Relativistic Geometrical Horizons in the $3D$ spherical Natario warp drive

Fernando Loup *†
independent researcher in warp drive spacetimes:Lisboa Portugal

July 16, 2026

Abstract

In 1994 Alcubierre developed the first warp drive work using the original $3+1$ ADM-MTW formalism.Seven years later in 2001 the same original $3+1$ ADM-MTW formalism appeared in the first part of the second warp drive work developed by Natario. But in 1997 the concept of the Relativistic Geometrical Horizon RGH was introduced by Hiscock in his work about the Alcubierre warp drive. According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00}=0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears. Hiscock computed the Relativistic Geometrical Horizon RGH for the Alcubierre warp drive.We present in this work the Relativistic Geometrical Horizon RGH for the $3D$ version of the Natario warp drive with both constant and variable speeds.Remember that a real spaceship is a $3D$ object and a spaceship must accelerate or de-accelerate in order to be accepted as a physical valid model. Due to the presence of three spatial dimensions the $g_{00}=0$ never appears however in a dimensional reduction from a $3+1$ spacetime to a $1+1$ spacetime the Relativistic Geometrical Horizon RGH $g_{00}=0$ appears again.

*spacetimeshortcut@yahoo.com,spacetimeshortcut@gmail.com
†https://independent.academia.edu/FernandoLoup,https://www.researchgate.net/profile/Fernando-Loup

1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates in [15]).(see Appendix A for the complete mathematical demonstration of the Natario calculations for the Hodge Star in [15]).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$ or better:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{1}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{2}$$

However Polar Coordinates are not real $3D$ coordinates since it uses only the two dimensional Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another alternative warp drive vector must be calculated.Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about $3D$ Spherical Coordinates in [15]).

Remember also that a spaceship must accelerate or de-accelerate in order to be accepted as a physical valid model.

This work introduces the concept of the Relativistic Geometrical Horizons(RGH) for the advanced Natario warp drive vectors in $3D$ Spherical Coordinates with constant or variable speeds.

For advanced versions of the $3D$ Spherical Coordinates modified Natario warp drive vectors with constant or variable velocities see [10],[11],[12],[13] and [14] but these works do not encompass the RGH.

In order to get an idea about how many advanced versions of the $3D$ Spherical Coordinates modified Natario warp drive vectors exists see section 4 in [12].see also section 4 in [13].

The warp drive work that predates Natario by 7 years was written by Alcubierre in 1994.(see [2])

Alcubierre([4]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([3]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pg 67 in [4],see also eq 1 pg 3 in [2]) and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.In this work concerning the ADM formalism we adopt the Alcubierre methodology.

The ADM equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq (21.40) pg 507 in [3]).[1]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{3}$$

In the equation above α is the so-called lapse function,γ_{ij} is the $3D$ diagonalized induced metric and β^i and β^j are the so-called shift vectors.

For a complete mathematical explanation of the $3+1$ ADM formalism see Appendix A

In this work we present the Relativistic Geometrical Horizons(RGH) concept as introduced by Hiscock in 1997.(see [20])

Hiscock in 1997 applied the RGH concept to the Alcubierre warp drive(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18])) but every time in any Relativistic metric if the $g_{00} = 0$ appears we will have a RGH Relativistic Geometrical Horizon.

According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00} = 0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears.(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18])

Alcubierre used a remote frame in which a remote observer outside the bubble takes the observations but Hiscock used a ship frame in which an observer inside the bubble takes the observations.

We will remain here in the remote frame in order to avoid co-moving coordinate transformations since our presentation about RGH is being given in an introductory level.The final results are similar.

We will demonstrate the Hiscock arguments for the RGH in the Alcubierre warp drive in a way more accessible to beginners or intermediate students avoiding the co-moving coordinate transformations used by Hiscock.

Alcubierre warp drive suffers from this pathology because in a given point of the Alcubierre warped region the $g_{00} = 0$ appears generating the Alcubierre Horizon equivalent to the Event Horizon of the Schwarzschild black hole.The point of the Alcubierre warped region where the Alcubierre Horizon is located is the Alcubierre radius similar to the Schwarzschild radius.

[1]we adopt the Alcubierre notation here

The Natario warp drive in a $1+1$ spacetime like its Alcubierre counterpart becomes singular in a given point of the Natario warped region where the $g_{00}=0$ also appears developing also the Natario Horizon as another case of a Relativistic Geometrical Horizon $g_{00}=0$.

However in the case of the Natario warp drive in a $2+1$ polar coordinates spacetime the $g_{00}=0$ never appears.Due to the presence of a second spatial dimension the $g_{00}=0$ can be avoided.The Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$,the shift vectors X^r,X^θ and the Natario vector $nX = X^r e_r + X^\theta e_\theta$ all in Polar Coordinates in the $2+1$ spacetime compensates each other to neutralize the $g_{00}=0$.

In order to keep a pedagogical sequence in this work we present the Hiscock RGH $g_{00}=0$ for the original versions of both Alcubierre and Natario warp drives following our work in [25].

We present in this work the Relativistic Geometrical Horizons(RGH) for the $3D$ version of the Natario warp drive in spherical coordinates in a $3+1$ spacetime with constant speed and in this case the $g_{00}=0$ never appears.Due to the presence of three spatial dimensions the $g_{00}=0$ can be avoided.The Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$,$\mathbf{e}_\phi$,the shift vectors X^r,X^θ,X^ϕ and the Natario vector $nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi$ all in Spherical Coordinates in the $3+1$ spacetime compensates each other to neutralize the $g_{00}=0$.However in a dimensional reduction from a $3+1$ spacetime to a $1+1$ spacetime the Hiscock RGH $g_{00}=0$ appears again.

We present also in this work the Relativistic Geometrical Horizons(RGH) for the $3D$ version of the Natario warp drive in spherical coordinates in a $3+1$ spacetime with variable speeds.

This presentation is mathematically more advanced and we provide all the calculations step by step.

In the equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{4}$$

The contravariant and the corresponding covariant shift vectors compensates each other in order to avoid the Natario Horizon RGH $g_{00}=0$.

The g_{00} spacetime metric tensor component is given by:

$$g_{00} = (1 - 2X_t + X_t X^t - X_r X^r - X_\theta X^\theta - X_\phi X^\phi) \tag{5}$$

Even in the range of the boundary conditions evaluated here the term $1 - 2X_t + X_t X^t$ compensates the term $X_r X^r + X_\theta X^\theta + X_\phi X^\phi$ and the RGH Natario Horizon $g_{00}=0$ never occurs.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical references required for the background needed to understand the process Hiscock used to generate the Relativistic Geometrical Horizons RGH.

We hope our paper is suitable to fill this proposed task.

This work was designed as a companion work to our works in [10],[11],[12],[13],[14] and [25].

2 The Relativistic Geometrical Horizons(RGH) concept as introduced by Hiscock applied to the original Alcubierre warp drive-A review

The warp drive metric proposed by Alcubierre using signature $(-,+,+,+)$ may be written as:(see eq 8 pg 4 in [2],eq 1 pg 3 in [20])

$$ds^2 = -dt^2 + (dx - vf(r)dt)^2 + dy^2 + dz^2 \tag{6}$$

Where $v = vs$ is the apparent velocity of the spaceship,(see pg 4 in [2],eq 2 pg 3 in [20])

$$v = vs = \frac{dx_s(t)}{dt} \tag{7}$$

$x_s(t)$ is the trajectory of the spaceship (chosen to be along the x direction), $r = rs$ is defined by:(see pg 4 in [2],eq 3 pg 3 in [20])

$$r = rs = \left[(x - x_s(t))^2 + y^2 + z^2\right]^{1/2} \tag{8}$$

And $f = f_{rs} = f(rs)$ is an arbitrary function which decreases from unity at $r = 0$ (the location of the spaceship) to zero at infinity. Alcubierre gave a particular example of such a function:(see eq 6 pg 4 in [2],eq 4 pg 3 in [20])

$$f = f_{rs}(r) = f(rs) = \frac{\tanh(\sigma(r+R)) - \tanh(\sigma(r-R))}{2\tanh(\sigma R)}, \tag{9}$$

$$f = f_{rs}(r) = f(rs) = \frac{\tanh(@(r+R)) - \tanh(@(r-R))}{2\tanh(@R)}, \tag{10}$$

The function above is the Alcubierre shape function where σ or @ and R are positive arbitrary constants.We outline the fact that σ or @ and R are positive arbitrary constants.R is the radius of the warp bubble while σ or @ are related to the thickness of the bubble.

