Cosmology from a Rastall-inspired variational theory of gravitation

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Abstract

We propose a novel gravitational theory inspired by Rastall's idea of non-conserved energy-momentum, but formulated through a consistent variational principle. After developing a Lagrangian density that naturally incorporates matter-geometry coupling, we study cosmological consequences and fit parameters using recent Hubble function $H(z)$ data. According to model selection criteria (AICc and BIC), we obtain competitive models relative to ΛCDM. In our framework, the cosmological constant emerges from energy exchange between spacetime geometry and matter content, providing an alternative to vacuum energy interpretations. Our formulation provides a self-consistent variational foundation for non-conservative gravity, offering a potential resolution to the cosmological constant problem through dynamical energy exchange rather than vacuum energy.

Summary for Physicists: We present a new theory of gravity where energy and momentum are not strictly conserved but exchanged between spacetime and matter. This exchange is controlled by a variational principle, making the theory self-consistent. When applied to cosmology, the model fits current data as well as the standard model, but interprets the cosmic acceleration as the result of this energy exchange, rather than a mysterious dark energy. This offers a new pathway to solving the long-standing cosmological constant problem.

Keywords. Rastall's theory; Variational Principles; Cosmological Constant.

1 Introduction

The discovery of the universe's late-time acceleration [1, 2] stands as one of the most significant challenges to theoretical physics in the past half-century. Within the framework of General Relativity (GR), the simplest explanation invokes Einstein's cosmological constant Λ [3], now reinterpreted not as a mechanism for staticity but as a source of repulsive gravity driving cosmic expansion [4]. This Λ-term, when combined with cold dark matter, constitutes the ΛCDM paradigm, which remains remarkably consistent with a vast array of cosmological observations [5].

The physical origin of Λ, however, is profoundly enigmatic. If interpreted as vacuum energy [6], quantum field theory predicts a value some 120 orders of magnitude larger than observed [7, 8]. This staggering discrepancy constitutes the infamous *cosmological constant problem* [7, 9, 10, 11, 12]. A second, related puzzle is the *cosmic coincidence problem*: why do the energy densities of matter and dark energy (Λ) happen to be comparable precisely in our present epoch [13, 14, 15]? This suggests a fine-tuning that lacks a compelling dynamical explanation within the standard model.

These conceptual crises have motivated explorations beyond GR. A promising direction involves modifying the coupling between spacetime geometry and the matter content it contains. A pioneering, though often overlooked, step in this direction was taken by Rastall [16], who questioned the standard assumption of energy-momentum conservation $\nabla^\nu T_{\mu\nu} = 0$ in curved spacetime. He proposed a non-minimal coupling where the divergence of $T_{\mu\nu}$ is proportional to the gradient of the Ricci scalar, $\nabla^\nu T_{\mu\nu} \propto \nabla_\mu R$. This simple modification leads to a theory where gravity is inherently non-conservative, suggesting a deeper, dynamic interplay between geometry and matter.

However, Rastall's original theory has long resisted a derivation from a fundamental action principle [17, 18], a feature often viewed as a theoretical shortcoming. This paper bridges that gap. We do not seek to force the exact 1972 field equations from a variational principle. Instead, we construct a new, self-consistent gravitational theory inspired by Rastall's core insight, formulated within the broader and well-defined class of $f(R, T)$ theories [19]. Our approach starts from a carefully chosen Lagrangian density, leading to field equations that embody a non-conservative coupling between matter and geometry.

The resulting framework offers a profound reinterpretation of the cosmological constant. In our theory, Λ is not a manifestation of vacuum energy but emerges naturally as an effective description of a continuous energy-momentum exchange between the gravitational field and the material content. This process can be viewed as a large-scale, classical analogue of particle production in strong gravitational fields, or more precisely, as a conversion between geometric and material degrees of freedom within a closed system. This perspective naturally reframes the cosmic coincidence not as an accident, but as a transient epoch marking the crossover where this energy transfer becomes dynamically significant.

In this work, we derive the complete cosmological equations from our variational principle for a FLRW universe. We fit a hierarchy of resulting models against recent Hubble parameter $H(z)$ data [20] and employ sophisticated model selection techniques (AICc and BIC) [21, 22] to compare their performance against ΛCDM. We find several competitive models, including one where the observed cosmic acceleration is driven entirely by the matter-geometry interaction, without a vacuum energy term. This suggests a potential pathway to alleviating the cosmological constant problem by recontextualizing dark energy as a dynamical, gravitational phenomenon.

This article is structured as follows: In Section 2, we review the original Rastall theory and the motivation for our generalization. Section 3 presents our novel variational formulation and derives the resulting field equations. Section 4 establishes the corresponding cosmological framework. Section 5 details our data analysis and model comparison. We discuss our findings in Section 6 and present our conclusions in Section 7.

2 Rastall's theory of gravitation

The standard formulation of General Relativity is built upon the Einstein-Hilbert action, leading to the celebrated field equativos:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{1}$$

where $\kappa = 8\pi G/c^4$, and $T_{\mu\nu}$ is the energy-momentum tensor of matter. A fundamental consequence of the Bianchi identities, $\nabla^\nu G_{\mu\nu} = 0$, is the covariant conservation of the energy-momentum tensor:

$$\nabla^\nu T_{\mu\nu} = 0 \tag{2}$$

This conservation law is typically interpreted as a geometric constraint on the matter sector, ensuring the consistency of the theory [23].

Rastall's seminal contribution [16] was to question whether this constraint is necessarily fundamental. He proposed a non-minimal coupling where the divergence of $T_{\mu\nu}$ is sourced by the gradient of the Ricci scalar:

$$\nabla^\nu T_{\mu\nu} = \lambda\nabla_\mu R \tag{3}$$

where λ is a dimensionless constant characterizing the strength of the matter-geometry interaction. This represents a radical departure: the conservation of energy-momentum is no longer an immutable law but becomes a dynamical equation intertwined with spacetime curvature.

