane, and everything is determined by the contribution from the pole at $k_0 = -\omega_{\mathbf{k}} + i\delta$. Then, by Cauchy's theorem, we have:

$$\begin{aligned}\Delta_F(x-x') = \int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{2\omega_{\mathbf{k}}}\left[i\theta(x_0-x_0')e^{-i\omega_{\mathbf{k}}(x_0-x_0')}\right] - \\ -\int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{2\omega_{\mathbf{k}}}\left[i\theta(x_0'-x_0)e^{i\omega_{\mathbf{k}}(x_0-x_0')}\right]\end{aligned} \tag{2.47}$$

After the substitution in the second integral $\mathbf{k} \to -\mathbf{k}$ we have:

$$\begin{aligned}\Delta_F(x-x') = -i\left\{\int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left[\theta(x_0-x_0')e^{-i\omega_{\mathbf{k}}(x_0-x_0')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right] + \right. \\ \left. +\int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left[\theta(x_0'-x_0)e^{i\omega_{\mathbf{k}}(x_0-x_0')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right]\right\}\end{aligned} \tag{2.48}$$

Finally we get

$$i\Delta_F(x-x') = \tau(x-x'). \tag{2.49}$$

3. Double stochastic quantization self-interacting scalar field $g\varphi^4$

As an example we consider the double stochastic self-interacting scalar field with lagrangian

$$\mathcal{L}_M = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}g\varphi^4. \tag{3.1}$$

The corresponding double stochastic Langevin equation in 4-D Minkowski spacetime is

$$\frac{\partial\varphi(x,\tau;\omega,\varpi)}{\partial\tau} = -i(\Box\varphi(x,\tau;\omega,\varpi) + m^2\varphi(x,\tau;\omega,\varpi) + g\varphi^3(x,\tau;\omega,\varpi)) + \eta(\omega) + \epsilon\tilde{\eta}(\varpi). \tag{3.2}$$

The random source η is a gaussian white noise with correlation function

$$\begin{aligned}\langle\eta(x,\tau)\eta(x',\tau')\rangle_\eta = 2(2\pi)^4\delta(\tau-\tau')\delta(x-x'), \\ \langle\eta(p,\tau)\eta(p',\tau')\rangle_\eta = 2(2\pi)^4\delta(\tau-\tau')\delta(p+p').\end{aligned} \tag{3.3}$$

By replacement $\varphi(x,\tau;\omega,\varpi) - a(\tau) = v_1(x,\tau,-a;\omega,\varpi)$, where $a(\tau) = 0$ if $\tau \le 0, a(\tau) = a$ if $\tau > 0$, from Eq.(3.2) we obtain

$$\begin{aligned}&\frac{\partial v_1(x,\tau,-a;\omega,\varpi)}{\partial \tau} = \\ &a\delta(\tau) - i\left\{\Box v_1(x,\tau,-a;\omega,\varpi) + m^2[v_1(x,\tau,-a;\omega,\varpi)+a] + g[v_1(x,\tau;\omega,\varpi)+a]^3\right\} + \\ &+\eta(\omega) + \epsilon\widetilde{\eta}(\varpi) = \\ &= a\delta(\tau) - i\{\Box v_1 + m^2 v_1 + m^2 a + g v_1^3 + 3gav_1^2 + 3ga^2 v_1 + ga^3\} + \eta(\omega) + \epsilon\widetilde{\eta}(\varpi) = \\ &a\delta(\tau) - i\{\Box v_1 + (m^2+3ga^2)v_1 + m^2 a + ga^3\} + \eta(\omega) + \epsilon\widetilde{\eta}(\varpi)\end{aligned} \quad . \quad (3.4)$$

