A Pedagogical Reconstruction of Maxwell's Equations: How Charge Conservation, Helmholtz's Theorem, and the Speed of Light Make Them Natural

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Maxwell's equations are traditionally introduced as four independent postulates, each grounded in a distinct experimental law. Although empirically sound, this approach can obscure the logical structure that unifies them. This paper offers a pedagogical reconstruction based on local charge conservation, Helmholtz's theorem, and the experimental fact that electromagnetic disturbances propagate in vacuum with speed c. A few natural auxiliary assumptions – linearity, locality, and the simplest possible coupling to sources – are stated explicitly. From the continuity equation and the physical necessity of a mediating field we argue that the electric and magnetic fields must be introduced. Helmholtz's theorem then demands that both divergences and curls of these fields be specified; coupling them to the available scalar and vector sources yields the inhomogeneous equations. Consistency with charge conservation forces the displacement current, making it a logical requirement rather than an empirical addition. The homogeneous equations follow from the same framework: $\nabla \cdot \mathbf{B} = 0$ is the most parsimonious choice consistent with experiment, and requiring wave propagation at speed c fixes Faraday's law uniquely as $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$. Once Maxwell's equations are established, the Lorentz force density $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$ emerges as the only dimensionally consistent expression compatible with the local conservation of momentum derived directly from the field equations. The argument employs only elementary vector calculus and is accessible to advanced undergraduates. We suggest that this unified narrative, which reveals the internal coherence of electromagnetism, can serve as a valuable complement to the traditional empirical presentation.

Keywords: Maxwell's equations, charge conservation, Helmholtz theorem, physics education, electromagnetism, Lorentz force

I. INTRODUCTION

Maxwell's equations unify electricity, magnetism, and optics into a single theoretical framework. In most textbooks, these equations are presented as four postulates, each motivated by a distinct experimental law: Gauss's law for electricity, Gauss's law for magnetism, Faraday's law of induction, and the Ampère-Maxwell law [1–3]. This empirical approach connects theory directly to observation and traces the historical development of electromagnetic ideas during the nineteenth century [4–6].

Nevertheless, the standard presentation can leave students with a feeling of arbitrariness. Why do the equations take this particular form? Why is the displacement current necessary, and why does the magnetic field have no monopole source? The answers often appear as facts to be memorized, rather than consequences that could be grasped from deeper principles [7, 8]. The four equations can seem like disconnected pieces, and their elegant unity may remain hidden until the student encounters tensor calculus or variational formulations in more advanced courses [9, 10].

Curricular constraints exacerbate the problem. In many introductory electromagnetism courses, particularly in engineering programs that devote only one semester to the subject, Maxwell's equations appear in the final weeks, after lengthy treatments of electrostatics, magnetostatics, and circuit theory [11, 12]. Many students thus complete the course without seeing the full theory applied to wave propagation, optics, or radiation [13, 14].

This paper presents an alternative narrative that can help overcome these limitations. Using only elementary vector calculus, we show how Maxwell's equations can be reconstructed from three fundamental ingredients together with a small set of natural auxiliary assumptions. The three ingredients are:

1. **Local charge conservation**, expressed by the continuity equation $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$, which embodies the empirical fact that electric charge cannot be created or destroyed.

2. **Helmholtz's theorem**, a mathematical result stating that any sufficiently well-behaved vector field that vanishes at infinity is uniquely determined by its divergence and curl.

3. The **experimental fact** that disturbances in the electromagnetic field propagate in vacuum with a finite and universal speed c.

The additional assumptions – linearity of the field equations, locality (first-order differential relations), and the use of the only available scalar and vector sources ρ and $\mathbf{J}$ – are made explicit at every step. We emphasize that this is a pedagogical reconstruction, not a purely logical deduction in the mathematical sense [15, 16]. The aim is not to derive Maxwell's equations from a minimal set of axioms without further input, but to build a transparent narrative that reveals the internal coherence of the theory. The central insight – that the displacement current is required by charge conservation – was first articulated by Maxwell himself. Our contribution is to embed this historical insight in a modern pedagogical framework that uses Helmholtz's theorem as an organizing principle, thereby reducing the sense of arbitrariness.