The corresponding field equations in the original Rastall theory are:

$$R_{\mu\nu} + \left(\kappa\lambda - \frac{1}{2}\right) g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{4}$$

General Relativity is recovered in the limit $\lambda \to 0$. A more useful parameterization, which includes a cosmological constant Λ, is given by:

$$G_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{\gamma-1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu} \tag{5}$$

$$\nabla^\nu T_{\mu\nu} = \frac{\gamma-1}{2}\nabla_\mu T \tag{6}$$

where γ is related to Rastall's original parameter by $\gamma = (6\kappa\lambda-1)/(4\kappa\lambda-1)$, and $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the energy-momentum tensor.

Despite its physical appeal, Rastall's theory has faced criticism for its lack of a Lagrangian formulation [17]. The relation $\nabla^\nu T_{\mu\nu} \propto \nabla_\mu R$ is non-holonomic—it cannot be derived as an Euler-Lagrange equation from a simple action principle without introducing constraints or modifying the variational procedure.

Our approach circumvents this historical impediment by recognizing that Rastall's core idea—a non-conservative coupling between matter and geometry—finds a natural home within the broader class of $f(R,T)$ gravity theories [19]. Instead of seeking a Lagrangian for the exact 1972 field equations, we construct a new, self-consistent theory that captures Rastall's physical insight while being fundamentally variational.

3 The Lagrangian and Equations of Motion

We begin with the following action:

$$\begin{aligned}\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[(1-\alpha)R + 2\Lambda + 8\pi\alpha\bar{T}\right] \\ + \int d^4x\sqrt{-g}\mathcal{L}_m\end{aligned} \tag{7}$$

where α is a new coupling constant, and $\bar{T}$ is the trace of a tensor $\bar{T}_{\mu\nu}$ that we will dynamically identify. This action can be seen as a specific case of $f(R,T)$ gravity with $f(R,\bar{T}) = (1-\alpha)R + 2\Lambda + 8\pi\alpha\bar{T}$.

The action in Eq. (7) constitutes the most general form linear in both R and $\bar{T}$ that incorporates a non-minimal coupling between geometry and matter. This choice is motivated by the intent to embed Rastall's core idea—a non-conservative interplay—within a rigorous variational framework, while maintaining tractability and reducing to General Relativity in the appropriate limit.

The variation of this action with respect to the metric $g^{\mu\nu}$, using the standard definitions [24]:

$$\delta(\sqrt{-g}R) = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} \tag{8}$$

$$\delta(\sqrt{-g}\bar{T}) = \sqrt{-g}\left(\bar{T}\mu\nu + \bar{\theta}\mu\nu - \frac{1}{2}\bar{T}g_{\mu\nu}\right)\delta g^{\mu\nu} \tag{9}$$

$$\bar{\theta}\mu\nu \equiv g^{\alpha\beta}\frac{\delta\bar{T}\alpha\beta}{\delta g^{\mu\nu}} \tag{10}$$

$$\delta(\sqrt{-g}\mathcal{L}m) = -\frac{1}{2}\sqrt{-g}T\mu\nu\delta g^{\mu\nu} \tag{11}$$

yields the field equations:

$$G_{\mu\nu} = \kappa T^{(\text{eff})}_{\mu\nu} \tag{12}$$

where the effective energy-momentum tensor is:

$$T^{(\text{eff})}_{\mu\nu} \equiv T_{\mu\nu} - \frac{\alpha}{2(1-\alpha)}\left(2\bar{T}\mu\nu + 2\bar{\theta}\mu\nu - \bar{T}g_{\mu\nu}\right) + \frac{\Lambda}{\kappa(1-\alpha)}g_{\mu\nu} \tag{13}$$

In this formulation, $\bar{T}_{\mu\nu}$ is an auxiliary tensor introduced in the variational action, whose physical identification will be determined by imposing the generalized conservation law. The tensor $\theta_{\mu\nu}$, defined by Eq. (10), arises naturally from the variation of the $\sqrt{-g}\bar{T}$ term with respect to the metric and captures the implicit dependence of the energy-momentum tensor on the metric.

The requirement that $\nabla^\nu T^{(\text{eff})}_{\mu\nu} = 0$ leads to a generalized conservation law:

$$\nabla^\nu T_{\mu\nu} = \frac{\alpha}{2(\alpha-1)}\left(\nabla_\mu\bar{T} - 2\nabla^\nu\bar{\theta}\mu\nu - 2\nabla^\nu\bar{T}\mu\nu\right) \tag{14}$$

To connect with the phenomenological structure of Rastall's theory, we impose that this conservation law reduces to the form of Eq. (6). This identifies the tensor $\bar{T}_{\mu\nu}$

and relates the coupling constants, leading to the final field equations of our variational Rastall-inspired theory:

$$G_{\mu\nu} = \kappa \left[T_{\mu\nu} - \frac{\gamma - 1}{2} g_{\mu\nu} T + (2 - \gamma) g_{\mu\nu} \frac{\Lambda}{\kappa} \right] \tag{15}$$

$$\nabla^{\nu} T_{\mu\nu} = \frac{\gamma - 1}{2} \nabla_{\mu} T \tag{16}$$

where $\alpha = (\gamma - 1)/(\gamma - 2)$. Crucially, comparing Eq. (14) with Rastall's original equation with a cosmological constant (Eq. 5) reveals a key distinction: the term multiplying the cosmological constant is $(2-\gamma)$ in our theory, versus 1 in the original formulation. This difference, arising directly from the variational principle, has profound implications for interpreting the nature of Λ, as we will explore in the cosmological context. The non-conservation equation (16) can be interpreted as embodying an irreversible energy transfer between geometry and matter, suggesting a fundamental thermodynamic aspect to gravity that warrants further exploration.

Thus, we have successfully embedded Rastall's fundamental insight—that gravity could be inherently non-conservative—within a rigorous variational framework. The price for this achievement is a slight modification of the original theory, but one that yields a more complete and interpretable physical picture.

4 Cosmological Framework

To explore the observational consequences of our variational Rastall-inspired theory, we derive the cosmological equations for a homogeneous and isotropic universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 + Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{17}$$

where $a(t)$ is the scale factor and $K = 0, \pm 1$ represents the spatial curvature.