By replacement $\varphi(x,\tau;\omega,\varpi) + a(\tau) = v_1(x,\tau,a;\omega,\varpi)$, where $a(\tau) = 0$ if $\tau \leq 0, a(\tau) = a$ if $\tau > 0$, from Eq.(3.2) we obtain

$$\begin{aligned}&\frac{\partial v_2(x,\tau,a;\omega,\varpi)}{\partial \tau} = \\ &-a\delta(\tau) - i\left\{\Box v_2(x,\tau,-a;\omega,\varpi) + m^2[v_2(x,\tau,-a;\omega,\varpi)-a] + g[v_2(x,\tau;\omega,\varpi)-a]^3\right\} + \\ &+\eta(\omega) + \epsilon\widetilde{\eta}(\varpi) = \\ &= -a\delta(\tau) - i\{\Box v_2 + m^2 v_2 - m^2 a + g v_2^3 - 3gav_2^2 + 3ga^2 v_2 + ga^3\} + \eta(\omega) + \epsilon\widetilde{\eta}(\varpi) = \\ &= -a\delta(\tau) - i\{\Box v_2 + (m^2+3ga^2)v_2 - m^2 a - ga^3\} + \eta(\omega) + \epsilon\widetilde{\eta}(\varpi).\end{aligned} \quad . \quad (3.5)$$

Therefore linear differential master equations reads

$$\begin{aligned}&\frac{\partial v_1(x,\tau,-a;\omega,\varpi)}{\partial \tau} = \\ &= a\delta(\tau) - i\{\Box v_1 + (m^2+3ga^2)v_1 + m^2 a + ga^3\} + \eta(\omega)\end{aligned} \quad (3.6)$$

and

$$\begin{aligned}&\frac{\partial v_2(x,\tau,a;\omega,\varpi)}{\partial \tau} = \\ &= -a\delta(\tau) - i\{\Box v_2 + (m^2+3ga^2)v_2 - m^2 a - ga^3\} + \eta(\omega).\end{aligned} \quad (3.7)$$

From Eqs.(3.6)-(3.7) we obtain

$$\begin{aligned}&\frac{\partial \widehat{v}_1(p,\tau,-a;\omega)}{\partial \tau} = \\ &= a\delta(\tau) - i\{-p^2\widehat{v}_1 + (m^2+3ga^2)\widehat{v}_1 + (m^2 a + ga^3)\delta(p)\} + \widehat{\eta}(\tau,p;\omega) = \\ &= a\delta(\tau) + i[p^2 - (m^2+3ga^2)]\widehat{v}_1 - i(m^2 a + ga^3)\delta(p) + \widehat{\eta}(\tau,p;\omega)\end{aligned} \quad (3.6)$$

and

$$\begin{aligned}&\frac{\partial \widehat{v}_2(p,\tau,-a;\omega)}{\partial \tau} = \\ &= -a\delta(\tau) - i\{-p^2\widehat{v}_2 + (m^2+3ga^2)\widehat{v}_2 - (m^2 a + ga^3)\delta(p)\} + \widehat{\eta}(\tau,p;\omega) = \\ &-a\delta(\tau) + i[p^2 - (m^2+3ga^2)]\widehat{v}_2 + i(m^2 a + ga^3)\delta(p) + \widehat{\eta}(\tau,p;\omega)\end{aligned} \quad (3.7)$$

where

$$\begin{aligned}
\mathcal{F}[1](p) &= (2\pi)^4\delta(p),\\
\mathcal{F}[v_1(\tau,x)](p) &= \widehat{\varphi}_\epsilon(\tau,p) = \int\limits_{\mathbb{R}^4} v_1(\tau,x)e^{ipx}d^4x,\\
\mathcal{F}^{-1}[\widehat{v}_1(\tau,p)](x) &= (2\pi)^{-4}\int\limits_{\mathbb{R}^4}\widehat{v}_1(\tau,p)e^{-ipx}d^4p,
\end{aligned} \tag{3.8}$$

see ref.[4].The unique solutions of Eqs.(3.6)-(3.7) reads

$$\begin{gathered}
\widehat{v}_1(\tau,p,-a;\omega) = (2\pi)^4 a\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\delta(t)\delta(p)dt+\\
+\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\widehat{\eta}(t,p;\omega)dt+\\
+i(m^2a+ga^3)(2\pi)^4\delta(p)\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]dt=\\
= a\delta(p)\exp[i(p^2-(m^2+3ga^2))\tau]+\\
\frac{i(m^2a+ga^3)(2\pi)^4\delta(p)}{-i(p^2-(m^2+3ga^2))}(1-\exp[i(p^2-(m^2+3ga^2))\tau])+\\
+\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\widehat{\eta}(t,p;\omega)dt=\\
=(2\pi)^4 a\delta(p)\exp[i(p^2-(m^2+3ga^2))\tau]-\\
-\frac{(m^2a+ga^3)(2\pi)^4\delta(p)}{(p^2-(m^2+3ga^2))}(1-\exp[i(p^2-(m^2+3ga^2))\tau])+\\
+\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\widehat{\eta}(t,p;\omega)dt
\end{gathered} \tag{3.9}$$