If $\sigma >> R$ or $@ >> R$ then we can use the following useful approximation:(see eq 14 pg 8 in [16],eq 16 pg 9 in [17],eq 8 pg 6 in [18],eq 12 pg 7 in [19])

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]] \tag{11}$$

Inside the bubble $f(rs) = 1$ and outside the bubble $f(rs) = 0$ as pointed out by Alcubierre(eq 7 pg 4 in [2]) and Hiscock(pg 3 in [20] before eq 4).

The region where $f(rs)$ starts to decrease from 1 to 0 is the Alcubierre warped region $1 > f(rs) > 0$.Numerical plots of the Alcubierre warped region for a warp bubble with 100 meters of radius $R = 100$ are presented in pg 9 in [16], pg 21 in [17],pg 13 in [18],pg 11 in [19].

The Alcubierre warped region $1 > f(rs) > 0$ is the region where the derivatives of the shape function $f(rs)$ are not null defining the walls of the warp bubble.

Working with signature $(+,-,-,-)$ we get:(see eq 6 pg 6 in [18])

$$ds^2 = dt^2 - (dx - vf(r)dt)^2 - dy^2 - dz^2 \tag{12}$$

$$ds^2 = dt^2 - (dx - vf(rs)dt)^2 - dy^2 - dz^2 \tag{13}$$

Expanding the squared terms we have:(see eq 17 pg 7 in [18])(see also eq 144 in Appendix A)

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{14}$$

Reducing to a $1+1$ dimensions we have:(compare with eq 7 pg 4 in [20])

$$ds^2 = dt^2 - (dx - vf(rs)dt)^2 \tag{15}$$

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 \tag{16}$$

Alcubierre used a remote frame in which a remote observer outside the bubble takes the observations but Hiscock used a ship frame in which an observer inside the bubble takes the observations.

The Alcubierre equations in a ship frame are given by:(see eqs 22 to 34 in [18])

We will remain here in the remote frame in order to avoid co-moving coordinate transformations since our presentation about RGH is being given in an introductory level.The final results are similar.

The spacetime metric components for the Alcubierre metric in a $1+1$ dimensions are given by:(see eq 18 pg 7,eq 21 pg 8 in [18])

$$g_{00} = (1 - [vsf(rs)]^2) \tag{17}$$

$$g_{01} = vsf(rs) \tag{18}$$

$$g_{11} = 1 \tag{19}$$

According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00} = 0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears.(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18]).[2]

Unfortunately the Alcubierre warp drive suffers from this pathology because in a given point of the Alcubierre warped region $1 > f(rs) > 0$ the $g_{00} = 0$ appears generating the Alcubierre Horizon equivalent to the Event Horizon of the Schwarzschild black hole.The point of the Alcubierre warped region where the Alcubierre Horizon is located is the Alcubierre radius similar to the Schwarzschild radius.

[2] we will discuss briefly in this work the the Event Horizon of the Schwarzschild black hole in order to compare with the Alcubierre Horizon

We will demonstrate the Hiscock arguments for the RGH in the Alcubierre warp drive in a way more accessible to beginners or intermediate students avoiding the co-moving coordinate transformations used by Hiscock.

Examining again the g_{00} in the Alcubierre metric:

$$g_{00} = (1 - [vsf(rs)]^2) \tag{20}$$

Inside the warp bubble $f(rs) = 1$ and $g_{00} = 1 - vs^2$ while outside the warp bubble $f(rs) = 0$ and $g_{00} = 1$.

Assuming a continuous decreasingly behavior of $f(rs)$ from 1 to zero implying in a decreasingly $vsf(rs)$ from vs inside the bubble to zero outside the bubble then in a given point of the Alcubierre warped region $1 > f(rs) > 0$ and in consequence $vsf(rs) = 1$ because $vs > vsf(rs) > 0$,$vs > 1 > 0$ implying in a $g_{00} = 0$.The Alcubierre metric becomes singular.(see eqs 19 and 20 pg 7 in [18]).This singularity is the Alcubierre Horizon similar to the Event Horizon of a Schwarzschild black hole.

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = \frac{1}{vs} \tag{21}$$

A better explanation:

- 1)-Inside the warp bubble:
  $g_{00} = (1 - [vsf(rs)]^2)$,$f(rs) = 1$,$vsf(rs) = vs$,$g_{00} = (1 - vs^2)$
- 2)-Outside the warp bubble:
  $g_{00} = (1 - [vsf(rs)]^2)$,$f(rs) = 0$,$vsf(rs) = 0$,,$g_{00} = 1$
- 2)-In the warp bubble walls(Alcubierre warped region):
  $g_{00} = (1 - [vsf(rs)]^2)$,$[1 > f(rs) > 0]$,$[vs > vsf(rs) > 0]$,$[vs > 1 > 0]$
  $[g_{00} = (1 - vs^2)] > [g_{00} = (1 - [vsf(rs)]^2)] > [g_{00} = 1]$

We are working with units $G = c = 1$ then a velocity two times light speed means a $vs = 2$,four times light speed means a $vs = 4$ and 1000 times light speed means a $vs = 1000$.

Inside the bubble $f(rs) = 1$,outside the bubble $f(rs) = 0$ and in the warp bubble walls(Alcubierre warped region) $[1 > f(rs) > 0]$.

When $vs = 2$,$f(rs) = \frac{1}{vs} = \frac{1}{2} = 0,5$ exactly in the middle of the Alcubierre warped region,when $vs = 4$,$f(rs) = \frac{1}{vs} = \frac{1}{4} = 0,25$ more close to the end of the Alcubierre warped region and when $vs = 1000$,$f(rs) = \frac{1}{vs} = \frac{1}{1000} = 0,0001$ even more close to the end of the Alcubierre warped region.

When $vs = 2$,$f(rs) = \frac{1}{vs} = \frac{1}{2} = 0,5$,$vsf(rs) = 2\frac{1}{2} = 1$ Alcubierre Horizon exactly in the middle of the Alcubierre warped region,when $vs = 4$,$f(rs) = \frac{1}{vs} = \frac{1}{4} = 0,25$,$vsf(rs) = 4\frac{1}{4} = 1$ Alcubierre Horizon more close to the end of the Alcubierre warped region and when $vs = 1000$,$f(rs) = \frac{1}{vs} = \frac{1}{1000} = 0,0001$,$vsf(rs) = 1000\frac{1}{1000} = 1$ Alcubierre Horizon even more close to the end of the Alcubierre warped region.

Our approximated Alcubierre shape function is:

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]] \tag{22}$$

We recall this function numerical plots of the Alcubierre warped region for a warp bubble with 100 meters of radius $R = 100$ presented in pg 9 in [16], pg 21 in [17],pg 13 in [18],pg 11 in [19].

In the Alcubierre Horizon

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]] = \frac{1}{vs} \tag{23}$$

The point rs in the Alcubierre warped region where the Alcubierre Horizon occurs is the Alcubierre radius similar to the Schwarzschild radius and is given by:(see Appendix G in [17])

$$rs = R + \frac{1}{@}arctanh(1 - \frac{2}{vs}) \tag{24}$$

Although we described the RGH Alcubierre Horizon $vsf(rs) = 1$ from the Hiscock point of view in which $g_{00} = 0$,the Alcubierre Horizon also appears in the study of Horizons of null-like geodesics when $ds^2 = 0$.(see pgs 9,10,11 and Appendix D in [17]),(see pg 16 in [18]),(see pg 17 in [19]).(see pg 34 in [5],pg 268 in [6])

However while in the study of Horizons of null-like geodesics when $ds^2 = 0$,the Horizon occurs if and only if an observer in the center of the bubble send a photon to the front of the bubble and if such this observer remains "quiet" we do not need to worry ourselves about Horizons,unfortunately the RGH Alcubierre Horizon $vsf(rs) = 1$ from the Hiscock point of view $g_{00} = 0$ always exists inside the metric wether we like it or not.