For the cosmic fluid, we consider a perfect fluid with energy-momentum tensor:

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - g_{\mu\nu} p \tag{18}$$

where ρ is the energy density, p is the pressure, and u^{μ} is the fluid's four-velocity satisfying $u^{\mu} u_{\mu} = 1$. The trace is given by $T = \rho - 3p$.

Substituting the FLRW metric and perfect fluid tensor into our field equations (14) and (15), we obtain the modified Friedmann equations:

$$3 \frac{\dot{a}^2}{a^2} = \kappa \left[\rho - \frac{\gamma - 1}{2} T \right] + \Lambda - \hat{\Lambda}(\gamma - 1) - 3 \frac{K}{a^2} \tag{19}$$

$$6 \frac{\ddot{a}}{a} = 2[\Lambda - (\gamma - 1)\hat{\Lambda}] - \kappa(6p + \gamma T) \tag{20}$$

where we have introduced $\hat{\Lambda}$ as an effective cosmological constant term that emerges distinctly from the matter-geometry interaction, in contrast to the geometric cosmological constant Λ. This separation is crucial for the physical interpretation of the cosmological constant's origin within our framework.

The non-conservation equation (15) yields the fluid evolution:

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = \frac{\gamma - 1}{2} \dot{T} \tag{21}$$

Introducing the equation of state $p = \omega \rho$, where ω is constant, we can rewrite these equations in a more familiar form. The Friedmann equation becomes:

$$3 \frac{\dot{a}^2}{a^2} = \kappa \rho \left[1 - \frac{\gamma - 1}{2} (1 - 3\omega) \right] + \Lambda + \hat{\Lambda}(1 - \gamma) - 3 \frac{K}{a^2} \tag{22}$$

The acceleration equation transforms to:

$$6 \frac{\ddot{a}}{a} = 2[\Lambda - (\gamma - 1)\hat{\Lambda}] - \kappa \rho [6\omega + \gamma(1 - 3\omega)] \tag{23}$$

And the conservation equation takes the form:

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (1 + \omega) \rho = \frac{\gamma - 1}{2} (1 - 3\omega) \dot{\rho} \tag{24}$$

Solving equation (22) yields the energy density evolution:

$$\rho = \rho_0 \left(\frac{a_0}{a} \right)^A, \quad A = \frac{6(1 + \omega)}{3 - 3\omega + \gamma(3\omega - 1)} \tag{25}$$

This represents a fundamental modification to the standard scaling of cosmic fluids. For dust matter $(\omega = 0)$:

$$\rho_m \sim a^{-\frac{6}{3-\gamma}} \tag{26}$$

For radiation $(\omega = 1/3)$:

$$\rho_r \sim a^{-4} \tag{27}$$

Remarkably, radiation maintains its standard evolution, while dust matter exhibits γ-dependent scaling. This selective modification suggests that the matter-geometry coupling primarily affects non-relativistic matter.

The acceleration equation can be expressed in terms of the density evolution:

$$\frac{6\ddot{a}}{a} = \kappa \rho_0 [3\omega(\gamma - 2) - \gamma] a^{-A} + 2\Lambda - 2(\gamma - 1)\hat{\Lambda} \tag{28}$$

From this, we identify the conditions for cosmic acceleration $(\ddot{a} > 0)$. Even in the absence of cosmological constants $(\Lambda = \hat{\Lambda} = 0)$, acceleration can occur for:

$$\omega < \frac{1}{3} \quad \text{and} \quad \gamma < \frac{6\omega}{3\omega - 1} \tag{29}$$

or

$$\omega > \frac{1}{3} \quad \text{and} \quad \gamma > \frac{6\omega}{3\omega - 1} \tag{30}$$

This reveals that our theory can generate late-time acceleration purely from the matter-geometry interaction, without invoking dark energy.

To connect with observations, we introduce standard cosmological parameters. The Hubble parameter is defined as:

$$H \equiv \frac{\dot{a}}{a} \tag{31}$$

with present value H_0. The density parameters are:

$$\Omega_i \equiv \frac{\kappa \rho_i}{3H^2} \tag{32}$$

for each fluid component i. The redshift is related to the scale factor by:

$$1 + z = \frac{a_0}{a} \tag{33}$$

Setting $a_0 = 1$ at present, the Friedmann equation becomes:

$$\begin{aligned} \left(\frac{H(z)}{H_0} \right)^2 &= \frac{3 - \gamma}{2} \Omega_m (1 + z)^{\frac{6}{3-\gamma}} \\ &+ \Omega_r (1 + z)^4 + \Omega_{\Lambda} \\ &+ (1 - \gamma)\Omega_{\hat{\Lambda}} - (\Omega_0 - 1)(1 + z)^2 \end{aligned} \tag{34}$$

where $\Omega_0 = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_{\hat{\Lambda}}$, and we have decomposed the cosmological constant contributions into geometric (Ω_Λ) and matter-creation ($\Omega_{\hat{\Lambda}}$) components.

Equation (31) represents the master equation for our cosmological analysis. It reduces to the standard ΛCDM case when $\gamma = 1$, and contains various interesting limits that we will explore through specific models in the next section.

The rich structure of this cosmological framework—with its modified density scaling, potential for self-acceleration, and interpretational flexibility regarding the cosmological constant—makes it an compelling arena for confronting our variational Rastall-inspired theory with observational data.

