$\tau\phi$-numbers

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Abstract

Let $\tau(n)$ be the number of divisors of n and let $\phi(n)$ denote the number of relatively prime numbers less than n. A number is called refactorable if and only if $\tau(n)$ divides n. It is known that $\phi(n)$ divides n if and only if n is 3-smooth. Furthermore, $n/\phi(n) = 2$ when n is a power of two and $n/\phi(n) = 3$ otherwise. Define $\lambda(n) = \operatorname{lcm}(\tau(n), \phi(n))$. Then $n/\lambda(n) \in \{1, 2, 3\}$ and n is called a $\tau\phi$-number. It is the purpose of this note to show when $\lambda(n) = n$, called the unital $\tau\phi$-numbers.

1 The functions τ, ϕ, and λ

Definition 1.1 (τ function [1], [6]). *Define $\tau(n)$ to be the number of divisors of the positive integer n. We set $\tau(1) = 1$. See Sequence 4.1.*

Theorem 1.2 ([6]). *The function τ has the following properties:*

1. *If p is prime, then*
$$\tau(p^k) = k + 1.$$
2. *The function τ is multiplicative, that is, if $n = p_1^{k_1} \cdots p_r^{k_r}$ is the prime factorization of n, then*
$$\tau(n) = \tau(p_1^{k_1}) \cdots \tau(p_r^{k_r}).$$

Definition 1.3 (ϕ function [5]). *Define $\phi(n)$ to be the number of integers less than n and relatively prime to n. We set $\phi(1) = 1$. See Sequence 4.2.*

Theorem 1.4 ([5]). *The function ϕ has the following properties:*

1. *If p is prime, then*
$$\phi(p^k) = \frac{p^{k+1} - 1}{p - 1}.$$

2. *The function ϕ is multiplicative, that is, if $n = p_1^{k_1} \cdots p_r^{k_r}$ is the prime factorization of n, then*
$$\phi(n) = \phi(p_1^{k_1}) \cdots \phi(p_r^{k_r}).$$

Definition 1.5 (λ function, [2]). *Let n be a positive integer and define $\lambda(n) = \operatorname{lcm}(\tau(n), \phi(n))$. See Sequence 4.6.*

It may be obvious that λ is not multiplicative, but here's an example: if $m = 2$ and $n = 5$, then $\lambda(10) = 4$ but $\lambda(2)\lambda(5) = 8$.

Definition 1.6 (ρ function). *Let n be a positive integer and define $\rho(n) = n/\lambda(n)$.*

In general, $\rho(n)$ is rational and $\rho(n) \geq 1$. Observe that $\rho(12) = 1$. If one considers when $\rho(n)$ jumps to a new maximum, then one finds the following:

Conjecture 1.7 (Primorial conjecture, [3]). *Let p be a prime greater than 2. If $n = p\#$, then $\rho(m) < \rho(n)$ whenever $m < n$. Furthermore, $\rho(p\#) \to \infty$ as $p \to \infty$. See Table 1.*

2 The $\tau\phi$-numbers

Definition 2.1. *A $\tau\phi$-number is a number such that both $\tau(n)$ and $\phi(n)$ are divisors of n, that is, such that $n \equiv 0 \pmod{\lambda(n)}$. See Table 2.*

If $\phi(n)$ is a divisor of n, then n is 3-smooth [4] and $n/\phi(n) = 2$ if n is a power of two, and $n/\phi(n) = 3$ otherwise. Suppose n is a $\tau\phi$-number. Then $n/\lambda(n) \in \{1, 2, 3\}$. If n is a power of two, then $n/\lambda(n) = 2$, so $n/\lambda(n) \in \{1, 3\}$ whenever n is not a power of 2. Let us determine when $n/\lambda(n) = 1$, called a unital $\tau\phi$-number.

Lemma 2.2. *If $n = 2^u \cdot 3^v$, then $\tau(n) = (u + 1)(v + 1)$ and $\phi(n) = 2^u \cdot 3^{v-1}$.*

Proof. An exercise for the reader. □

Observe that if $n = 2^3 \cdot 3^7$, then $\tau(n) = 2^5$ does not divide n. However, if $n = 2^5 \cdot 3^7$, then $\tau(n) = 2^4 \cdot 3$ does divide n. By Lemma 2.2, we need only verify that $\tau(n)$ divides n. Thus,

Lemma 2.3. *If $n = 2^u \cdot 3^v$ and $u + 1$ and $v + 1$ are 3-smooth numbers such that $\tau(n) = (u + 1)(v + 1)$ divides n, then n is a $\tau\phi$-number.*

Definition 2.4. *A $\tau\phi$-number such that $\lambda(n) = n$ is called a unital $\tau\phi$-number. See Table* **??**.