Independently from "quiet" or "photon sender" observers inside the bubble "throwing powerful laser beams to the front of the bubble (or not)" in the system of units $G = c = 1$ and assuming a continuous growth of the speed vs from 0 to 1000 times light speed $vs = 1000$ then in a given moment of time $vs = 1$ because $0 <= vs <= 1000$ and $0 < 1 < 1000$ and when $vs = 1$(light speed) even inside the bubble where $f(rs) = 1$ we have:

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = 1 \dashrightarrow vs = 1 \dashrightarrow g_{00} = (1 - [1 \times 1]^2) = 0 \tag{25}$$

The Alcubierre Horizon similar to the Event Horizon of a Schwarzschild black hole according to Hiscock appears every time a spaceship reaches the speed of light $vs = 1$ and the Alcubierre metric becomes singular.$g_{00} = 0$

We are now ready to compare the Alcubierre warp drive with the Schwarzschild black hole.

The Schwarzschild metric is described by the following equation:(see eqs 5.1 and 5.2 pg 193 in [7])(see also eq 6.1.44 pg 124 in [8])(see also eq 11.13 pg 229 in [9])

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{26}$$

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \tag{27}$$

The equations in [7] are being given with signature $(-, +, +, +)$ in the system of units $c = 1$.

The term that interest to ourselves is the $g_{00} = g_{tt}$ given by:(see eq 5.22 pg 196 in [7])

$$g_{00} = g_{tt} = -(1 - \frac{2GM}{r}) \tag{28}$$

Passing to the signature $(+, -, -, -)$ we get:

$$ds^2 = (1 - \frac{2GM}{r})dt^2 - (1 - \frac{2GM}{r})^{-1}dr^2 - r^2 d\Omega^2 \tag{29}$$

$$g_{00} = g_{tt} = (1 - \frac{2GM}{r}) \tag{30}$$

Rewriting the term $g_{00} = g_{tt}$ in normal SI units MKS we have:

$$g_{00} = g_{tt} = (1 - \frac{2GM}{c^2 r}) \tag{31}$$

Hiscock applied the point of view of $g_{00} = 0$ to the Alcubierre warp drive.The same point of view $g_{00} = 0$ applied to the Schwarzschild metric gives:(see eq 6.1.45 pg 124 in [8])

$$g_{00} = (1 - \frac{2GM}{c^2 r}) = 0 \dashrightarrow 1 = \frac{2GM}{c^2 r} \dashrightarrow r = \frac{2GM}{c^2} \tag{32}$$

The Schwarzschild metric becomes singular $g_{00} = 0$ when $r = \frac{2GM}{c^2}$.The term r is the Schwarzschild radius.For our Sun a star with about 1 million kilometers in diameter the corresponding Schwarzschild radius is approximately 3 kilometers which means to say that if we could shrink the volume of the Sun to a small size of about 3 kilometers the Sun would become a black hole.The Sun will never be a black hole because it do not possesses enough mass but stars with more than 3 solar masses cannot withstand their structures against the gravitational pull when the nuclear furnace ceases the production of energy and then these stars starts to shrink in catastrophic gravitational collapse to very small volumes compatible with their Schwarzschild radius becoming black holes in the process generating Event Horizons in the point of the Schwarzschild radius.(see pgs 124,125 in [8])(see pg 231 in [9]).For details about black holes see [7],[8] and [9].

The Schwarzschild metric:

$$ds^2 = (1 - \frac{2GM}{r})dt^2 - (1 - \frac{2GM}{r})^{-1}dr^2 - r^2 d\Omega^2 \tag{33}$$

The Schwarzschild Event Horizon:$g_{00} = 0$

$$g_{00} = (1 - \frac{2GM}{c^2 r}) = 0 \dashrightarrow 1 = \frac{2GM}{c^2 r} \dashrightarrow r = \frac{2GM}{c^2} \tag{34}$$

The Alcubierre metric:

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{35}$$

The Alcubierre Horizon:$g_{00} = 0$

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = \frac{1}{vs} \tag{36}$$

Both Horizons are RGH Relativistic Geometrical Horizons because in both cases the $g_{00} = 0$ appears.

Hiscock in 1997 applied the RGH concept to the Alcubierre warp drive(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18])) but every time in any Relativistic metric if the $g_{00} = 0$ appears we will have a RGH Relativistic Geometrical Horizon.

Although we described the RGH Alcubierre Horizon $vsf(rs) = 1$ from the Hiscock point of view in which $g_{00} = 0$,the Alcubierre Horizon also appears in the study of Horizons of null-like geodesics when $ds^2 = 0$.(see pgs 9,10,11 and Appendix D in [17]),(see pg 16 in [18]),(see pg 17 in [19]).(see pg 34 in [5],pg 268 in [6])

In the case of null-like geodesics $ds^2 = 0$ a photon stops and cannot reach the outermost layers of the warp bubble which remains "causally disconnected" from the "photon sender" observer in the center of the bubble.

The solution that allows contact with the bubble walls was presented in pg 83 in [22].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [22] to be sent to the large warp bubble keeping it in causal contact.This idea seems to be endorsed by pg 34 in [5],pg 268 in [6] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

But unfortunately the Alcubierre Horizon $g_{00} = 0$ cannot be circumvented.

Metrics of the form given below appears many times in General Relativity.Applyimg the Einstein Summing Convention we have:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \tag{37}$$

$$ds^2 = g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + ... + g_{33} dx^3 dx^3 \tag{38}$$

General Relativity metrics are physically consistent only when $g_{00} \neq 0$.A given spacetime metric cannot have RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The Alcubierre warp drive metric is not physically consistent because it cannot avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The RGH and therefore the inconsistency of the metric appears only when the ship reaches the speed of light $vs = 1$

The quest of the metric inconsistency when the RGH appears $g_{00} = 0$ and how this affects the whole spacetime metric is still an open issue.

3 The Relativistic Geometrical Horizons(RGH) concept applied to the original Natario warp drive-Simplest case-A review

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates in [15]).(see Appendix A for the complete mathematical demonstration of the Natario calculations for the Hodge Star in [15]).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$ or better:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{39}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{40}$$

We prefer the second equation above.

The equation of the Natario warp drive vector nX in polar coordinates with a constant speed is given by(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{41}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [15])

$$X^r = 2v_s f(r) \cos\theta \tag{42}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta \tag{43}$$

Considering a valid $f(r) = f(rs)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large $r = rs$(outside the warp bubble) and $f(r) = 0$ for small $r = rs$(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1]).The Natario warp drive was computed in ship frame coordinates.See Appendices G and H in [15].

The generic warp drive equation is given by:(see Appendix A)

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{44}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{45}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{46}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ for the original Natario warp drive are given by:

$$X_i = \gamma_{ii} X^i \tag{47}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r) \cos\theta = X^r \tag{48}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s (2f(r) + (r)f'(r)) \sin\theta \tag{49}$$

Then the equation of the Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{50}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{51}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{52}$$

In order to simplify our analysis we consider first motion in the $x - axis$ only (like Natario did) or the equatorial plane $x - y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1])

The Natario warp drive equation in the $1+1$ spacetime is then given by:

$$ds^2 = (1 - X_r X^r)dt^2 + 2X_r dr dt - dr^2 \tag{53}$$

In order to avoid confusion between the shape functions of Natario and Alcubierre, $f(r) = f(rs)$ is the Alcubierre shape function while the Natario shape functions are represented by the letters $n(r) = n(rs)$ and $N(r) = N(rs)$,$v_s = vs$,$r = rs$.Then we have:

$$ds^2 = (1 - X_r X^r)dt^2 + 2X_r dr dt - dr^2 \tag{54}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s N(r) = X^r \tag{55}$$

$$ds^2 = (1 - [2vsN(rs)][2vsN(rs)])dt^2 + 2[2vsN(rs)]dr dt - dr^2 \tag{56}$$

$$ds^2 = (1 - [2vsN(rs)]^2)dt^2 + 4[vsN(rs)]drdt - dr^2 \tag{57}$$

$$ds^2 = (1 - 4[vsN(rs)]^2)dt^2 + 4[vsN(rs)]drdt - dr^2 \tag{58}$$

Compare with the Alcubierre metric:

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 \tag{59}$$

$$ds^2 = (1 - X_r^2)dt^2 + 2X_r drdt - dr^2 \tag{60}$$

The spacetime metric tensor component g_{00} is given by:

$$g_{00} = (1 - X_r^2) \dashrightarrow X_r = 2v_s N(r) \tag{61}$$

A Relativistic Geometrical Horizon RGH occurs when $g_{00} = 0$ and this means the following result:

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2v_s N(r) = 1 \tag{62}$$

We are ready to introduce now the Natario shape functions $n(r) = n(rs)$ and $N(r) = N(rs)$.The first Natario shape function $n(rs)$ is given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)] \tag{63}$$

According with Natario(pg 5 in [1]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive. Another Natario shape function can be defined by:

$$N(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{64}$$

This shape function gives the result of $N(rs) = 0$ inside the warp bubble and $N(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < N(rs) < \frac{1}{2}$ in the Natario warped region. Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $N(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $N(rs) = \frac{1}{2}$(exterior of the Natario bubble).

The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.We prefer the second Natario shape function because it behaves much better in the negative energy density requirements to sustain a warp bubble.See section 3 in [16].See section 2 in [24]

Considering a valid $N(rs)$ as a Natario shape function being $N(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $N(rs) = 0$ for small rs(inside the warp bubble) while being $0 < N(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]) and assuming a continuous behavior for $N(rs)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2v_sN(rs)$ from 0 to vs we can clearly see that inside the bubble $2v_sN(rs) = 0$ because $N(rs) = 0$ and outside the bubble $2v_sN(rs) = vs$ because $N(rs) = \frac{1}{2}$ and assuming also continuous values from 0 to vs with $vs > 1$[3] then somewhere in the Natario warped region where $0 < N(rs) < \frac{1}{2}$ we have the situation in which $2v_sN(rs) = 1$ because 1 lies in the continuous interval from 0 to vs and in consequence:

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2v_sN(r) = 1 \dashrightarrow g_{00} = 0!!! \tag{65}$$

The Natario warp drive like its Alcubierre counterpart becomes singular $g_{00} = 0$ developing also the Natario Horizon as another case of a Relativistic Geometrical Horizon $g_{00} = 0$.

This situation resembles the descriptions in pg 9 in [10] and pg 11 in [23].In the case of null-like geodesics $ds^2 = 0$ a photon stops and cannot reach the outermost layers of the warp bubble which remains "causally disconnected" from the "photon sender" observer in the center of the bubble.If the "photon sender" remains "quiet" and do not send photons to the front of the bubble then the null-like geodesics $ds^2 = 0$ do not occurs but the Natario Horizon $g_{00} = 0$ unfortunately cannot be avoided.

What happens when a Natario warp drive reaches the speed of light(vs=1)?Look to the region outside the bubble where $N(rs) = \frac{1}{2}$

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2N(r) = 1 \dashrightarrow X_r = 2\frac{1}{2} = 1 \dashrightarrow g_{00} = 0!!! \tag{66}$$

The Natario warp drive metric in the $1 + 1$ spacetime becomes singular $g_{00} = 0$

Metrics of the form given below appears many times in General Relativity.Applyimg the Einstein Summing Convention we have:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu \tag{67}$$

$$ds^2 = g_{00}dx^0dx^0 + g_{01}dx^0dx^1 + ... + g_{33}dx^3dx^3 \tag{68}$$

General Relativity metrics are physically consistent only when $g_{00} \neq 0$.A given spacetime metric cannot have RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The Natario warp drive metric in $1 + 1$ polar coordinates is not physically consistent because it cannot avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The RGH and therefore the inconsistency of the metric appears only when the ship reaches the speed of light $vs = 1$

The quest of the metric inconsistency when the RGH appears $g_{00} = 0$ and how this affects the whole spacetime metric is still an open issue.

[3]Remember that we are working with Geometrized Units in which $G = c = 1$

4 The Relativistic Geometrical Horizons(RGH) concept applied to the original Natario warp drive-Complete case-A review

The Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original 3+1 ADM formalism is given by:(see Appendix A)

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{69}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{70}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{71}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [15])

$$X^r = 2v_s f(r)\cos\theta \tag{72}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{73}$$

Remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ for the original Natario warp drive are given by:

$$X_i = \gamma_{ii} X^i \tag{74}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r)\cos\theta = X^r \tag{75}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s(2f(r) + (r)f'(r))\sin\theta \tag{76}$$

The g_{00} for the complete Natario warp drive in polar coordinates is given by:

$$g_{00} = (1 - X_r X^r - X_\theta X^\theta) \tag{77}$$

$$g_{00} = (1 - [X_r X^r + X_\theta X^\theta]) \tag{78}$$

$$X_r X^r = [2v_s f(r)\cos\theta]^2 \tag{79}$$

$$X_\theta X^\theta = (r^2)[v_s(2f(r) + (r)f'(r))\sin\theta]^2 \tag{80}$$

Now we have motion in two dimensions r and θ and this affects the Relativistic Geometrical Horizon $g_{00} = 0$

$$g_{00} = (1 - [[2v_s f(r)\cos\theta]^2 + (r^2)[v_s(2f(r) + (r)f'(r))\sin\theta]^2]) \tag{81}$$

$$g_{00} = (1 - [[2v_s f(r)\cos\theta]^2 + (r^2)[v_s(2f(r) + (r)f'(r))\sin\theta]^2]) \tag{82}$$

The Relativistic Geometrical Horizon $g_{00} = 0$ never occurs in the complete case of the Natario warp drive in polar coordinates in the $2+1$ spacetime due to the terms r^2 and $(r)f'(r)$.Also the terms $\cos\theta$ and $\sin\theta$ complements each other except when $\theta = 0$,$\sin\theta = 0$ and $\cos\theta = 1$.When $\theta = 90$,$\sin\theta = 1$,$\cos\theta = 0$ but the term $(r)f'(r)$ avoids the Natario Horizon.When $\theta = 0$,$\sin\theta = 0$ and $\cos\theta = 1$ we recover the $1+1$ spacetime and the RGH Natario Horizon.

Due to the presence of 2 dimensions the RGH can be avoided.This resembles a situation described in section 6 and specially eqs 76 and 77 pg 16 all in [10] for null-like geodesics $ds^2 = 0$ Horizons.

In the previous presented cases of the Schwarzschild metric,the Alcubierre warp drive metric and the Natario warp drive metric in the $1+1$ spacetime the Relativistic Geometrical Horizon $g_{00} = 0$ appears and all these metrics becomes singular.

$$g_{00} = (1 - \frac{2GM}{c^2 r}) = 0 \dashrightarrow 1 = \frac{2GM}{c^2 r} \dashrightarrow r = \frac{2GM}{c^2} \tag{83}$$

In the case of the Alcubierre warp drive the RGH Alcubierre Horizon $g_{00} = 0$ appears even inside the bubble when the velocity reaches the speed of light $vs = 1$.

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = 1 \dashrightarrow vs = 1 \dashrightarrow g_{00} = (1 - [1 \times 1]^2) = 0 \tag{84}$$

But in the case of the Natario warp drive in the $1+1$ spacetime the RGH Natario Horizon $g_{00} = 0$ appears outside the bubble when the velocity reaches the speed of light $vs = 1$.

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2vsN(r) = 1 \dashrightarrow vs = 1 \dashrightarrow N(r) = \frac{1}{2} \dashrightarrow X_r = 2\frac{1}{2} = 1 \dashrightarrow g_{00} = 0!!! \tag{85}$$

Metrics of the form given below appears many times in General Relativity.Applyimg the Einstein Summing Convention we have:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu \tag{86}$$

$$ds^2 = g_{00}dx^0dx^0 + g_{01}dx^0dx^1 + ... + g_{33}dx^3dx^3 \tag{87}$$

General Relativity metrics are physically consistent only when $g_{00} \neq 0$.A given spacetime metric cannot have RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The Natario warp drive in $2+1$ polar coordinates is physically consistent because it can avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$.