5 Fitting data

Using the cosmological framework developed in Section 4, particularly the Hubble function expressed in Eq. (34), we fit the parameters of several specific models against recent observational data [20]. Such data are shown below and are plotted in Fig. (1) along with the fitted curve of the standard model ΛCDM.

z	$H(z)$
0.07	69 ± 19.6
0.09	69 ± 12
0.12	68.6 ± 26.2
0.17	83 ± 8
0.179	75 ± 4
0.199	75 ± 5
0.2	72.9 ± 29.6
0.24	79.69 ± 2.65
0.27	77 ± 14
0.28	88.8 ± 36.6
0.35	82.1 ± 4.9
0.35	84.4 ± 7
0.352	83 ± 14
0.3802	83 ± 13.5
0.4	95 ± 17
0.4004	77 ± 10.2
0.4247	87.1 ± 11.2
0.43	86.45 ± 3.68
0.44	82.6 ± 7.8
0.4497	92.8 ± 12.9
0.4783	80.9 ± 9
0.48	97 ± 62
0.57	92.4 ± 4.5
0.593	104 ± 13
0.6	87.9 ± 6.1
0.68	92 ± 8
0.73	97.3 ± 7
0.781	105 ± 12
0.875	125 ± 17
0.88	90 ± 40
0.9	117 ± 23
1.037	154 ± 20
1.3	168 ± 17
1.363	160 ± 33.6
1.43	177 ± 18

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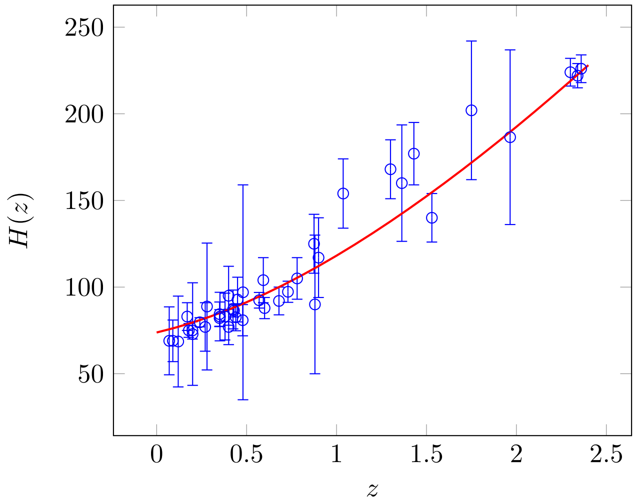

Figure 1: $H(z)$ data with error bars and the fitted ΛCDM model $H(z) = H_0\sqrt{(1-\Omega) + \Omega(1+z)^3}$, with $H_0 = 73.8$ $kms^{-1}Mpc^{-1}$ [25], $\Omega = 0.223$, $\chi^2_{\min} = 22.79$, $AICc = 25.11$ and $BIC = 26.51$.

Continued from previous column

z	$H(z)$
1.53	140 ± 14
1.75	202 ± 40
1.965	186.5 ± 50.4
2.3	224 ± 8
2.34	222 ± 7
2.36	226 ± 8

Concluded

We will fit the $H(z)$ data using some cosmological models proposed below, many of them from Eq. (34) and two from purely phenomenological origin proposed in [26]. These two purely phenomenological models are oscillatory models that best fit the data proposed in [26, 27]. They are included here for purposes of comparison with the models motivated by Eq. (34) and for an update of the conclusions obtained in [26, 27] with more recent data available.

Table (1) shows 12 models studied plus ΛCDM model considered as benchmark. Of the 12 proposed models, 10 are particular cases of Eq. (34) and 2 are the oscillatory models mentioned. For example, the most complex model is the model (H) where Ω_m represents baryonic matter, Ω_K represents "matter associated with the curvature of spacetime," Ω_r is associated with radiation and $\Omega_{\hat{\Lambda}}$ represents the matter associated with the cosmological constant due to matter creation. Another example is the model (L) where we leave ω as free parameter and we made $\Omega \equiv \Omega_x = 1$. The table (1) is arranged in order to display the models in ascending order of "discrepancy from $\chi^2/d.o.f - 1$". This preliminary quality of fit was measured in terms of the χ^2 analysis, where χ^2 is defined by

$$\chi^2(\text{parameters}) = \sum_{i=1}^{41} \frac{[H_{\text{mod}}(\text{parameters}; z_i) - H_{\text{obs}}(z_i)]^2}{\sigma^2(z_i)}, \tag{35}$$

where H_{mod} is the predicted value for the Hubble parameter in the assumed model, H_{obs} is the observed value, σ corresponds to 1σ uncertainty, and the summation is over the 41 observational $H(z)$ data points at redshift z_i. The parameters are estimated by the minimization of χ^2. What actually orders the models with respect to their quality of fit is χ^2 divided by the degrees of freedom (d.o.f). Values of $\chi^2/d.o.f$ close to 1 ($\chi^2/d.o.f \approx 1$) represent better fittings [28].

All models in the table (1) that do not have ΛCDM in their names are models based on our variational framework and therefore in Eq. (34). This fact is evidenced by the presence of γ in the equations that define them. In the next section we will consider better strategies for selecting models based on information criteria and Bayesian analysis.

5.1 Model selection

Some model selection methods are defined in terms of an appropriate *information criterion*, a mechanism well-founded theoretically that uses data to give a model a score, which leads to a ranked list of candidate models from the best to the worst [21, 29, 22]. Two of the most important of these information criteria are the Akaike information criterion (AIC) and Bayesian information criterion (BIC). The general formulas for models with vector parameter $\hat{\theta}$ are

$$AIC = -2\log \mathcal{L}(\hat{\theta}|\text{data}) + 2p \tag{36}$$

and

$$BIC = -2\log \mathcal{L}(\hat{\theta}|\text{data}) + \log(n)p, \tag{37}$$

where $\log \mathcal{L}(\hat{\theta}|\text{data})$ is the log-likelihood of the model, p is the number of parameters and n is the sample size. For models with normally distributed residuals, $\log \mathcal{L} = -\frac{n}{2}\log \chi^2$ with χ^2 given by Eq. (35).

The AIC and BIC criteria act as a penalized log-likelihood criterion, providing a balance between good fit (high value of log-likelihood) and complexity (complex models are penalized more that simple ones). These criteria punishes the models for being too complex in the sense of containing many parameters. The models with the lowest AIC and BIC scores are selected.

