The Einstein and Landau-Lifshitz pseudotensors – a mathematical note on existence [a]

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Abstract: For a closed system, the conservation of energy and momentum has been affirmed through a vast array of experiments. In an attempt to reconcile the General Theory of Relativity with these findings, Einstein constructed, *ad hoc*, his so-called pseudotensor [A. Einstein, Ann. Phys. 49, 769 (1916)]. Yet this solution fell outside the tensorial mathematical structure of his theory. Landau and Lifshitz also constructed, *ad hoc*, an even more complex pseudotensor, as a proposed improvement upon the work of Einstein [The Classical Theory of Fields (Addison-Wesley Press, Inc., Cambridge, MA, 1951)]. Their pseudotensor is symmetric, unlike that proposed by Einstein. They advance that their pseudotensor yields a conservation law which also included angular momentum. However, once again, this approach leads to a mathematical construct which is not a tensor and thereby falls outside the very mathematical structure of Einstein's theory. Both pseudotensors, whether that advanced by Einstein or by Landau and Lifshitz, violate the rules of pure mathematics and therefore can hold no place in physics.

Key words: General relativity, pseudotensor; conservation of energy and momentum; first-order intrinsic differential invariant

I. INTRODUCTION

The usual conservation of energy and momentum for a closed system has been well established by a vast array of experiments. In an attempt to reconcile the General Theory of Relativity with the conservation laws, Einstein constructed his pseudotensor, t_σ^α, for the energy and momentum of his gravitational field alone. To this quantity, he added the energy-momentum tensor, T_σ^α, for the material sources of his gravitational field to obtain the total energy and momentum. *It must be remembered that besides the energy density of the matter there must also be given an energy density of the gravitational field, so that there can be no talk of principles of conservation of energy and momentum for matter alone.* [1] Accordingly, the total energy and momentum, E_σ^α, of Einstein's gravitational field and its material sources is,

$$E_\sigma^\alpha = T_\sigma^\alpha + t_\sigma^\alpha . \tag{1}$$

This is not a tensor equation, and therefore, a tensor divergence cannot be taken. The pseudotensor is constructed such that the ordinary divergence of E_σ^α is zero, to imply an alleged conservation law:

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$$\frac{\partial\left(T_{\sigma}^{\alpha}+t_{\sigma}^{\alpha}\right)}{\partial x^{\alpha}}=0\,. \tag{2}$$

> *Thus it results from our field equations of gravitation that the laws of conservation of momentum and energy are satisfied.* [2]

Einstein's pseudotensor is not symmetric. Landau and Lifshitz [3] proposed their alternative pseudotensor, which is symmetric, to purportedly include conservation of angular momentum. Both pseudotensors are also utilised to determine the energy and momentum of Einstein's alleged gravitational waves. These pseudotensors were constructed *ad hoc* simply to satisfy Eq.(2). However, not only are they not tensors, they violate the rules of pure mathematics. Consequently, they lack all scientific merit.

II. EINSTEIN'S PSEUDOTENSOR

The Riemann-Christoffel symbol of the second kind, denoted $\Gamma_{\beta\gamma}^{\alpha}$, is defined by,

$$\Gamma_{\beta\gamma}^{\alpha}=\frac{1}{2}g^{\alpha\omega}\left(\frac{\partial g_{\omega\gamma}}{\partial x^{\beta}}-\frac{\partial g_{\gamma\beta}}{\partial x^{\omega}}+\frac{\partial g_{\beta\omega}}{\partial x^{\gamma}}\right). \tag{3}$$

$\Gamma_{\beta\gamma}^{\alpha}$ is not a tensor and is composed solely of the components of the metric tensor and their first-derivatives. Einstein's pseudotensor, denoted t_{σ}^{α}, is defined by [2],

$$t_{\sigma}^{\alpha}=\frac{1}{\kappa}\left(\frac{1}{2}\delta_{\sigma}^{\alpha}g^{\mu\nu}\Gamma_{\mu\beta}^{\lambda}\Gamma_{\nu\lambda}^{\beta}-g^{\mu\nu}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\sigma}^{\beta}\right), \tag{4}$$

where κ is a constant. Although t_{σ}^{α} is not a tensor, it is claimed that it acts "like a tensor" under linear transformations of coordinates. Since it allegedly acts like a tensor, it can be contracted like a tensor, to produce an invariant t, thus,

$$t=t_{\alpha}^{\alpha}=\frac{1}{\kappa}g^{\mu\nu}\Gamma_{\mu\beta}^{\lambda}\Gamma_{\nu\lambda}^{\beta}\,. \tag{5}$$

From Eq.(3) it is easily seen that Eq.(5) is a first-order intrinsic differential invariant; that is, it is an invariant composed solely of the components of the metric tensor and their first derivatives. But the pure mathematicians proved, in 1901 [4], that first-order intrinsic differential invariants do not exist. Thus, Einstein's pseudotensor is incapable of contributing to physics. Nevertheless, astronomers and cosmologists use this pseudotensor to represent the energy-momentum of Einstein's gravitational field and hence, his gravitational waves. Strikingly, they implement this pseudotensor in their calculations, which, as a result, can have no validity.