5 Natario Vectors and Natario vectors

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{88}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result all in [12])(see Section 5 in [15])(see Section 3 in [12])(see Appendices K and L for details in [15])

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{89}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{90}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{91}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{92}$$

The equation of the Natario warp drive vector nX in polar coordinates with a variable speed vs due to a constant acceleration a is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{93}$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector are defined by(see Appendices A and B for pedagogical purposes and C for the final result all in [12])(see Section 3 in [15])(see Section 2 in [12])(see Appendices B and C for details in [15]) :

$$X^t = 2f(r)rcos\theta a \tag{94}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{95}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{96}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates.

Considering a valid $f(r) = f(rs)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large $r = rs$(outside the warp bubble) and $f(r) = 0$ for small $r = rs$(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]:

The term a is the acceleration of the ship and therefore we must have $a \gg 1$

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a constant speed vs nX is given by::

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{97}$$

With the contravariant shift vector components X^r, X^θ and X^ϕ given by:
(see Appendices F and G for details all in [12])(see Section 4 in [15])(see Section 3 in [10])(see Appendix J for details in [15])

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{98}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{99}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{100}$$

The equation of the Natario warp drive vector nX in polar coordinates with a constant speed is given by(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{101}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [12])(see Section 2 in [15])(see Section 2 in [10])(see Appendix A for details in [15])

$$X^r = 2v_s f(r)\cos\theta \tag{102}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{103}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates

Considering a valid $f(r) = f(rs)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large $r = rs$(outside the warp bubble) and $f(r) = 0$ for small $r = rs$(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]:

The term v_s is the velocity of the ship and therefore we must have $v_s \gg 0$

For more details about Natario Vectors and Natario vectors see section 4 in [12]).

6 Natario Warp Drive and Natario warp drive

The Natario warp drive equation for variable velocities in the original $3+1$ ADM formalism in real $3D$ spherical coordinates is given by:(see Appendix J in [12] for details)

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (104)$$

$$ds^2 = ((1-2X_t+X_t X^t) - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (105)$$

The equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration in the original $3+1$ ADM formalism in polar coordinates is given by:(see Appendix J in [12] for details)

$$ds^2 = (1 - 2X_t + X_t X^t - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_{rs} dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \quad (106)$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \quad (107)$$

The equation of the Natario warp drive spacetime in $3D$ spherical coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:(see Appendix I in [12] for details)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (108)$$

The equation of the Natario warp drive spacetime in polar coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:(see Appendix I in [12] for details)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \quad (109)$$

The difference between variable and constant velocity warp drives is due to the term α^2 that affect the geometrical structure of the whole spacetimes.The term α behaves as a lapse function.

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt} X^t + \gamma_{tt} X^t X^t) = (1 - 2X_t + X_t X^t) \quad (110)$$

For more details about Natario Warp Drive and Natario warp drive, differences between variable and constant velocities and the term α see Appendices I,J and section 5 all in [12].

7 The Relativistic Geometrical Horizons(RGH) concept applied to the $3D$ spherical Natario warp drive-Constant speeds case

The equation of the Natario warp drive spacetime in $3D$ spherical coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:(see Appendix I in [12] for details)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (111)$$

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a constant speed vs nX is given by::

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \quad (112)$$

With the contravariant shift vector components X^r, X^θ and X^ϕ given by:
(see Appendices F and G for details all in [12])(see Section 4 in [15])(see Section 3 in [10])

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \quad (113)$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \quad (114)$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \quad (115)$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ with $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2\sin^2\theta$. Then the covariant shift vector components X_r,X_θ and X_ϕ are given by:

$$X_i = \gamma_{ii} X^i \quad (116)$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \quad (117)$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \quad (118)$$

$$X_\phi = \gamma_{\phi\phi} X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \quad (119)$$

The g_{00} spacetime metric tensor component is given by:

$$g_{00} = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi) \quad (120)$$

$$X_r X^r = (vs(t)[\sin\phi][2f(r)\ cos\theta])^2 \quad (121)$$

$$X_\theta X^\theta = r^2(vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta])^2 \quad (122)$$

$$X_\phi X^\phi = r^2 sin^2\theta([vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]])^2 \quad (123)$$

$$g_{00} = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi) \tag{124}$$

$$X_r X^r = (vs(t)[\sin\phi][2f(r)\ cos\theta])^2 \tag{125}$$

$$X_\theta X^\theta = r^2(vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta])^2 \tag{126}$$

$$X_\phi X^\phi = r^2 sin^2\theta([vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]])^2 \tag{127}$$

The Relativistic Geometrical Horizon $g_{00} = 0$ never occurs in the case of the Natario warp drive $3D$ spherical coordinates with constant speeds due to the terms r^2 and $(r)f'(r)$.Also the terms $\cos\theta$ and $\sin\theta$ complements each other except when $\phi = 90$,$\sin\phi = 1$,$\cos\phi = 0$,$\theta = 0$,$\sin\theta = 0$ and $\cos\theta = 1$.The RGH do not occurs unless $\theta = 0$,$\cos\theta = 1$,$\sin\theta = 0$,$\phi = 90$,$\sin\phi = 1$,$\cos\phi = 0$,$r^2 d\theta^2 = 0$ and $r^2\sin^2\theta d\phi^2 = 0$ and we recover in this case the $1+1$ spacetime and the RGH Natario Horizon.When $\phi = 90$,$\sin\phi = 1$,$\cos\phi = 0$,$\theta = 90$,$\sin\theta = 1$,$\cos\theta = 0$ but the term $(r)f'(r)$ avoids the Natario Horizon.When $\phi = 90$,$\sin\phi = 1$,$\cos\phi = 0$,$\theta = 0$,$\sin\theta = 0$ and $\cos\theta = 1$ we recover the $1+1$ spacetime and the RGH Natario Horizon.

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$ we have the following situation:

$$g_{00} = (1 - X_\phi X^\phi) \tag{128}$$

$$X_\phi X^\phi = r^2 sin^2\theta([vs(t)][cot\theta[2(f(r)) + (rf'(r))]])^2 \tag{129}$$

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$,$\theta = 90$,$\sin\theta = 1$,$\cos\theta = 0$ or when $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$,$\theta = 0$,$\sin\theta = 0$,$\cos\theta = 1$ we have only a $g_{00} = 1$!!!! but fortunately no RGH Natario Horizon.

Due to the presence of 3 dimensions the RGH can be avoided.This resembles a situation described in section 8 and eqs 99 to 105 in [10] for null-like geodesics $ds^2 = 0$ Horizons.

Metrics of the form given below appears many times in General Relativity.Applyimg the Einstein Summing Convention we have:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu \tag{130}$$

$$ds^2 = g_{00}dx^0 dx^0 + g_{01}dx^0 dx^1 + ... + g_{33}dx^3 dx^3 \tag{131}$$

General Relativity metrics are physically consistent only when $g_{00} \neq 0$.A given spacetime metric cannot have RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The Natario warp drive in $3D$ spherical coordinates with constant speeds is physically consistent because it can avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$.