and

$$
\begin{aligned}
\hat{v}_2(\tau,p,a;\omega) &= -(2\pi)^4 a \int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\delta(t)\delta(p)dt + \\
&+\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\hat{\eta}(t,p;\omega)dt - \\
&-i(m^2+3ga^2)(2\pi)^4 a\delta(p)\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]dt = \\
&= -a(2\pi)^4\delta(p)\exp[i(p^2-(m^2+3ga^2))\tau] + \\
&\frac{-i(m^2a+ga^3)(2\pi)^4\delta(p)}{-i(p^2-(m^2+3ga^2))}(1-\exp[i(p^2-m^2)\tau]) + \\
&+\int_0^\tau \exp[i(p^2-m^2)(\tau-t)]\hat{\eta}(t,p;\omega)dt = \\
&-(2\pi)^4 a\delta(p)\exp[i(p^2-(m^2+3ga^2))\tau] + \\
&+\frac{(m^2a+ga^3)(2\pi)^4\delta(p)}{(p^2-(m^2+3ga^2))}(1-\exp[i(p^2-(m^2+3ga^2))\tau]) + \\
&+\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\hat{\eta}(t,p;\omega)dt,
\end{aligned} \tag{3.10}
$$

where we have used

$$
\begin{aligned}
&\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]dt = \\
&\exp[i(p^2-(m^2+3ga^2))\tau]\int_0^\tau \exp[-i(p^2-(m^2+3ga^2))t]dt = \\
&\frac{1}{-i(p^2-(m^2+3ga^2))}\exp[i(p^2-(m^2+3ga^2))\tau]\times \\
&\times\{\exp[-i(p^2-(m^2+3ga^2))\tau]-1\} = \\
&\frac{1}{-i(p^2-(m^2+3ga^2))}\{1-\exp[i(p^2-(m^2+3ga^2))\tau]\}.
\end{aligned} \tag{3.11}
$$

From Eqs.(3.9)-(3.10) we obtain

$$\mathcal{F}^{-1}[\hat{v}_1(\tau,p,-a;\omega)] = \int_{\mathbb{R}^4} \hat{v}_1(\tau,p,-a;\omega)e^{-ipx}d^4p = v_1(\tau,x,-a;\omega) =$$

$$a\int_{\mathbb{R}^4}\delta(p)\exp[i(p^2-(m^2+3ga^2))\tau]e^{-ipx}d^4p -$$

$$-(m^2a+ga^3)\times$$

$$\times\int_{\mathbb{R}^4}\frac{1}{(p^2-(m^2+3ga^2))}\delta(p)(1-\exp[i(p^2-(m^2+3ga^2))\tau])e^{-ipx}d^4p +$$

$$+(2\pi)^{-4}\int_{\mathbb{R}^4}e^{-ipx}d^4p\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\hat{\eta}(t,p;\omega)dt = \tag{3.12}$$

$$= a\exp[-i(m^2+3ga^2)\tau] +$$

$$+\frac{(2\pi)^{-4}(m^2a+ga^3)}{(m^2+3ga^2)}(1-\exp[-i(m^2+3ga^2)\tau]) +$$

$$+(2\pi)^{-4}\int_{\mathbb{R}^4}e^{-ipx}d^4p\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\hat{\eta}(t,p;\omega)dt$$

and

$$\mathcal{F}^{-1}\left[\hat{v}_2\left(\tau,p,\ a;\omega\right)\right] = (2\pi)^{-4}\int_{\mathbb{R}^4}\hat{v}_2\left(\tau,p,\ a;\omega\right)e^{-ipx}d^4p = v_2\left(\tau,x,\ a;\omega\right) =$$