A further strength of the reconstruction is that the Lorentz force law, rather than being introduced as an independent postulate, emerges naturally once Maxwell's equations are in place. By combining a well-known vector identity that follows from the field equations with elementary dimensional analysis, one obtains the force density $\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$ as the only consistent coupling between fields and charges. This closes the logical circle: the same principles that make Maxwell's equations natural also select the force law.

We do not advocate replacing the traditional empirical presentation. Both approaches have merit [21, 22]. The empirical narrative grounds the theory in experiment and shows the creative process of discovery; the logical reconstruction reveals the internal economy of the finished theory. Ideally, students should encounter both perspectives, with the logical approach serving as a unifying summary after the empirical development.

The paper is organized as follows. Section II recalls the continuity equation and discusses why charge conservation, together with causality and locality, leads naturally to a field description. Section III applies Helmholtz's theorem to the fields, couples them to their sources, and shows that consistency with charge conservation forces the displacement current. Section IV completes the set by determining the divergence of $\mathbf{B}$ and the curl of $\mathbf{E}$ using parsimony and the wave propagation requirement. Section V demonstrates how the Lorentz force emerges from the field equations and dimensional analysis. Section VI examines the pedagogical implications, lists all assumptions, explains why alternative theories are excluded, and acknowledges limitations. An appendix provides a consistency check by counting degrees of freedom.

II. LOCAL CHARGE CONSERVATION AND THE NECESSITY OF FIELDS

A. The continuity equation

Local charge conservation is an experimental fact of great precision. It states that the total charge inside any volume can change only by a net flow of charge across the boundary. In differential form, this is the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{1}$$

where $\rho(\mathbf{x}, t)$ is the charge density and $\mathbf{J}(\mathbf{x}, t)$ is the current density. This equation is not specific to electromagnetism; it is a universal kinematic constraint for any locally conserved quantity. It links ρ and $\mathbf{J}$ through time evolution.

B. The physical need for a mediating field

The continuity equation describes local conservation: a change of charge in a region is compensated by a flux through the surface. This description immediately raises the question of how the information about a moving charge reaches other charges. Relativistic causality forbids instantaneous action at a distance [23, 24]. When a charge accelerates, its influence cannot be felt elsewhere at the same instant; some physical entity must propagate the interaction at a finite speed.

The electromagnetic field is precisely this mediator. A field is a quantity defined at every point in space and time, capable of carrying energy, momentum, and information. Charge conservation alone does not logically compel the introduction of fields, but the combination of conservation, causality, and locality makes the field description practically inevitable. The differential form of the continuity equation is essential here: it enforces conservation at every space-time point, a requirement that becomes mandatory in a relativistic setting where covariance demands local laws [2, 25].

We therefore assume the existence of two fundamental vector fields, the electric field $\mathbf{E}(\mathbf{x}, t)$ and the magnetic induction $\mathbf{B}(\mathbf{x}, t)$. Their dimensions are $[\mathbf{E}] = \mathrm{N\,C^{-1}} = \mathrm{V\,m^{-1}}$ and $[\mathbf{B}] = \mathrm{N\,A^{-1}\,m^{-1}} = \mathrm{T}$. At this stage we defer the question of how these fields act on charges; as will be shown in Sec. V, the Lorentz force law follows directly from the field equations themselves once they have been established. Our immediate task is to find the equations that govern the dynamics of $\mathbf{E}$ and $\mathbf{B}$ and their coupling to the sources ρ and $\mathbf{J}$.

III. HELMHOLTZ'S THEOREM AND THE INHOMOGENEOUS EQUATIONS

A. Helmholtz's theorem as a completeness requirement

Helmholtz's theorem – the fundamental theorem of vector calculus – states that any sufficiently smooth vector field that vanishes at infinity is uniquely determined, up to an additive constant, by its divergence and curl [1, 17, 30]. The field can be decomposed as $\mathbf{F} = -\nabla\Phi + \nabla \times \mathbf{A}$, where Φ is fixed by $\nabla \cdot \mathbf{F}$ and $\mathbf{A}$ by $\nabla \times \mathbf{F}$. For time-dependent fields, this decomposition holds at each instant.

When we seek a complete field theory of electromagnetism, the electric and magnetic fields are the fundamental dynamical objects. To determine them fully we must specify four quantities at every point: $\nabla\cdot\mathbf{E}$, $\nabla\times\mathbf{E}$, $\nabla\cdot\mathbf{B}$, and $\nabla\times\mathbf{B}$. This requirement is not a choice but a mathematical necessity. The theory must provide these four pieces of information; otherwise the fields are not well defined.