III. THE LANDAU-LIFSHITZ PSEUDOTENSOR

The Riemann-Christoffel symbol of the first kind, denoted $\Gamma_{\alpha\beta\gamma}$, is defined by,

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial g_{\gamma\alpha}}{\partial x^{\beta}}\right). \quad (6)$$

$\Gamma_{\alpha\beta\gamma}$ is not a tensor and is composed solely of the first derivatives of the components of the metric tensor. The Landau-Lifshitz pseudotensor, denoted t^{ik}, is defined by [3],

$$t^{ik} = \frac{c^4}{16\pi k}\left\{\begin{array}{l} \left(g^{il}g^{km} - g^{ik}g^{lm}\right)\left(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{\ln}\Gamma^p_{mp}\right) \\ + g^{il}g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{lp} - \Gamma^k_{np}\Gamma^p_{lm} - \Gamma^k_{lm}\Gamma^p_{np}\right) \\ + g^{kl}g^{mn}\left(\Gamma^i_{lp}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{lp} - \Gamma^i_{np}\Gamma^p_{lm} - \Gamma^i_{lm}\Gamma^p_{np}\right) \\ + g^{lm}g^{np}\left(\Gamma^i_{\ln}\Gamma^k_{mp} - \Gamma^i_{lm}\Gamma^k_{np}\right) \end{array}\right\}, \quad (7)$$

where k in the opening denominator is a constant, c the speed of light in vacuum.

Although t^{ik} is not a tensor, it is claimed that it acts "like a tensor" under linear transformations of coordinates. Since it acts like a tensor, a superscript can be lowered, thus,

$$t^i_r = g_{rk}t^{ik} = \frac{c^4}{16\pi k}\left\{\begin{array}{l} \left(g_{rk}g^{il}g^{km} - g_{rk}g^{ik}g^{lm}\right)\left(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{\ln}\Gamma^p_{mp}\right) \\ + g^{il}g^{mn}\left(\Gamma_{rlp}\Gamma^p_{mn} + \Gamma_{rmn}\Gamma^p_{lp} - \Gamma_{rnp}\Gamma^p_{lm} - \Gamma_{rlm}\Gamma^p_{np}\right) \\ + g_{rk}g^{kl}g^{mn}\left(\Gamma^i_{lp}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{lp} - \Gamma^i_{np}\Gamma^p_{lm} - \Gamma^i_{lm}\Gamma^p_{np}\right) \\ + g^{lm}g^{np}\left(\Gamma^i_{\ln}\Gamma_{rmp} - \Gamma^i_{lm}\Gamma_{rnp}\right) \end{array}\right\}, \quad (8)$$

which can then be contracted like a tensor, to produce an invariant t, thus,

$$t = t^i_i = \frac{c^4}{16\pi k}\left\{\begin{array}{l} \left(\delta^l_k g^{km} - \delta^i_i g^{lm}\right)\left(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{nm}\Gamma^n_{\ln}\Gamma^p_{mp}\right) \\ + g^{il}g^{mn}\left(\Gamma_{ilp}\Gamma^p_{mn} + \Gamma_{imn}\Gamma^p_{lp} - \Gamma_{inp}\Gamma^p_{lm} - \Gamma_{ilm}\Gamma^p_{np}\right) \\ + g^{kl}g^{mn}\left(\Gamma_{klp}\Gamma^p_{mn} + \Gamma_{kmn}\Gamma^p_{lp} - \Gamma_{knp}\Gamma^p_{lm} - \Gamma_{klm}\Gamma^p_{np}\right) \\ + g^{lm}g^{np}\left(\Gamma^i_{\ln}\Gamma_{imp} - \Gamma^i_{lm}\Gamma_{inp}\right) \end{array}\right\}, \quad (9)$$

where δ^l_k is the Kronecker delta[b].

[b] $\delta^l_k = 1$ when $k = l$, but $= 0$ when $k \neq l$.

From Eqs. (3) and (6) it is easily seen that Eq.(9) is a first-order intrinsic differential invariant. Thus, the Landau-Lifshitz pseudotensor is mathematically invalid. Yet, once again, astronomers and cosmologists use it to represent the energy-momentum of Einstein's gravitational field and his gravitational waves. Any ensuing calculations are without merit.

V. CONCLUSION

First-order intrinsic differential invariants do not exist. Einstein's pseudotensor and the Landau-Lifshitz pseudotensor are invalid combinations of mathematical symbols because they produce, upon contraction, first-order intrinsic differential invariants.

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