It is useful to briefly mention how these criteria behave in relation to their properties of *consistency* and *efficiency.* The comparison is based in the study of the *penalty* applied to the maximized log-likelihood value in a framework with *increasing sample size.* If we make the assumption that there are one *true model* that generates the data and that this model *is one of the candidate models*, we would want the model selection method to identify this true model. This is related to consistency. From [30] (our emphasis):

> A model selection method is weakly consistent if, with probability tending to one *as the sample size tends to infinity*, the selection method is able to select the true model from the candidate models. Strong consistency is obtained when the selection of the true model happens almost surely.

When we do not assume that the true model is present among the proposed models, as we believe to be the case in most practical situations, including cosmology, we can assume that a candidate model is the closest to the true model in the Kullback-Leibler distance sense (see bellow) [29]. In this case [30],

> $\cdots$ we can state weak consistency as the property that, with probability tending to one [*as the sample size tends to infinity*], the model selection method picks such a closest model.

From the foregoing we see that a strongly desirable condition for the use of a consistent criterion is to have a sufficiently large sample size. This fact, among others that will be mentioned below, makes us less likely to take the results very strictly for BIC, at least with the amount of data currently available for $H(z)$.

Another desirable property of selection methods, efficiency, can be roughly described by [30]:

> $\cdots$ [W]e might want an information criterion to posses is that it behaves "almost as well", in terms of mean square error, or squared error loss.

In [30] it is proved that

> $\cdots$ AIC is not strongly consistent, though it is efficient, while the opposite is true for the BIC.

Given the above assertion, it is useful to question whether a model selection method that contemplates both desirable properties, BIC consistency with AIC efficiency, exists. The answer is *not*, as proven in [31]. This means that when selecting a method to choose the best models, we are necessarily making a choice between consistency and efficiency. The optimal or most convenient choice is conditioned to the size of the sample n as well as to the *world view* of the researcher, as will be discussed in the section (6).

We have seen that for BIC, the properties of consistency are defined (rigorously in [30]) in terms of asymptotic properties related to sample size n. If this size is not "large enough", the practical validity of the theorems weaken.

For AIC something analogous occurs. In its derivation, an approximation is made that is valid for large sample sizes. A rule of thumb was proposed in [29] to define what would be a "sample size not large enough". Such a rule states that if $n/p \leq 40$, the results for the AIC method may be less accurate. To mitigate this problem, a correction was proposed for the Akaike method, the so-called *corrected Akaike information criterion*, AICc, whose formula is given, under certain assumptions, by [29]

$$AICc = AIC + \frac{2(p+1)(p+2)}{n-p-2}. \tag{38}$$

AICc is essentially AIC with a greater penalty for extra parameters. Using AIC, instead of AICc, when n is not many times larger than p^2, increases the probability of selecting models that have too many parameters, *i.e.*, of overfitting. Due to the fact that we are using data with $n = 41$, we will use AICc as the criterion for selecting models based on Akaike.

AIC, AICc and BIC (collectively dubbed IC, from Information Criteria) are absolute numbers that have no significance when evaluated in isolation. What matters is the value of the difference of IC from two different models. In order to better characterize this fact, it is common to calculate $\Delta_i IC \equiv IC_i - IC_*$, where we calculate the difference of IC values for two models, i and $*$, where the model $*$ is established as the reference model. In this work we consider as reference model the standard model of cosmology, the ΛCDM model. These $\Delta_i IC$ are easy to interpret and allow a quick strength-of-evidence comparison of candidate models. According to [21]:

> Some simple rules of thumb are often useful in assessing the relative merits of models in the set: Models having $\Delta_i \leq 2$ have substantial support

(evidence), those in which $4 \leq \Delta_i \leq 7$ have considerably less support, and models having $\Delta_i > 10$ have essentially no support.

Another useful tool for providing weights of evidence for each of the $R = 11$ (phenomenological models were excluded) models considered in the analysis are the Akaike weights w_i given by [21]

$$w_i = \frac{e^{-\frac{\Delta_i}{2}}}{\sum_{r=1}^{R} e^{-\frac{\Delta_r}{2}}}. \tag{39}$$

where Δ_i represents $\Delta_i AICc$ or $\Delta_i BIC$. The w_i from AICc can be interpreted as the probability of model i being, in fact, the best model in the sense of the Kulback-Leibler's (K-L) distance [21]. K-L information $I(f, g)$ is the *information lost* when model $g(x, \theta)$ is used to approximate the "full reality or truth", f; this is defined for continuous functions as the integral

$$I(f, g) = \int f(x) \log\left(\frac{f(x)}{g(x|\theta)}\right) dx. \tag{40}$$

The best model loses the minimum amount of information possible and Akaike's criterion seeks precisely this by minimizing the distance of K-L.

All our data analysis results are shown in tables (1), (2) and (3), the latter in the appendix, and in figure (2). Table (1) shows all models with their parametrizations, table (2) presents the AICc and BIC measurements and their variants, and figure (2) presents a column chart to facilitate the general understanding of the hierarchy of models with regard to the rule of thumb. For the calculation of w_i in table (2) we do not consider the weights relative to the phenomenological models (B) and (C). In the next section we will discuss the results.

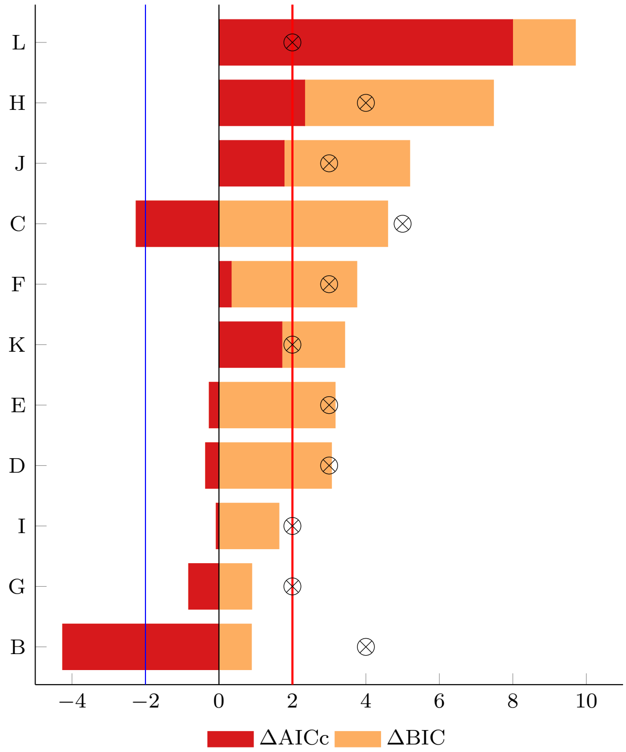

Figure 2: Model selection bar chart. The bars are ordered according to the classification by ΔBIC in table (2). Model (A) is not present due to poor fit distorting the horizontal scale. The red vertical line represents the threshold above which the models do not show statistical support according to the standard rule of thumb. Analogous reasoning holds for the blue line, with the rule *against* the standard model. Number of parameters of each model (collected from horizontal axis) are represented by ⊗.