B. Coupling fields to their sources

The only localized scalar and vector quantities that can serve as sources are the charge density ρ and the current density $\mathbf{J}$. The question is how to relate these sources to the divergences and curls of the fields.

We invoke three principles: linearity, locality, and first differential order. Linearity is suggested by the superposition principle observed in electrostatics and magnetostatics. Locality means the equations involve only the fields and their derivatives at the same point, without integrals or infinite series. First differential order is the simplest choice compatible with a second-order wave equation; higher derivatives would introduce extra degrees of freedom.

With these constraints, the only way to obtain a scalar from a vector field using first derivatives is the divergence, and the only way to obtain a vector from another vector is the curl. The natural couplings are therefore proportionalities between the divergence of one field and the scalar source ρ, and between the curl of the other field and the vector source $\mathbf{J}$. The association of $\mathbf{E}$ with ρ and $\mathbf{B}$ with $\mathbf{J}$ is guided by the fact that $\mathbf{E}$ exerts a force on stationary charges (as will emerge in Sec. V) while $\mathbf{B}$ acts only on moving charges. Thus we write

$$\nabla\cdot\mathbf{E} = \alpha\,\rho, \tag{2}$$

$$\nabla\times\mathbf{B} = \beta\,\mathbf{J}. \tag{3}$$

The constants α and β carry dimensions: $[\alpha] = \mathrm{N\,m^2\,C^{-2}}$ and $[\beta] = \mathrm{N\,A^{-2}}$. Their numerical values are not fixed by the internal logic of the theory; they must come from experiment.

C. Consistency with charge conservation: the displacement current

Equations (2) and (3) cannot be the complete set. Taking the divergence of Eq. (3) yields

$$\nabla\cdot(\nabla\times\mathbf{B}) = \beta\,\nabla\cdot\mathbf{J}, \tag{4}$$

and the left-hand side is identically zero for any smooth field. Thus $\nabla\cdot\mathbf{J} = 0$, which contradicts the continuity equation (1) whenever the charge density varies with time. The static Ampère law is incompatible with charge conservation.

The most natural repair, preserving linearity, locality, and first order, is to add a term involving the time derivative of the only other vector field present in the theory:

$$\nabla\times\mathbf{B} = \beta\,\mathbf{J} + \gamma\,\frac{\partial\mathbf{E}}{\partial t}. \tag{5}$$

The new constant γ has dimensions $[\gamma] = \mathrm{s^2\,m^{-2}}$. Taking the divergence of Eq. (5) and using Eqs. (2) and (1) gives

$$0 = \beta\nabla\cdot\mathbf{J} + \gamma\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) = -\beta\frac{\partial\rho}{\partial t} + \gamma\alpha\frac{\partial\rho}{\partial t} = (\gamma\alpha-\beta)\frac{\partial\rho}{\partial t}. \tag{6}$$

For this to hold for arbitrary charge distributions, we must have

$$\gamma\alpha = \beta, \tag{7}$$

a consistency condition that links the three constants. Substituting $\gamma = \beta/\alpha$ back into Eq. (5) yields the inhomogeneous Maxwell equations in their most general linear form:

$$\nabla\cdot\mathbf{E} = \alpha\,\rho, \tag{8}$$

$$\nabla\times\mathbf{B} = \beta\,\mathbf{J} + \frac{\beta}{\alpha}\frac{\partial\mathbf{E}}{\partial t}. \tag{9}$$

The term proportional to $\partial_t\mathbf{E}$ is the displacement current. It emerges not as an empirical discovery but as a mathematical necessity: without it, Ampère's law cannot be reconciled with charge conservation. This argument was first given by Maxwell himself [4, 5] and remains one of the most compelling illustrations of theoretical consistency in physics.

D. Experimental determination of the constants

The constants α and β encode the coupling strengths between fields and sources. They are fixed by static experiments. Coulomb's law for the force between point charges in vacuum gives $\alpha = 1/\varepsilon_0$, where ε_0 is the permittivity of free space, with dimensions $[\varepsilon_0] = \mathrm{C^2\,N^{-1}\,m^{-2}}$. The Biot-Savart law (or Ampère's force law) for steady currents yields $\beta = \mu_0$, the permeability of free space, with $[\mu_0] = \mathrm{N\,A^{-2}}$. Inserting these values we obtain

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{10}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}. \tag{11}$$

The product $\mu_0\varepsilon_0$ will shortly acquire an important physical meaning related to the speed of electromagnetic waves.