$$-a\int_{\mathbb{R}^4}\delta(p)\exp[i(p^2-(m^2+3ga^2))\tau]e^{-ipx}d^4p +$$

$$+(m^2a+ga^3)\times$$

$$\times\int_{\mathbb{R}^4}\frac{1}{(p^2-(m^2+3ga^2))}\delta(p)(1-\exp[i(p^2-(m^2+3ga^2))\tau])e^{-ipx}d^4p +$$

$$+(2\pi)^{-4}\int_{\mathbb{R}^4}e^{-ipx}d^4p\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\hat{\eta}(t,p;\omega)dt = \tag{3.13}$$

$$= -a\exp[-i(m^2+3ga^2)\tau] -$$

$$-\frac{(m^2a+ga^3)}{(m^2+3ga^2)}(1-\exp[-i(m^2+3ga^2)\tau]) +$$

$$+(2\pi)^{-4}\int_{\mathbb{R}^4}e^{-ipx}d^4p\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\hat{\eta}(t,p;\omega)dt$$

From Eqs.(3.12)-(3.13) we obtain

$$\begin{aligned}
&\langle v_1(\tau,x,-a;\omega)v_2(\tau',x',a;\omega)\rangle_\eta = \\
&\left\langle \left\{ a\exp[-i(m^2+3ga^2)\tau] + \frac{(m^2a+ga^3)}{(m^2+3ga^2)}(1-\exp[-i(m^2+3ga^2)\tau]) + \right.\right.\\
&\left. +(2\pi)^{-4}\int_{\mathbb{R}^4} e^{-ipx}d^4p\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\widehat{\eta}(t,p;\omega)dt \right\} \times \\
&\left\{ -a\exp[-i(m^2+3ga^2)\tau] - \frac{(m^2a+ga^3)}{(m^2+3ga^2)}(1-\exp[-i(m^2+3ga^2)\tau]) + \right.\\
&\left.\left. +(2\pi)^{-4}\int_{\mathbb{R}^4} e^{-ip'x}d^4p\int_0^\tau \exp[i(p'^2-(m^2+3ga^2))(\tau-t)]\widehat{\eta}(t,p';\omega)dt \right\} \right\rangle_\eta .
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&w\text{-}\lim_{\tau=\tau'\to\infty}\langle v_1(\tau,x,-a;\omega)v_2(\tau',x',a;\omega)\rangle_\eta = \\
&-\frac{(m^2a+ga^3)^2}{(m^2+3ga^2)^2} + \\
&(2\pi)^{-8}\left\langle \int_{\mathbb{R}^4} e^{-ipx}d^4p\int_0^\tau \exp[i(p^2-(m^2+3ga^2))(\tau-t)]\widehat{\eta}(t,p;\omega)dt \right.\\
&\left. \int_{\mathbb{R}^4} e^{-ip'x}d^4p\int_0^\tau \exp[i(p'^2-(m^2+3ga^2))(\tau-t)]\widehat{\eta}(t,p';\omega)dt \right\rangle_\eta .
\end{aligned} \tag{3.15}$$

Thus transcendental master equation corresponding to two point function $a^2(x,x') = a^2(x-x') = \Delta(x-x')$ reads

$$\begin{gathered}
\frac{(m^2a+ga^3)^2}{(m^2+3ga^2)^2} - i(2\pi)^{-4}\int_{\mathbb{R}^4}\frac{e^{-ip(x-x')}d^4p}{p^2-m^2+i\varepsilon+3ga^2} = 0. \\
\frac{(m^2a+ga^3)^2}{(m^2+3ga^2)^2} - i\Delta_F(x-x';a) = 0 \\
\Delta_F(x-x';a^2) = (2\pi)^{-4}\int_{\mathbb{R}^4}\frac{e^{-ip(x-x')}d^4p}{p^2-m^2+i\varepsilon+3ga^2}
\end{gathered} \tag{3.16}$$

Note that

$$\frac{(m^2a+ga^3)^2}{(m^2+3ga^2)^2} = \left(\frac{m^2a+ga^3}{m^2+3ga^2}\right)^2 = \frac{a+\frac{ga^3}{m^2}}{1+\frac{3ga^2}{m^2}} = a^2\frac{\left(1+\frac{ga^2}{m^2}\right)^2}{\left(1+\frac{3ga^2}{m^2}\right)^2} \tag{3.17}$$