6 Discussion

The analysis presented in the previous sections reveals several compelling features of our variational Rastall-inspired framework. The model selection criteria, while providing quantitative measures of performance, point toward deeper theoretical insights when interpreted through the lens of our reformulated theory.

The most promising models emerging from our analysis—particularly models (G) and (K)—deserve reexamination in light of the variational formulation. Model (G), characterized by parameters $(\Omega_m, \Omega_\Lambda, \gamma)$, represents the direct analog of ΛCDM within our framework. Its strong performance according to both AICc and BIC criteria suggests that the standard cosmological model can be naturally embedded within a theory featuring non-minimal matter-geometry coupling. However, the optimal value $\gamma \approx 1.14$ (Table 3) indicates a detectable departure from pure general relativity ($\gamma = 1$), hinting at an underlying interaction between spacetime geometry and material content even in this "conservative" extension.

More profoundly, Model (K) $(\Omega_m, \Omega_{\hat{\Lambda}}, \gamma)$ demonstrates that cosmic acceleration can be achieved without invoking vacuum energy. Here, the cosmological constant $\hat{\Lambda}$ emerges not as energy of empty space, but as an effective description of the energy-momentum exchange between gravitational field and matter. The favorable ΔAICc value of 1.71 suggests this model is competitive with ΛCDM, while offering a radical reinterpretation: the observed acceleration may be a thermodynamic consequence of ongoing energy transfer rather than a fundamental vacuum energy. This provides a potential pathway to addressing the cosmological constant problem, as the extremely small observed value of Λ would not require fine-tuning of quantum vacuum contributions, but would instead reflect the current rate of energy exchange at cosmic scales.

While the purely phenomenological oscillatory models (B) and (C) provide an excellent fit to the $H(z)$ data, as indicated by their low χ^2 values, they lack a fundamental motivation within our proposed variational framework. Consequently, although they serve as useful benchmarks for goodness-of-fit, our physical interpretation focuses on the models derived from our action principle, which offer a compelling theoretical narrative for the observed cosmic acceleration.

The divergence between AICc and BIC criteria in model selection reveals a fundamental philosophical tension in cosmological model building. AICc, favoring models (D), (E), (F), (G), and (K), prioritizes predictive accuracy and is appropriate if one believes the true model is complex and possibly outside our candidate set—a perspective particularly relevant given the unknown nature of dark energy. BIC, with its stronger penalty for complexity, approves only models (B), (G), and (I), reflecting a view that a simple underlying reality exists and should be identifiable with sufficient data. This methodological dichotomy echoes the broader tension in theoretical cosmology between pursuing the most accurate empirical descriptions versus seeking the most fundamental

Model	$(H(z)/H_0)^2$, $H_0 = 73,8\ km\ s^{-1}\ Mpc^{-1}$ [25]	$\chi^2/d.o.f$
$\Omega_m + \Omega_r$	$\frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}} + (1-\Omega_m)(z+1)^4$	1.764 (A)
ΛCDM Oscillation	$a_1 \cos\left(a_2 z^2 + a_3\right) + 1 - a_1\cos(a_3) - \Omega_m + \Omega_m(z+1)^3$	0.3388 (B)
Oscillation	$a_1 \cos\left(a_2 z^2 + a_3\right) + 1 - a_1\cos(a_3) - \Omega_m + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}}$	0.3482 (C)
$\Omega_m + \Omega_\Lambda + \Omega_r$	$1 - \Omega_m - \Omega_r + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}} + \Omega_r(z+1)^4$	0.4726 (D)
$\Omega_m + \Omega_\Lambda + \Omega_{\hat{\Lambda}}$	$1 - \Omega_\Lambda - \Omega_m + (1-\gamma)\Omega_\Lambda + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}}$	0.4876 (E)
$\Omega_m + \Omega_{\hat{\Lambda}} + \Omega_r$	$(1-\gamma)(1-\Omega_m-\Omega_r) + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}} + \Omega_r(z+1)^4$	0.5032 (F)
$\Omega_m + \Omega_\Lambda$	$1 - \Omega_m + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}}$	0.5120 (G)
$\Omega_m + \Omega_K + \Omega_r + \Omega_{\hat{\Lambda}}$	$(1-\gamma)(1-\Omega-\Omega_r) + (K-1)(z+1)^2 + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}} + \Omega_r(z+1)^4$	0.5168 (H)
$\Lambda CDM + \Omega_r$	$1 - \Omega_m - \Omega_r + \Omega_m(z+1)^3 + \Omega_r(z+1)^4$	0.5312 (I)
$\Omega_m + \Omega_K + \Omega_r$	$(K-1)(z+1)^2 + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}} + (1-\Omega_m)(z+1)^4$	0.5411 (J)
$\boldsymbol{\Lambda CDM}$	$\boldsymbol{(1-\Omega) + \Omega(z+1)^3}$	**0.5698 (*)**
$\Omega_m + \Omega_{\hat{\Lambda}}$	$(1-\gamma)(1-\Omega_m) + \frac{1}{2}(3-\gamma)\Omega_m(z+1)^{\frac{6}{3-\gamma}}$	0.5769 (K)
$\Omega_x = 1,\ \omega$	$\left[1 - \frac{1}{2}(\gamma-1)(1-3\omega)\right](z+1)^{\frac{6(\omega+1)}{\gamma(3\omega-1)-3\omega+3}}$	0.7379 (L)

Table 1: All functions $H(z)$ fitted, ordered in terms of $\chi^2/d.o.f$ from (A) to (L), from $\chi^2/d.o.f$ furthest from 1 to $\chi^2/d.o.f$ closer to 1, with ΛCDM, our benchmark, emphasized. All models without ΛCDM in front of the name refer to the models that originated in our variational framework, as verified by the presence of γ in their respective equations.