IV. COMPLETING THE SET: THE HOMOGENEOUS EQUATIONS

A. The remaining divergence and curl

Equations (10) and (11) provide $\nabla\cdot\mathbf{E}$ and $\nabla\times\mathbf{B}$. According to Helmholtz's theorem, specifying the divergence and curl of $\mathbf{B}$ and $\mathbf{E}$ is mandatory to determine the fields completely. The missing pieces are $\nabla\cdot\mathbf{B}$ and $\nabla\times\mathbf{E}$, the so-called homogeneous Maxwell equations.

B. The divergence of the magnetic field

What should $\nabla\cdot\mathbf{B}$ be? No scalar magnetic source analogous to the electric charge has ever been observed, despite extensive experimental searches [26–28]. The most economical choice consistent with all available data is

$$\nabla\cdot\mathbf{B}=0. \tag{12}$$

This equation is not a logical consequence of the preceding ones; it is a separate physical input justified by parsimony and by the absence of magnetic monopoles. A non-zero divergence would require introducing a new independent source, which finds no experimental support. In geometric language, Eq. (12) states that magnetic field lines have no sources or sinks; they form closed loops or extend to infinity.

C. Faraday's law from wave propagation

The final equation is $\nabla\times\mathbf{E}$. We again invoke linearity, locality, and first differential order. The only natural vector that can appear on the right-hand side, using only the fields already present, is the time derivative of the magnetic field. We therefore postulate

$$\nabla\times\mathbf{E}=\lambda\frac{\partial\mathbf{B}}{\partial t}, \tag{13}$$

where λ is a dimensionless constant to be determined.

To fix λ we require that the theory support wave solutions propagating in vacuum with the experimentally measured speed c. Consider a region free of charges and currents ($\rho=0$, $\mathbf{J}=0$). The equations reduce to

$$\nabla\cdot\mathbf{E}=0, \tag{14}$$
$$\nabla\cdot\mathbf{B}=0, \tag{15}$$
$$\nabla\times\mathbf{E}=\lambda\frac{\partial\mathbf{B}}{\partial t}, \tag{16}$$
$$\nabla\times\mathbf{B}=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}. \tag{17}$$

Taking the curl of Eq. (16) and using Eq. (17) gives

$$\nabla\times(\nabla\times\mathbf{E})=\lambda\frac{\partial}{\partial t}(\nabla\times\mathbf{B})=\lambda\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{18}$$

With the vector identity $\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$ and Eq. (14), we obtain a wave equation

$$\nabla^2\mathbf{E}=-\lambda\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{19}$$

This equation describes waves propagating with speed $v=1/\sqrt{-\lambda\mu_0\varepsilon_0}$. For a real speed we must have $\lambda<0$.

Experiment provides two crucial facts. First, electromagnetic waves in vacuum travel at speed c, so $v=c$. Second, precision measurements over more than a century have established the relation

$$\mu_0\varepsilon_0=\frac{1}{c^2}. \tag{20}$$

This connection between the static constants of electricity and magnetism and the speed of light was recognized by Maxwell as a deep link between apparently disparate phenomena. Substituting $v=c$ and Eq. (20) into the wave speed expression yields

$$\begin{aligned} c&=\frac{1}{\sqrt{-\lambda\mu_0\varepsilon_0}}=\frac{1}{\sqrt{-\lambda/c^2}}\\ &\implies c=\frac{c}{\sqrt{-\lambda}}\implies\sqrt{-\lambda}=1, \end{aligned} \tag{21}$$

so $\lambda=-1$. Faraday's law therefore takes the standard form

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}. \tag{22}$$

In this reconstruction, Faraday's law is not an independent postulate. It is the unique linear, first-order relation between $\nabla\times\mathbf{E}$ and $\partial_t\mathbf{B}$ that, together with the inhomogeneous equations already established, yields wave propagation at the observed speed c.