We calculate now the integral $\Delta_F(x-x';a)$ defined in Eq.(3.16):

$$\begin{gathered}\Delta_F(x-x';a^2) = \int\limits_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}d^4k}{k^2-(m^2+3ga^2)+i\varepsilon} = \\ = \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k_0^2-(\mathbf{k}^2+m^2)+i\varepsilon-3ga^2} = \\ = \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)\left(1-\dfrac{3ga^2}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon}\right)}, \\ \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}.\end{gathered} \tag{3.18}$$

$$\left(1-\frac{3ga^2}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon}\right)^{-1} = 1+\sum_{n=1}^{\infty}(-1)^n\left(\frac{3ga^2}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon}\right)^n \tag{3.19}$$

From Eqs.(3.18)-(3.19) we obtain

$$\begin{gathered}\Delta_F(x-x';a^2) = \\ = \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)\left(1-\dfrac{3ga^2}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon}\right)} = \\ = \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon} + 3ga^2 \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \left\{\frac{1}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon} \sum_{n=1}^{\infty}(-1)^n\left(\frac{e^{-ik(x-x')}}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon}\right)^n\right\} = \\ = \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon} + \sum_{n=1}^{\infty}(-1)^n(3g)^n a^{2n} \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)^{n+1}}\end{gathered} \tag{3.20}$$

.

$$\begin{gathered}\Delta_F(x-x') = \Delta_F^+(x-x')-\Delta_F^-(x-x') \\ \Delta_F^+(x-x') = \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{2\omega_{\mathbf{k}}} \frac{1}{k_0-\omega_{\mathbf{k}}+i\delta}, \\ \Delta_F^-(x-x') = \int\limits_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{2\omega_{\mathbf{k}}} \frac{1}{k_0+\omega_{\mathbf{k}}-i\delta},\end{gathered} \tag{3.(3.21)}$$

The integral over k_0 here, as always, can be easily calculated by contour integration. The exponent contains $e^{-ik_0(x_0-x_0')}$, so for $x_0-x_0'>0$, we close the integration contour in the lower half-plane of k_0 , and the integral is determined by the contribution from the pole at $k_0=\omega_{\mathbf{k}}-i\delta$. For $x_0-x_0'<0$, we close the integration contour in the upper half-plane, and everything is determined by the contribution from the pole at $k_0=-\omega_{\mathbf{k}}+i\delta$. Then, by Cauchy's theorem, we have:

$$\Delta_F(x-x') = \int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{2\omega_{\mathbf{k}}} \left[-i\theta(x_0-x_0')e^{-i\omega_{\mathbf{k}}\left(x_0-x_0'\right)}\right] - \\ -\int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{2\omega_{\mathbf{k}}} \left[i\theta(x_0'-x_0)e^{i\omega_{\mathbf{k}}\left(x_0-x_0'\right)}\right]. \tag{3.22}$$

After the substitution in the second integral $\mathbf{k} \to -\mathbf{k}$ we have:

$$\Delta_F(x-x') = -i\left\{\int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\theta(x_0-x_0')e^{-i\omega_{\mathbf{k}}\left(x_0-x_0'\right)+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right] + \right. \\ \left. + \int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\theta(x_0'-x_0)e^{i\omega_{\mathbf{k}}\left(x_0-x_0'\right)+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right]\right\} \tag{3.23}$$

Finally we get transcendental master equation corresponding to two point function $a^2(x-x';a^2) = \Delta(x-x';a^2)$:

$$a^2(x-x')\frac{(m^2+ga^2(x-x'))^2}{(m^2+3ga^2(x-x'))^2} = \\ = -i\left\{\int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\theta(x_0-x_0')e^{-i\omega_{\mathbf{k},a}\left(x_0-x_0'\right)+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right] + \right. \\ \left. + \int_{\mathbb{R}^4} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\theta(x_0'-x_0)e^{i\omega_{\mathbf{k},a}\left(x_0-x_0'\right)+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right]\right\} + \\ + \sum_{n=1}^{\infty}(-1)^n(3g)^n a^{2n}(x-x') \int_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)^{n+1}} = \\ = \int_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon} + \sum_{n=1}^{\infty}(-1)^n \int_{\mathbb{R}^4} \frac{d^3\mathbf{k}dk_0}{(2\pi)^4} \frac{(3g)^n a^{2n}(x-x')e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)^{n+1}} \tag{3.24}$$