8 The Relativistic Geometrical Horizons(RGH) concept applied to the $3D$ spherical Natario warp drive-Variable speeds case

The Natario warp drive equation for variable velocities in the original $3+1$ ADM formalism in real $3D$ spherical coordinates is given by:(see Appendix J in [12] for details)

$$ds^2 = ((1-2X_t+X_tX^t)-X_rX^r-X_\theta X^\theta-X_\phi X^\phi)dt^2+2(X_rdr+X_\theta d\theta+X_\phi d\phi)dt-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 \tag{132}$$

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX is given by:

$$nX = X^te_t + X^re_r + X^\theta e_\theta + X^\phi e_\phi \tag{133}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result all in [12])(see Section 5 in [15])(see Section 3 in [12])(see Appendices K and L for details in [15])

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{134}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{135}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{136}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{137}$$

But remember that $dl^2 = \gamma_{ii}dx^idx^i = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$ with $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2\sin^2\theta$.Remember also that $\gamma_{tt} = 1$. Then the covariant shift vector components X_t,X_r,X_θ and X_ϕ are given by:

$$X_i = \gamma_{ii}X^i \tag{138}$$

$$X_t = \gamma_{tt}X^t \tag{139}$$

$$X_t = \gamma_{tt}X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) = X^t \tag{140}$$

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = X^r \tag{141}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2X^\theta = -r^2(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{142}$$

$$X_\phi = \gamma_{\phi\phi}X^\phi = r^2sin^2\theta X^\phi = r^2sin^2\theta(2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{143}$$

The g_{00} spacetime metric tensor component is given by:

$$g_{00} = ((1 - 2X_t + X_t X^t) - X_r X^r - X_\theta X^\theta - X_\phi X^\phi) \tag{144}$$

$$g_{00} = (1 - 2X_t + X_t X^t - X_r X^r - X_\theta X^\theta - X_\phi X^\phi) \tag{145}$$

$$2X_t = 4(rf(r)a))(\sin\phi)(\cos\theta) \tag{146}$$

$$X_t X^t = [2(rf(r)a))(\sin\phi)(\cos\theta)]^2 \tag{147}$$

$$X_r X^r = [(2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta)]^2 \tag{148}$$

$$X_\theta X^\theta = r^2[(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta)]^2 \tag{149}$$

$$X_\phi X^\phi = r^2 sin^2\theta[(2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta)]^2 \tag{150}$$

Natario used Polar Coordinates(See pg 4 in [1]) in his original work but Polar Coordinates are not real $3D$ coordinates since it uses only the two dimensional Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.(see Appendix D about Polar Coordinates in [15]).For a real $3D$ Spherical Coordinates another alternative warp drive vectors were presented in this work.Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about $3D$ Spherical Coordinates in [15]).

Remember also that a real spaceship must accelerate or de-accelerate

In order to make easy our analysis we must decompose the g_{00} spacetime metric tensor component into:

$$g_{00} = (1 + A - B) \tag{151}$$

$$A = -2X_t + X_t X^t \dashrightarrow A = X_t X^t - 2X_t \dashrightarrow X_t X^t \gg -2X_t \tag{152}$$

$$B = X_r X^r + X_\theta X^\theta + X_\phi X^\phi \tag{153}$$

In order to avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$ the condition in which $1 + A = B$ can never occurs.

We must evaluate the terms $1 + A$ and B separately.

- 1)-Evaluating first the term $1 + A$:

$$1 + A = 1 - 2X_t + X_t X^t \dashrightarrow 1 + X_t X^t - 2X_t \tag{154}$$

$$X_t X^t = [2(rf(r)a))(\sin\phi)(\cos\theta)]^2 \tag{155}$$

$$2X_t = 4(rf(r)a))(\sin\phi)(\cos\theta) \tag{156}$$

$$1 + A = 1 + [2(rf(r)a))(\sin\phi)(\cos\theta)]^2 - 4(rf(r)a))(\sin\phi)(\cos\theta) \tag{157}$$

When $\phi = 90$,$\sin\phi = 1$ and we have:

$$1 + A = 1 + [2(rf(r)a))(\cos\theta)]^2 - 4(rf(r)a))(\cos\theta) \tag{158}$$

In the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 0$,$\cos\theta = 1$ and we are left with:

$$1 + A = 1 + [2(rf(r)a))]^2 - 4(rf(r)a)) \tag{159}$$

In the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 90$,$\cos\theta = 0$ and we are left with:

$$1 + A = 1 \dashrightarrow A = 0!!!! \tag{160}$$

When $\phi = 0$,$\sin\phi = 0$ regardless of any value for θ and we have:

$$1 + A = 1 \dashrightarrow A = 0!!!! \tag{161}$$

The "pilot" angle is ϕ and since an accelerated warp drive must possesses acceleration then it is desirable that $\phi \neq 0$ and $\theta \neq 90$.

If $\phi = 0$ or $\theta = 90$ then $1 + A = 1$ and $g_{00} = (1 + A - B)$,$g_{00} = (1 - B)$.This is not a problem.The term B can avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$ because $B \neq 1$ and is the next evaluation.

Back to the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 0$,$\cos\theta = 1$:

$$1 + A = 1 + [2(rf(r)a))]^2 - 4(rf(r)a)) \dashrightarrow [2(rf(r)a))]^2 \gg 4(rf(r)a)) \dashrightarrow 1 + A \gg 1 \dashrightarrow A \gg 1 \tag{162}$$

The term a is the acceleration of the ship and therefore we must have $a \gg 1$

- 2)-Evaluating now the term B:

$$B = X_r X^r + X_\theta X^\theta + X_\phi X^\phi \tag{163}$$

$$X_r X^r = [(2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta)]^2 \tag{164}$$

$$X_\theta X^\theta = r^2[(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta)]^2 \tag{165}$$

$$X_\phi X^\phi = r^2 sin^2\theta[(2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta)]^2 \tag{166}$$

When $\phi = 90$,$\sin\phi = 1$,$\cos\phi = 0$ and we have:

$$X_r X^r = [(2at)[2f(r)^2 + (rf'(r))](\cos\theta)]^2 \tag{167}$$

$$X_\theta X^\theta = r^2[(2f(r)at)[2f(r) + rf'(r)](\sin\theta)]^2 \tag{168}$$

$$B = [(2at)[2f(r)^2 + (rf'(r))](\cos\theta)]^2 + r^2[(2f(r)at)[2f(r) + rf'(r)](\sin\theta)]^2 \tag{169}$$

In the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 0$,$\cos\theta = 1$,$\sin\theta = 0$ and we are left with:

$$X_r X^r = [(2at)[2f(r)^2 + (rf'(r))]]^2 \tag{170}$$

$$B = [(2at)[2f(r)^2 + (rf'(r))]]^2 \tag{171}$$

In the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 90$,$\cos\theta = 0$,$\sin\theta = 1$ and we are left with:

$$X_\theta X^\theta = r^2[(2f(r)at)[2f(r) + rf'(r)]]^2 \tag{172}$$

$$B = r^2[(2f(r)at)[2f(r) + rf'(r)]]^2 \tag{173}$$

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$ and we have:

$$X_\phi X^\phi = r^2 sin^2\theta[(2f(r)at)[2f(r) + (rf'(r))](cot\theta)]^2 \tag{174}$$

$$B = r^2 sin^2\theta[(2f(r)at)[2f(r) + (rf'(r))](cot\theta)]^2 \tag{175}$$

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$,$\theta = 0$,$\cos\theta = 1$,$\sin\theta = 0$ and we have a $B = 0$.

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$,$\theta = 90$,$\cos\theta = 0$,$\sin\theta = 1$ and we also have a $B = 0$.

Now we can place altogether our analysis of the separate independent terms $1 + A$ and B for the spacetime metric tensor g_{00}:

$$g_{00} = (1 + A - B) \tag{176}$$

We must concentrate ourselves in the boundary conditions:

When $\phi = 90$,$\sin\phi = 1$ and we have:

$$1 + A = 1 + [2(rf(r)a))(\cos\theta)]^2 - 4(rf(r)a))(\cos\theta) \tag{177}$$

In the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 0$,$\cos\theta = 1$ and we are left with:

$$1 + A = 1 + [2(rf(r)a))]^2 - 4(rf(r)a)) \tag{178}$$

When $\phi = 90$,$\sin\phi = 1$,$\cos\phi = 0$ and we have:

$$X_r X^r = [(2at)[2f(r)^2 + (rf'(r))](\cos\theta)]^2 \tag{179}$$

$$X_\theta X^\theta = r^2[(2f(r)at)[2f(r) + rf'(r)](\sin\theta)]^2 \tag{180}$$

$$B = [(2at)[2f(r)^2 + (rf'(r))](\cos\theta)]^2 + r^2[(2f(r)at)[2f(r) + rf'(r)](\sin\theta)]^2 \tag{181}$$