D. The complete set of Maxwell's equations

Collecting Eqs. (10), (22), (12), and (11), we have the full set of Maxwell's equations in SI units:

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0}, \tag{23}$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{24}$$
$$\nabla\cdot\mathbf{B}=0, \tag{25}$$
$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}. \tag{26}$$

The inhomogeneous equations (23) and (26) describe how charges and currents generate fields; the homogeneous equations (24) and (25) describe the intrinsic structure of the fields, valid even in vacuum. Together they form a consistent, complete, and elegant system [31, 32].

V. THE LORENTZ FORCE FROM MAXWELL'S EQUATIONS

With Maxwell's equations established, we can now determine how the fields act back on the charges that generate them. Good and Nelson showed that a local conservation law for momentum emerges directly from the field equations [29]. Taking the vector product of $\mathbf{E}$ with

Eq. (24) and the vector product of $\mathbf{B}$ with Eq. (26), and adding the results, yields

$$\epsilon_0\mathbf{E}\times(\nabla\times\mathbf{E})+\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B}) \\ =-\epsilon_0\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}+\mathbf{B}\times\mathbf{J}+\epsilon_0\mathbf{B}\times\frac{\partial\mathbf{E}}{\partial t}. \tag{27}$$

Using the vector identity $\mathbf{F}\times(\nabla\times\mathbf{F})=\frac{1}{2}\nabla(\mathbf{F}\cdot\mathbf{F})-(\mathbf{F}\cdot\nabla)\mathbf{F}$ for each field, together with the divergence equations (23) and (25), Eq. (27) can be rewritten after some algebra in the compact form

$$\frac{\partial}{\partial t}(\epsilon_0\mathbf{E}\times\mathbf{B})+\nabla\cdot\mathbf{T}=-(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}). \tag{28}$$

Here $\mathbf{T}$ is the Maxwell stress tensor, whose precise form is not needed for the present argument. Equation (28) has the structure of a local conservation law: the time derivative of a density plus the divergence of a flux equals a source term.

Dimensional analysis confirms the physical interpretation. The quantity $\epsilon_0\mathbf{E}\times\mathbf{B}$ has dimensions

$$[\epsilon_0][E][B]=\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}\times\mathrm{N}\,\mathrm{C}^{-1}\times\mathrm{N}\,\mathrm{A}^{-1}\,\mathrm{m}^{-1}=\mathrm{N}\,\mathrm{s}\,\mathrm{m}^{-3},$$

which are the dimensions of a momentum density. The divergence term $\nabla\cdot\mathbf{T}$ has dimensions of momentum flux density, $\mathrm{N}\,\mathrm{m}^{-2}$. Consequently, the right-hand side must represent a source of momentum, i.e., a force per unit volume. Indeed,

$$[\rho\mathbf{E}]=\mathrm{C}\,\mathrm{m}^{-3}\times\mathrm{N}\,\mathrm{C}^{-1}=\mathrm{N}\,\mathrm{m}^{-3},$$

and

$$[\mathbf{J}\times\mathbf{B}]=\mathrm{A}\,\mathrm{m}^{-2}\times\mathrm{N}\,\mathrm{A}^{-1}\,\mathrm{m}^{-1}=\mathrm{N}\,\mathrm{m}^{-3}.$$

Both terms have the dimensions of force density.

The identification of the right-hand side of Eq. (28) as the force density exerted by the field on the charges is logically compelling: it is the term that appears in the momentum balance derived from Maxwell's equations themselves, and it is the only combination of the available fields and sources with the correct dimensions. For a point charge q moving with velocity $\mathbf{v}$, the densities are $\rho=q\delta(\mathbf{r})$ and $\mathbf{J}=q\mathbf{v}\delta(\mathbf{r})$. Integrating the force density over the volume surrounding the charge yields the Lorentz force law

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}). \tag{29}$$

Thus, the Lorentz force is not an independent postulate appended to Maxwell's theory. It is the necessary expression of momentum conservation hidden within the field equations themselves, revealed by a straightforward vector manipulation and identified through elementary dimensional analysis. The same guiding principles – linearity, locality, and the simplest admissible structure – that shaped Maxwell's equations also select the force law. This derivation offers a satisfying closure to the logical reconstruction: the fields are generated by charges and currents according to Maxwell's equations, and those same equations imply that the fields act back on the charges through the Lorentz force.