Model	χ^2_{min}	AICc	BIC	$\Delta AICc$	ΔBIC	$w_i AICc$	$w_i BIC$
$\Omega_m + \Omega_r$	68.80 (A)	73.12 (13)	76.22 (13)	48.01	49.71	0.00	0.00
$\Omega_x = 1,\ \omega$	28.78 (L)	33.10 (12)	36.21 (12)	7.99	9.70	0.00	0.00
$\boldsymbol{\Lambda CDM}$	**22.79 (*)**	**25.11 (7)**	**26.51 (1)**	**0**	**0**	0.13	0.34
$\Omega_m + \Omega_{\hat{\Lambda}}$	22.50 (K)	26.82 (9)	29.93 (7)	1.71	3.42	0.05	0.06
$\Lambda CDM + \Omega_r$	20.72 (I)	25.04 (6)	28.14 (4)	-0.07	1.63	0.13	0.15
$\Omega_m + \Omega_K + \Omega_r$	20.56 (J)	26.88 (10)	31.70 (10)	1.77	5.19	0.05	0.03
$\boldsymbol{\Omega_m + \Omega_\Lambda}$	**19.97 (G)**	**24.29 (3)**	**27.40 (3)**	**-0.82**	**0.89**	0.19	0.22
$\Omega_m + \Omega_K + \Omega_r + \Omega_{\hat{\Lambda}}$	19.13 (H)	27.44 (11)	33.98 (11)	2.33	7.47	0.04	0.01
$\Omega_m + \Omega_{\hat{\Lambda}} + \Omega_r$	19.13 (F)	25.44 (8)	30.26 (8)	0.33	3.75	0.11	0.05
$\Omega_m + \Omega_\Lambda + \Omega_{\hat{\Lambda}}$	18.53 (E)	24.85 (5)	29.67 (6)	-0.26	3.16	0.14	0.07
$\Omega_m + \Omega_\Lambda + \Omega_r$	18.43 (D)	24.75 (4)	29.57 (5)	-0.36	3.06	0.15	0.07
ΛCDM Oscillation	**12.54** (B)	**20,86** (1)	**27.39** (2)	**-4.25**	**0.88**	–	–
Oscillation	12.54 (C)	22.86 (2)	31.1 (9)	-2.25	4.59	–	–

Table 2: All functions $H(z)$ fitted, ordered in terms of χ^2_{min}, showing the valus of $AICc$, BIC, $\Delta AICc$, ΔBIC, $w_i AICc$ and $w_i BIC$. Models that pass in the BIC criterion are evidenced in orange color and models that pass in the AICc criterion are evidenced in light cyan. Phenomenological models were excluded in the calculations of w_i.

physical explanations.

Our framework naturally addresses the cosmic coincidence problem by reinterpreting it as a transient epoch in the dynamical evolution of matter-geometry interaction. The approximate equality $\rho_m \sim \rho_\Lambda$ observed today need not represent a special moment in cosmic history, but rather marks the crossover point where the energy transfer rate between spacetime and matter becomes comparable to the matter density itself. In this picture, the "coincidence" emerges as a natural consequence of the scaling laws governing the interaction, rather than requiring anthropic or multiverse explanations.

Several limitations and open questions deserve mention. The present formulation leaves the microscopic mechanism of energy-momentum exchange unspecified—while we describe this phenomenon at the cosmological fluid level, its origin in fundamental particle physics remains to be elucidated. Additionally, the precise physical interpretation of the γ parameter warrants further investigation; it may represent a fundamental constant or an emergent property of collective gravitational behavior. Future work should explore the implications of our framework for structure formation, gravitational waves, and early universe cosmology, where the non-conservative coupling may yield distinctive signatures.

Future work should explore the implications of this framework for structure formation and the growth of perturbations, where the non-conservative coupling could leave distinctive signatures. Furthermore, the behavior of gravitational waves in this theory and potential constraints from upcoming missions (e.g., LISA) present a promising avenue for testing its validity. Investigating the theory's predictions in the early universe, particularly during inflation and reheating, could also reveal novel phenomenology and connections to quantum gravity.

The variational formulation has proven essential in revealing these insights. Unlike the original Rastall theory, our action-based approach naturally distinguishes between geometric and emergent cosmological constants, clarifies the conservation properties of the theory, and provides a consistent foundation for exploring thermodynamic aspects of the matter-geometry interaction. This demonstrates that the quest for variational consistency is not merely mathematical pedantry, but a powerful guiding principle that reveals deeper physical structure.

7 Conclusion

In this work, we have successfully developed a self-consistent variational formulation for a gravitational theory that captures the essential physical insight of Rastall's original proposal—that gravity may be fundamentally non-conservative—while resolving long-standing theoretical limitations. Our approach demonstrates that the core idea of matter-geometry interaction can be naturally embedded within the broader framework of $f(R,T)$ gravity, yielding a theory that is both mathematically rigorous and physically compelling.

The key achievement of this formulation is the emergence of a modified coupling to the cosmological constant—encoded in the $(2-\gamma)$ factor in Eq. (14)—that distinguishes our theory from previous non-conservative gravity models. This subtle but profound difference, which arises directly from the variational principle, allows for a novel interpretation of the cosmological constant not as vacuum energy, but as an effective description of energy-momentum exchange between spacetime geometry and material content.