In the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 0$,$\cos\theta = 1$,$\sin\theta = 0$ and we are left with:

$$X_r X^r = [(2at)[2f(r)^2 + (rf'(r))]]^2 \tag{182}$$

$$B = [(2at)[2f(r)^2 + (rf'(r))]]^2 \tag{183}$$

$$g_{00} = 1 + [2(rf(r)a))]^2 - 4(rf(r)a)) - [(2at)[2f(r)^2 + (rf'(r))]]^2 \tag{184}$$

In the case of $\phi = 90$,$\sin\phi = 1$,$\theta = 90$,$\cos\theta = 0$ and we are left with:

$$1 + A = 1 \dashrightarrow A = 0!!!! \tag{185}$$

$$B = r^2[(2f(r)at)[2f(r) + rf'(r)]]^2 \tag{186}$$

$$g_{00} = 1 - r^2[(2f(r)at)[2f(r) + rf'(r)]]^2 \tag{187}$$

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$ and we have:

$$X_\phi X^\phi = r^2 sin^2\theta[(2f(r)at)[2f(r) + (rf'(r))](cot\theta)]^2 \tag{188}$$

$$B = r^2 sin^2\theta[(2f(r)at)[2f(r) + (rf'(r))](cot\theta)]^2 \tag{189}$$

When $\phi = 0$,$\sin\phi = 0$ regardless of any value for θ and we have:

$$1 + A = 1 \dashrightarrow A = 0!!!! \tag{190}$$

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$,$\theta = 0$,$\cos\theta = 1$,$\sin\theta = 0$ and we have a $B = 0$.

When $\phi = 0$,$\sin\phi = 0$,$\cos\phi = 1$,$\theta = 90$,$\cos\theta = 0$,$\sin\theta = 1$ and we also have a $B = 0$.

$$g_{00} = (1 + A - B) \dashrightarrow g_{00} = 1!!!!! \tag{191}$$

In the $3D$ accelerated Natario warp drive the RGH Natario Horizon never occurs !!!!

We will never have a $g_{00} = 0!!!!!$

This resembles section 8 eqs 162 to 170 about Horizons in null-like geodesics $ds^2 = 0$ in [12].

Due to the presence of 3 dimensions the RGH can be avoided.

In the equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{192}$$

The contravariant and the corresponding covariant shift vectors compensates each other in order to avoid the Natario Horizon RGH $g_{00} = 0$.

The g_{00} spacetime metric tensor component is given by:

$$g_{00} = (1 - 2X_t + X_t X^t - X_r X^r - X_\theta X^\theta - X_\phi X^\phi) \tag{193}$$

$$2X_t = 4(rf(r)a))(\sin\phi)(\cos\theta) \tag{194}$$

$$X_t X^t = [2(rf(r)a))(\sin\phi)(\cos\theta)]^2 \tag{195}$$

$$X_r X^r = [(2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta)]^2 \tag{196}$$

$$X_\theta X^\theta = r^2[(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta)]^2 \tag{197}$$

$$X_\phi X^\phi = r^2 sin^2\theta[(2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta)]^2 \tag{198}$$

Even in the range of the boundary conditions evaluated here the term $1 - 2X_t + X_t X^t$ compensates the term $X_r X^r + X_\theta X^\theta + X_\phi X^\phi$ and the RGH Natario Horizon $g_{00} = 0$ never occurs.

Metrics of the form given below appears many times in General Relativity.Applyimg the Einstein Summing Convention we have:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \tag{199}$$

$$ds^2 = g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + ... + g_{33} dx^3 dx^3 \tag{200}$$

General Relativity metrics are physically consistent only when $g_{00} \neq 0$.A given spacetime metric cannot have RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The Natario warp drive in $3D$ spherical coordinates with variable speeds is physically consistent because it can avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$.

9 Negative energy density in the Natario warp drive

The negative energy density according to Natario in Polar Coordinates is given by(see pg 5 in [1])[4]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2\cos^2\theta + \left(f'(rs) + \frac{r}{2}f''(rs)\right)^2\sin^2\theta\right] \tag{201}$$

In the bottom of pg 4 in [1] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = r\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{r}$ and in $\sin(\theta) = \frac{y}{r}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2(\frac{x}{rs})^2 + \left(f'(rs) + \frac{r}{2}f''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{202}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $r^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $r^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2\right] \tag{203}$$

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [21])

The distribution of energy presented here is valid only for the Natario warp drive vector in Polar Coordinates.

But for the case of the warp drive vector in $3D$ Spherical Coordinates equations we can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pg 342 in [3], pg 276 in [26],pgs $454, 457, 560$ in [27] pg 98 in [28],pg 178 in [29]).

Appendix C pgs $551 - 555$ in [27] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3 + 1$ spacetime metric using $Mathematica$.

[4] $f(r)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

Unfortunately we dont have access to these programs *Maple* or *Mathematica* so we have our hands "tied up".

For negative energy densities in the Natario warp drive and how these levels of energy can be ameliorated see section 3 in [16] and section 2 in [24].

10 Conclusion

In 1994 Alcubierre developed the first warp drive work using the original $3+1$ ADM-MTW formalism.[2].

Seven years later in 2001 the same original $3+1$ ADM-MTW formalism appeared in the first part of the second warp drive work developed by Natario.[1].

But in 1997 the concept of the Relativistic Geometrical Horizon RGH was introduced by Hiscock in his work about the Alcubierre warp drive.[20].

According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00} = 0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears.

Unfortunately the Alcubierre warp drive suffers from this pathology because in a given point of the Alcubierre warped region the $g_{00} = 0$ appears generating the Alcubierre Horizon equivalent to the Event Horizon of the Schwarzschild black hole.The point of the Alcubierre warped region where the Alcubierre Horizon is located is the Alcubierre radius similar to the Schwarzschild radius.

The Natario warp drive was formulated in polar coordinates in a $2+1$ spacetime.A dimensional reduction of the Natario warp drive from a $2+1$ to a $1+1$ spacetimes in a given point of the Natario warped region the $g_{00} = 0$ appears generating the Natario Horizon in a $1+1$ spacetime.

But in the original Natario warp drive in polar coordinates in a $2+1$ spacetime the $g_{00} = 0$ never appears.Due to the presence of a second spatial dimension the $g_{00} = 0$ can be avoided.The Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$,the shift vectors X^r,X^θ and the Natario vector $nX = X^r e_r + X^\theta e_\theta$ all in Polar Coordinates in the $2+1$ spacetime compensates each other to neutralize the $g_{00} = 0$.

In this work we presented the original Alcubierre warp drive and the original Natario warp drive with the Natario case in Polar Coordinates in the $2+1$ spacetime.Both warp drives were presented with constant speeds.The presentation of the RGH Horizons $g_{00} = 0$ as proposed by Hiscock applied to the original works of Alcubierre and Natario was given only for pedagogical purposes.

Natario used Polar Coordinates however Polar Coordinates are not real $3D$ coordinates since it uses only the two dimensional Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.Also a real warp drive must accelerate and de-accelerate in order to be accepted as a physical valid model.

In this work we presented the RGH Horizons $g_{00} = 0$ using more advanced versions of the Natario vectors in $3D$ spherical coordinates with constant or variable speeds as depicted in [10],[11],[12],[13] and [14] specially in section 4 in [12].

In the $3D$ version of the Natario warp drive in spherical coordinates in a $3+1$ spacetime with constant speed the $g_{00} = 0$ never appears.Due to the presence of three spatial dimensions the $g_{00} = 0$ can be avoided.The Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$,$\mathbf{e}_\phi$,the shift vectors X^r,X^θ,X^ϕ and the Natario vector $nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi$ all in Spherical Coordinates in the $3+1$ spacetime compensates each other to neutralize the $g_{00} = 0$.

However like in the case of a dimensional reduction from a $3+1$ to a $1+1$ spacetime the Natario RGH Horizons $g_{00} = 0$ appears again.

In the equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{204}$$

The contravariant and the corresponding covariant shift vectors compensates each other in order to avoid the Natario Horizon RGH $g_{00} = 0$.