VI. DISCUSSION: SCOPE, ASSUMPTIONS, AND PEDAGOGICAL IMPLICATIONS

A. What the reconstruction offers

The narrative constructed above offers a unified alternative to the traditional empirical presentation. By starting from the continuity equation, the physical need for a mediating field, and Helmholtz's theorem, and by making all supplementary assumptions transparent, we have shown that Maxwell's equations can be viewed as the natural outcome of a few deep principles.

A student following this argument can appreciate that the equations are not arbitrary but are linked by consistency requirements. The displacement current appears as a necessity forced by charge conservation, not as an ad hoc modification. Faraday's law emerges as the unique companion of Ampère's law that allows wave propagation at the known speed of light. Helmholtz's theorem clarifies why four equations are needed: two divergences and two curls. This demystifies the degree-of-freedom count and addresses the recurrent question of whether the system is overdetermined (see Appendix A).

The Lorentz force, far from being an extra postulate, flows naturally from the field equations. The vector manipulation of Good and Nelson, combined with dimensional analysis, shows that the momentum balance derived from Maxwell's equations forces the source term to be $\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}$. This closes the logical circle: the fields and their action on matter are two sides of the same theoretical coin.

Because the argument uses only elementary vector calculus, it is accessible to undergraduates early in their study of electromagnetism. It can be delivered as a single unifying lecture after students have been exposed to vector calculus and the basic experimental laws, or as a capstone summary at the end of a course. Instructors may adapt it to their own curricular needs; the reconstruction does not require adding a full extra lecture, but reorganizes material that is often already present in a more fragmented form [33, 34].

B. Explicit assumptions and the question of uniqueness

The reconstruction does not claim to derive Maxwell's equations from three principles alone. Several auxiliary assumptions are needed, and we list them here for transparency.

1. **Existence of fields.** Electromagnetic interactions are mediated by the vector fields $\mathbf{E}$ and $\mathbf{B}$ defined at every point in space and time. This is motivated by the need for local, causal interactions (Sec. II).

2. **Linearity.** The equations relating the fields to the sources and to each other are linear. This is the

simplest possibility and is strongly supported by the superposition principle observed in electrostatics and magnetostatics.

3. **Locality and first differential order.** The equations involve only the fields and their first derivatives evaluated at the same space-time point. This precludes non-local integral formulations or the appearance of higher-order derivatives, which would introduce new independent degrees of freedom.

4. **Sources.** The only independent scalar and vector sources are the electric charge density ρ and the electric current density $\mathbf{J}$. No other physical quantities (such as scalar magnetic charge or intrinsic spin densities) appear as sources.

5. **Helmholtz's theorem.** The fields are assumed to be sufficiently smooth and to vanish at infinity, so that they are uniquely determined by their divergences and curls.

6. **Charge conservation.** The sources satisfy the continuity equation (1).

7. **Experimental values of ε_0, μ_0, and c.** The static constants are determined by Coulomb and Ampère/Biot-Savart experiments, and the speed of light c and the relation $\mu_0\varepsilon_0 = 1/c^2$ are taken as empirical facts.

8. **Absence of magnetic monopoles.** $\nabla \cdot \mathbf{B} = 0$ is chosen as the simplest possibility consistent with all experiments.

The Lorentz force is not listed as an independent assumption because, as shown in Sec. V, it is a direct consequence of the field equations and dimensional analysis. Within this tightly constrained framework, Maxwell's equations are essentially unique. Theories that introduce, for example, a scalar potential with a non-Maxwellian constitutive relation involving $\partial\rho/\partial t$ are excluded because they violate one or more of the stated assumptions – typically by breaking linearity between the physical fields and their sources, by introducing higher-order derivatives, or by requiring extra independent fields beyond $\mathbf{E}$ and $\mathbf{B}$. The reconstruction therefore demonstrates not absolute logical inevitability but *near-inevitability*: given the simplest mathematical structure that can accommodate the known physics, Maxwell's equations are the natural outcome.

C. Relation to Maxwell's historical argument

The key step in this reconstruction – adding the displacement current to restore consistency with the continuity equation – was first taken by Maxwell. We claim no originality for this insight. Our contribution is to embed Maxwell's argument in a framework organized around Helmholtz's theorem and the explicit physical need for fields. This makes the overall structure more transparent and reduces the apparent arbitrariness of the homogeneous equations. The derivation of the Lorentz force via the Good-Nelson identity is also not new, but its integration into the pedagogical narrative shows that the force law is already encoded in the field equations. The reconstruction is intended to complement, not replace, the empirical narrative that connects the equations to their experimental roots [4, 5].