$$
\begin{aligned}
&a^2(x-x') = \\
&= -i\frac{(m^2+3ga^2(x-x'))^2}{(m^2+ga^2(x-x'))^2}\left\{\int_{\mathbb{R}^4}\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left[\theta(x_0-x'_0)e^{-i\omega_{\mathbf{k}}\left(x_0-x'_0\right)+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right]+\right. \\
&\left.+\int_{\mathbb{R}^4}\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left[\theta(x'_0-x_0)e^{i\omega_{\mathbf{k}}\left(x_0-x'_0\right)+i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\right]\right\}+ \\
&+\sum_{n=1}^{\infty}(-1)^n(3g)^n a^{2n}(x-x')\int_{\mathbb{R}^4}\frac{d^3\mathbf{k}dk_0}{(2\pi)^4}\frac{e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)^{n+1}} = \\
&= \frac{(m^2+3ga^2(x-x'))^2}{(m^2+ga^2(x-x'))^2}\left\{\int_{\mathbb{R}^4}\frac{d^3\mathbf{k}dk_0}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon}+\right. \\
&\left.+\sum_{n=1}^{\infty}(-1)^n(3g)^n a^{2n}(x-x')\int_{\mathbb{R}^4}\frac{d^3\mathbf{k}dk_0}{(2\pi)^4}\frac{e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)^{n+1}}\right\}.
\end{aligned}
\tag{3.25}
$$

In the neighborhood of the light cone

$$
\begin{aligned}
&\Delta_F(x-x';\mu,\eta) = \frac{i}{4\pi^2\lambda}-\frac{im^2}{8\pi^2}\ln\sqrt{|\lambda|}-\frac{m^2}{16\pi}, \\
&1>\mu\geq\lambda=(x_0-x'_0)^2-(\mathbf{x}-\mathbf{x}')^2\geq\eta>0.
\end{aligned}
\tag{3.26}
$$

We rewrite Eq.(3.25) of the form corresponding to the Picard iterates $a_i = T(a_{i-1})$, (see Apendix.1)

$$
\begin{aligned}
&i = 1,2,\ldots, \\
&a_i^2(x-x') = \\
&= \frac{(m^2+3ga_{i-1}^2(x-x'))^2}{(m^2+ga_i^2(x-x'))^2}\left\{\int_{\mathbb{R}^4}\frac{d^3\mathbf{k}dk_0}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon}+\right. \\
&\left.+\sum_{n=1}^{\infty}(-1)^n(3g)^n a_{i-1}^{2n}(x-x')\int_{\mathbb{R}^4}\frac{d^3\mathbf{k}dk_0}{(2\pi)^4}\frac{e^{-ik(x-x')}}{(k_0^2-\omega_{\mathbf{k}}^2+i\varepsilon)^{n+1}}\right\}, \\
&1>\mu\geq(x_0-x'_0)^2-(\mathbf{x}-\mathbf{x}')^2\geq\eta>0, 0<|x_0-x'_0|<1, \\
&a_0^2(x-x') = \Delta_F(x-x';\mu,\eta).
\end{aligned}
\tag{3.27}
$$

We assume now that

$$
g\lambda^2 < 1. \tag{3.28}
$$

Note that under condition (3.28) Generalized Banach fixed-point theorem holds,see Apendix.1.

.

Apendix. 1.Generalized Banach fixed-point theorem.

Theorem 1. [6] Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping with the Lipschitz constant $\alpha \in [0,1)$. Then:

(1) T has a unique fixed point $\bar{x}$ in X.

(2) For an *arbitrary* point $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$generated by the Picard iteration process (defined by $x_{n+1} = T(x_n), n \in \mathbb{N} \cup \{0\}$) converges to $\bar{x}$.

(3) $d(x_n,\bar{x}) \leq \dfrac{\alpha^n}{1-\alpha}\, d(x_0,x_1)$ for all $n \in \mathbb{N}$.

Definition 1. [6] We say that $\bar{x} \in X$ is a contractive fixed point (abbr. CFP) of T if $\bar{x} = T(\bar{x})$ and the Picard iterates $T_n(x)$ converge to $\bar{x}$ as $n \to \infty$ for all $x \in X$.