The g_{00} spacetime metric tensor component is given by:

$$g_{00} = (1 - 2X_t + X_t X^t - X_r X^r - X_\theta X^\theta - X_\phi X^\phi) \tag{205}$$

Even in the range of the boundary conditions evaluated here the term $1 - 2X_t + X_t X^t$ compensates the term $X_r X^r + X_\theta X^\theta + X_\phi X^\phi$ and the RGH Natario Horizon $g_{00} = 0$ never occurs.

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel due to its negative energy density distribution.(see Appendices K,M and N all in [17]).Remember that negative energy density have repulsive gravitational behavior deflecting hazardous objects(asteroids comets photons etc) that may be in front of the ship.(see pg 116 in [21]).

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pg 98 in [8] and pg 377 in [9]).

For the experimental verification of the acceleration of the Universe see for example the bottom of pg 355 and top of pg 356 eq 8.155 in [7].

11 Appendix A:mathematical demonstration of the original Alcubierre and Natario warp drive equations for a constant speed vs in the original $3+1$ ADM-MTW Formalism

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg 64 in [4])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg 65) in [4].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg 65 in [4])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg 66 in [4])
(see also fig 21.2 pg 506 in [3] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg 507 in [3])[5]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[5] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs 507 and 508 in [3]
with the eqs (2.2.5) and (2.2.6) pg 67 in [4] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{206}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{207}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{208}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{209}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{210}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{211}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{212}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{213}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{214}$$

$$(dx^i)^2 = dx^i dx^i \tag{215}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{216}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{217}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{218}$$

Note that the expression above is exactly the eq (2.2.4) pg 67 in [4].It also appears as eq 1 pg 3 in [2].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{219}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{220}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{221}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{222}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{223}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{224}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{225}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{226}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{227}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{228}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{229}$$

The generic warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [1])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{230}$$

The generic Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{231}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{232}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i \beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{233}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i \beta^i \end{pmatrix} \tag{234}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{235}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{236}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i \beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have the generic warp drive equation:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{237}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{238}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{239}$$

Looking to the equation of the original Natario vector nX in Polar Coordinates(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{240}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])(see also Appendices A and B in [15] for details)

$$X^r = 2v_s f(r) \cos\theta \tag{241}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta \tag{242}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ are given by:

$$X_i = \gamma_{ii} X^i \tag{243}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r) \cos\theta = X^r \tag{244}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s(2f(r) + (r)f'(r)) \sin\theta \tag{245}$$

The generic equations of the warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{246}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{247}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{248}$$

Then the equation of the Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original 3 + 1 ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{249}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{250}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{251}$$

The Alcubierre warp drive can also be defined in a Natario-like vectorial form $nA = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.In the case of the Alcubierre warp drive (pg 3 in [1]) the Coordinates System is the Cartezian one with the original Alcubierre vector given by:

$$nA = X^i e_i = X^x e_x + X^y e_y + X^z e_z \tag{252}$$

$X^i = X^x = vsf(rs)$,$X^y = X^z = 0$ is the Alcubierre shift vector and $e_i = e_x$ is the Canonical Basis of the x-axis where the motion occurs.The Alcubierre vector is then given by:

$$nA = vsf(rs)e_x \tag{253}$$

Back again to the generic equations of the warp drive in the $3+1$ ADM formalism with signature $(+,-,-,-)$ given by:

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{254}$$

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{255}$$

In Cartezian Coordinates all the terms of the induced metric components $\gamma_{ii} = 1$ and therefore $X_i = \gamma_{ii} X^i = X_i = X^i$.Inserting the Alcubierre shift vector $X^i = X^x = vsf(rs)$,$X^y = X^z = 0$ we get:

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 = dt^2 - (dx - vsf(rs)dt)^2 - dy^2 - dz^2 \tag{256}$$

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{257}$$

$$ds^2 = dt^2 - (dx - vsf(rs)dt)^2 - dy^2 - dz^2 \tag{258}$$

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{259}$$

And we recover the original equation of the Alcubierre warp drive with signature $(+, -, -, -)$.Compare with eq 8 pg 4 in [2] and eq 7 pg 4 in [20].

Note that in the Alcubierre case the motion occurs only over the x-axis implying in an Alcubierre shift vector defined by $X^i = X^x = vsf(rs)$,$X^y = X^z = 0$,$X^y e_y + X^z e_z = 0$.

The equation of the original Natario warp drive vector nX in polar coordinates with a constant speed is given by(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{260}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [12])

$$X^r = 2v_s f(r) \cos\theta \tag{261}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta \tag{262}$$

The equation of the original Alcubierre warp drive vector nA in cartezian coordinates with a constant speed is given by:

$$nA = X^i e_i = X^x e_x + X^y e_y + X^z e_z \tag{263}$$

$$nA = vsf(rs)e_x \tag{264}$$

With the contravariant shift vector components X^i and given by $X^i = X^x = vsf(rs)$,$X^y = X^z = 0$.

The Natario original warp drive vector have 2 non-null contravariant shift vector components in 2 corresponding canonical basis while the Alcubierre original warp drive vector have only one non-null contravariant shift vector component in one corresponding canonical basis.

References

[1] *Natario J.,(*2002*). Classical and Quantum Gravity. 19 1157-1166,arXiv gr-qc/0110086*

[2] *Alcubierre M., (*1994*). Classical and Quantum Gravity. 11 L73-L77,arXiv gr-qc/0009013*

[3] *Misner C.W.,Thorne K.S.,Wheeler J.A,(Gravitation) (W.H.Freeman* 1973*)*

[4] *Alcubierre M.,(Introduction to* $3+1$ *Numerical Relativity) (Oxford University Press* 2008*)*

[5] *Lobo F.,(*2007*).,arXiv:0710.4474v1 (gr-qc)*

[6] *Lobo F.,(Wormholes Warp Drives and Energy Conditions) (Springer International Publishing AG* 2017*)*

[7] *Carroll S.,(Spacetime and Geometry:An Introduction to General Relativity) (Pearson Education and Addison Wesley Publishing Company Inc.*2004*)*

[8] *Wald R.,(General Relativity) (The University of Chicago Press* 1984*)*

[9] *Rindler W.,(Relativity Special General and Cosmological - Second Edition) (Oxford University Press* 2006*)*

[10] *Loup F.,(viXra:2506.0148v1)(*2025*)*

[11] *Loup F.,(viXra:2507.0068v1)(*2025*)*

[12] *Loup F.,(viXra:2509.0064v1)(*2025*)*

[13] *Loup F.,(viXra:2605.0014v1)(*2026*)*

[14] *Loup F.,(viXra:2605.0094v1)(*2026*)*

[15] *Loup F.,(viXra:2408.0126v1)(*2024*)*

[16] *Loup F.,(viXra:1702.0110v1)(*2017*)*

[17] *Loup F.,(viXra:1311.0019v1)(*2013*)*

[18] *Loup F.,(viXra:1308.0104v1)(*2013*)*

[19] *Loup F.,(viXra:1308.0033v1)(*2013*)*

[20] *Hiscock W., (*1997*). Classical and Quantum Gravity. 14 L183-L188,arXiv gr-qc/9707024*

[21] *Everett A.,Roman T.,(Time Travel and Warp Drives) (The University of Chicago Press* 2012*)*

[22] *Krasnikov S.,(Back in Time and Faster Than Light Travel in General Relativity) (Springer International Publishing AG* 2018*)*

[23] *Loup F.,(viXra:2004.0503v1)(*2020*)*

[24] *Loup F.,(viXra:1712.0348v1)(*2017*)*

[25] *Loup F.,(viXra:2607.0024v1)(*2026*)*

[26] *Schutz B.F.,(A First Course in General Relativity . Second Edition) (Cambridge University Press* 2009*)*

[27] *Hartle J.B.,(Gravity:An Introduction to Einstein General Relativity) (Pearson Education Inc. and Addison Wesley* 2003*)*

[28] *Plebanski J.,Krasinski A.,(Introduction to General Relativity and Cosmology) (Cambridge University Press* 2006*)*

[29] *DInverno R.,(Introducing Einstein Relativity) (Oxford University Press* 1998*)*

.