D. Connections to other areas of physics

The same structural pattern – a continuity equation for a conserved quantity combined with Helmholtz's theorem – appears in other classical field theories. In hydrodynamics, mass conservation leads to the Navier-Stokes equations, which are nonlinear due to convective terms [35, 36]. In gravitoelectromagnetism, the weak-field limit of general relativity produces equations closely analogous to Maxwell's, with mass density and current playing the roles of charge and current [37, 38]. Electromagnetism is distinguished by its linearity in vacuum and by its gauge invariance, properties intimately connected to the masslessness of the photon [19, 20]. Exploring these analogies can deepen students' appreciation of the special role played by Maxwell's equations within the broader landscape of field theories.

E. Limitations

Despite its pedagogical appeal, the reconstruction has clear boundaries. It does not derive the existence of fields from a deeper action principle or from gauge invariance [39, 40]. The argument works in vacuum; material media require additional constitutive relations that go beyond the scope of this discussion. Special relativity is not explicitly incorporated, yet it provides an even deeper unification of electric and magnetic fields through Lorentz transformations [41, 42]. The assumption that fields vanish at infinity is a boundary condition that may not apply in cosmological contexts, but it is sufficient for the vast majority of laboratory problems [43, 44]. These limitations do not diminish the value of the reconstruction as a pedagogical tool; they mark the points at which a more advanced treatment becomes necessary.

VII. CONCLUSION

We have presented a pedagogical reconstruction of Maxwell's equations that highlights their internal logic. Starting from the continuity equation and the physical need for a mediating field, we used Helmholtz's theorem to demand that all four divergences and curls be specified. Coupling the fields to the available sources in the simplest linear, local, first-order manner gave the inhomogeneous equations. Consistency with charge conservation forced the displacement current. The homogeneous equations were completed by the most parsimonious choice for the divergence of $\mathbf{B}$ and by requiring wave propagation at the experimental speed c, which uniquely fixed Faraday's law. The Lorentz force emerged not as an additional postulate but as the necessary expression of momentum conservation contained in the field equations, identified through the Good-Nelson identity and dimensional analysis.

This narrative is offered as a complement to the traditional empirical presentation. Together, the two approaches can give students a richer understanding of one of the cornerstones of modern physics. For instructors, the reconstruction can be used as a unifying summary after the experimental laws have been covered, or as an alternative pathway for students who benefit from a more deductive perspective.

Whether every instructor or student finds this particular narrative persuasive is less important than the broader lesson it conveys: Maxwell's equations and the Lorentz force possess a deep internal coherence that can be uncovered through careful reasoning. We hope this approach will help students appreciate the beauty and unity of electromagnetism and inspire further exploration of the connections among conservation laws, geometry, and physical fields.

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Appendix A: Degrees of Freedom and Consistency of Maxwell's Equations

A common question from attentive students concerns the apparent overdetermination of Maxwell's equations. There are eight scalar equations (two scalar and two vector equations) for six unknown field components. How can the system be consistent?

The resolution lies in recognizing that not all equations are dynamical. Equations (23) and (25) contain no time derivatives; they are constraints that must hold at each instant. Equations (24) and (26) govern the time evolution. The constraints are preserved by the dynamics if charge is conserved. Taking the divergence of Eq. (24) gives $\partial_t(\nabla\cdot\mathbf{B}) = 0$; if the divergence of $\mathbf{B}$ is initially zero, it remains so. Taking the divergence of Eq. (26) and using the continuity equation yields $\partial_t(\nabla\cdot\mathbf{E} - \rho/\varepsilon_0) = 0$; if Gauss's law holds initially, it continues to hold.

Thus, the six field components are subject to two constraints, leaving four independent dynamical variables. In potential language, the scalar potential ϕ and vector potential $\mathbf{A}$ obey four equations (one scalar, one vector) for four unknowns, with gauge freedom reducing the physical degrees of freedom to two, corresponding to the transverse polarizations of light [1, 2]. The counting is consistent, and the system is neither overdetermined nor underdetermined.

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