The operator T from Definition 1.1 became known as a Picard operator (shortly PO),

Definition 2. We say that $\bar{x} \in X$ is a quasy contractive fixed point (abbr. QCFP) of T if $\bar{x} = T(\bar{x})$ and there is an $x_0 \in X$ such that the Picard iterates $T_n(x_0)$ converge to $\bar{x}$ as $n \to \infty$.

Definition 3.The operator T from Definition 2 became known as a quasy Picard operator (abbr. QPO)

Theorem 2.(Generalized Banach fixed-point theorem). Let (X,d) be a complete metric space and $T : X \to X$ is continuous.Assume that there is an $y_0 \in X$ such that the inequalities are satisfied

$$\begin{gathered} d(y_1,y_0) = d(T(y_0),y_0) \leq \alpha \times const, \\ d(y_2,y_1) = d(T(y_1),T(y_0)) \leq \alpha d(y_1,y_0), \\ d(y_3,y_2) = d(T(y_2),T(y_1)) \leq \alpha d(y_2,y_1), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ d(y_{n+1},y_n) = d(T(y_n),T(y_{n-1})) \leq \alpha d(y_n,y_{n-1}), \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{gathered} \tag{1.1}$$

where $\alpha < 1$, $y_n = T(y_{n-1})$. Ten there is a quasy contractive fixed point $\bar{x}$ of T, such that the Picard iterates $T_n(y_0)$ converge to $\bar{y} \in X$ as $n \to \infty$.

Proof. From inequalities (1.1) we obtain

$$\begin{gathered} d(y_1,y_0) = d(T(y_0),y_0) \leq \alpha \times const, \\ d(y_2,y_1) = d(T(y_1),T(y_0)) \leq \alpha d(y_1,y_0), \\ d(y_3,y_2) = d(T(y_2),T(y_1)) \leq \alpha d(y_2,y_1) \leq \alpha^2 d(y_1,y_0), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ d(y_{n+1},y_n) = d(T(y_n),T(y_{n-1})) \leq \alpha d(y_n,y_{n-1}) \leq \alpha^n d(y_1,y_0), \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{gathered} \tag{1.2}$$

From inequalities (1.2) by using triangle inequality we obtain for some $\varepsilon \ll 1$

$$\begin{gathered} d(y_n,y_{n+m}) \leq d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \ldots + d(y_{n+m-1},y_{n+m}) \leq \\ \leq (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+m-1}) d(y_1,y_0) = \alpha^n \frac{1-\alpha^{n+m}}{1-\alpha} d(y_1,y_0) < \\ < \frac{\alpha^n}{1-\alpha} d(y_1,y_0) < \varepsilon. \end{gathered} \tag{1.3}$$

Thus a sequence $\{y_n\}_{n=0}^{\infty} \subset X$ is a fundamental sequence in X and there is exists

$\lim_{n\to\infty} y_n = \bar{y} \in X.$ We assume now that $T(\bar{y}) = \bar{\bar{y}},$ by using triangle inequality we obtain

$$d(\bar{y},\bar{\bar{y}}) \leq d(\bar{y},y_n) + d(y_n,y_{n+1}) + d(y_{n+1},\bar{\bar{y}}). \tag{1.4}$$

For any $\varepsilon > 0$ we can choose $N = N(\varepsilon)$ such that for any $n \geq N$

(i) $d(\bar{y},y_n) < \varepsilon/3,$ since that is $\bar{y} = \lim_{n\to\infty} y_n;$

(ii) $d(y_n,y_{n+1}) < \varepsilon/3,$ since that $\{y_n\}_{n=0}^{\infty} \subset X$ is a fundamental sequence;

(iii) $d(y_{n+1},\bar{\bar{y}}) = d(T(y_n),T(\bar{y})) \leq \varepsilon/3,$ since that is $\lim_{n\to\infty} T(y_n) = T(\lim_{n\to\infty} y_n) = T(\bar{y}).$

Therefore for any ϵ such that $0 < \epsilon < \varepsilon : d(\bar{y},\bar{\bar{y}}) \leq \epsilon$ it follows that $d(\bar{y},\bar{\bar{y}}) = 0$ and consequently $\bar{\bar{y}} = \bar{y}.$

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