Cosmologically, our framework generates a rich phenomenology that remains competitive with ΛCDM while offering conceptually attractive alternatives. Models such as (K), which achieve cosmic acceleration without vacuum energy, suggest that the observed expansion history may be explained by ongoing gravitational creation of effective dark energy from the time-varying curvature of spacetime. The fact that such models perform favorably under information-theoretic criteria indicates that this reinterpretation is not merely philosophically appealing, but empirically viable.

The methodological tension between AICc and BIC in model selection reflects a deeper epistemological choice in theoretical cosmology. By favoring predictive accuracy over asymptotic consistency, we align with a perspective that acknowledges the possible complexity of dark energy and the limitations of our current model space. This pragmatic approach has revealed promising avenues for addressing both the cosmological constant and cosmic coincidence problems through dynamical mechanisms rather than fine-tuning.

Looking forward, this work opens several productive research directions: investigating the implications of our framework for structure formation and cosmic perturbations; exploring thermodynamic aspects of the matter-geometry interaction; and developing more fundamental microphysical interpretations of the energy-exchange process. The variational formulation provides a solid foundation for these future explorations, ensuring consistency and interpretability.

In conclusion, we have not merely found "a" Lagrangian for Rastall-like gravity—we have constructed a new theoretical framework that transforms a phenomenological idea into a fundamental principle. The resulting theory suggests that the accelerated expansion of the universe may be the macroscopic manifestation of a deep-seated interaction between matter and geometry, offering a promising pathway toward resolving some of the most persistent puzzles in modern cosmology. Future work will incorporate additional cosmological probes such as supernovae and baryon acoustic oscillations to further test the framework.

A Appendix

In this appendix we present in table (3) all the models fitted with the values of their respective parameters as well as the value of the root mean square RMSE.

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Model	**Parameters**	Values	Standard error	Confidence Interval	RMSE
(A)	Ω_m, γ	$\Omega_m = 2.52326$	0.0456223	$\{2.47704, 2.56947\}$	13.61
		$\gamma = 1.42942$	0.00469647	$\{1.42466, 1.43417\}$	
(L)	γ, ω	$\gamma = 1.27113$	0.0189034	$\{1.25198, 1.29028\}$	10.79
		$\omega = -0.527717$	0.0123655	$\{-0.540243, -0.515191\}$	
(*)	$\boldsymbol{\Omega}$	$\boldsymbol{\Omega = 0.223236}$	**0.00873264**	$\boldsymbol{\{0.214393, 0.23208\}}$	**12.10**
(K)	Ω_m, γ	$\Omega_m = 0.377151$	0.0113577	$\{0.365646, 0.388657\}$	10.84
		$\gamma = 0.541202$	0.0389177	$\{0.501778, 0.580626\}$	
(I)	Ω_m, Ω_r	$\Omega_m = 0.160909$	0.0437373	$\{0.116603, 0.205216\}$	12.71
		$\Omega_r = 0.0202814$	0.0139837	$\{0.00611585, 0.034447\}$	
(J)	Ω_m, γ, K	$\Omega_m = 0.941743$	0.0157855	$\{0.925747, 0.957739\}$	12.31
		$K = 0.0521157$	0.0614141	$\{-0.0101186, 0.11435\}$	
		$\gamma = -0.60197$	0.278139	$\{-0.883823, -0.320116\}$	
(G)	Ω_m, γ	$\Omega_m = 0.187023$	0.0237533	$\{0.162961, 0.211086\}$	12.54
		$\gamma = 1.1427$	0.0867217	$\{1.05485, 1.23055\}$	
(H)	$\Omega_m, \gamma, K, \Omega_r$	$\Omega_m = 0.249322$	284.314	$\{-287.966, 288.464\}$	11.88
		$\gamma = 0.290255$	291.403	$\{-295.111, 295.692\}$	
		$K = 0.988712$	428.966	$\{-433.862, 435.84\}$	
		$\Omega_r = 0.0324707$	0.729464	$\{-0.707001, 0.771943\}$	
(F)	$\Omega_m, \gamma, \Omega_r$	$\Omega_m = 0.24182$	0.0731334	$\{0.16771, 0.31593\}$	11.89
		$\gamma = 0.297836$	0.135457	$\{0.16057, 0.435101\}$	
		$\Omega_r = 0.0324613$	0.0144455	$\{0.0178229, 0.0470998\}$	
(E)	$\Omega_m, \gamma, \Omega_{\hat{\Lambda}}$	$\Omega_m = 0.245662$	0.0609698	$\{0.183878, 0.307446\}$	11.68
		$\gamma = 0.95634$	0.194274	$\{0.759471, 1.15321\}$	
		$\Omega_{\hat{\Lambda}} = 0.11527$	0.128262	$\{-0.0147047, 0.245245\}$	
(D)	$\Omega_m, \gamma, \Omega_r$	$\Omega_m = 0.52637$	0.282041	$\{0.240562, 0.812178\}$	11.28
		$\gamma = 1.37401$	0.0547329	$\{1.31854, 1.42947\}$	
		$\Omega_r = -0.225615$	0.188242	$\{-0.41637, -0.0348587\}$	
(B)	Ω_m, a_1, a_2, a_3	$\Omega_m = 0.230154$	0.0112571	$\{0.218743, 0.241566\}$	10.00
		$a_1 = -0.689671$	0.265715	$\{-0.959032, -0.420311\}$	
		$a_2 = -2.48397$	0.239258	$\{-2.72652, -2.24143\}$	
		$a_3 = 7.01061$	0.183163	$\{6.82493, 7.19629\}$	
(C)	$\Omega, a_1, a_2, a_3, \gamma$	$\Omega_m = 0.231265$	0.065034	$\{0.165314, 0.297216\}$	10.00
		$a_1 = 0.690777$	0.277451	$\{0.409412, 0.972141\}$	
		$a_2 = 2.48847$	0.319206	$\{2.16476, 2.81218\}$	
		$a_3 = 2.40863$	0.353219	$\{2.05042, 2.76683\}$	
		$\gamma = 0.996315$	0.211654	$\{0.781675, 1.21095\}$	

Table 3: Parameters of the $H(z)$ fitted. RMSE is the Root Mean Square Error [32].

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