The following theorem characterizes unital $\tau\phi$-numbers.

Theorem 2.5. *The number $n = 2^u \cdot 3^v$ is a unital $\tau\phi$-number if and only if $v + 1 = 2^x \cdot 3^s$ and $u + 1 = 2^y \cdot 3^{v-s}$.*

Proof. Necessity. Suppose that n is a unital $\tau\phi$-number. Then n is 3-smooth and $n = 2^u \cdot 3^v$. Furthermore, both $u + 1$ and $v + 1$ are 3-smooth and $(u + 1)(v + 1)$ divides n. Let $v + 1 = 2^x \cdot 3^s$, $u + 1 = 2^y \cdot 3^t$. Then $\tau(n) = 2^{x+y} \cdot 3^{s+t}$. Since $\phi(n) = 2^u \cdot 3^{v-1}$, we impose $s + t = v$ so that $t = v - s$, and we have the required conditions.

Sufficiency. An exercise for the reader. $\square$

Remark 2.6. *The unital $\tau\phi$-numbers have the full form*

$$2^{(2^y \cdot 3^{(2^x \cdot 3^s - s - 1)} - 1)} \cdot 3^{2^x \cdot 3^s - 1},$$

where $x, y, s \geq 0$. They can be computed only for $s = 0, 1, 2$. At $s = 3$ with $x = y = 0$ we obtain

$$2^{3^{23} - 1} \cdot 3^{26},$$

with approximately 2.834×10^{10} digits, or about 28 billion digits. Clearly, there are infinitely many unital $\tau\phi$-numbers.

Remark 2.7 (Cototient). *If one does something similar with the cototient, $\phi'(n) = n - \phi(n)$, then one obtains the following: Let $p > 2$ be a prime and let $n = p^{p^k - 1}$. Then $\tau(n) = p^k$, $\phi'(n) = p^{p^k - 2}$, $\lambda'(n) = \mathrm{lcm}(\tau(n), \phi'(n)) = p^{p^k - 2}$, and $n/\lambda'(n) = p$. See Table 4. If one has both $\phi(n)$ and $\phi'(n)$ divide n, then one obtains that n is a prime power.*

3 Tables

3.1 Table of maxima of $\rho(n)$

$p\#$	$\rho(p\#)$
2#	1.0000
3#	1.5000
5#	3.7500
7#	4.3750
11#	4.8125
13#	5.2135
17#	5.5394
19#	5.8471
23#	6.1129
29#	6.3312
31#	6.5423
37#	6.7240
41#	6.8921
43#	7.0562

Table 1: Table of values of $\rho(n)$ where $\rho(n)$ jumps to a new maximum, namely at $n = p\#$. See Conjecture 1.7.

3.2 Table of $\tau\phi$-numbers

$n = 2^u \cdot 3^v$	$\tau(n)$	$\phi(n)$	$\lambda(n)$	$n/\lambda(n)$
1	1	1	1	1
(2)	(2)	1	(2)	1
$(2)^3$	$(2)^2$	$(2)^2$	$(2)^2$	2
$(2)^7$	$(2)^3$	$(2)^6$	$(2)^6$	2
$(2)^{15}$	$(2)^4$	$(2)^{14}$	$(2)^{14}$	2
$(2)^{31}$	$(2)^5$	$(2)^{30}$	$(2)^{30}$	2
$(2)^{63}$	$(2)^6$	$(2)^{62}$	$(2)^{62}$	2
$(2)^{127}$	$(2)^7$	$(2)^{126}$	$(2)^{126}$	2
$(2)^2(3)$	$(2)(3)$	$(2)^2$	$(2)^2(3)$	1
$(2)^3(3)$	$(2)^3$	$(2)^3$	$(2)^3$	3
$(2)^5(3)$	$(2)^2(3)$	$(2)^5$	$(2)^5(3)$	1
$(2)^7(3)$	$(2)^4$	$(2)^7$	$(2)^7$	3
$(2)^{11}(3)$	$(2)^3(3)$	$(2)^{11}$	$(2)^{11}(3)$	1
$(2)^{15}(3)$	$(2)^5$	$(2)^{15}$	$(2)^{15}$	3
$(2)^{23}(3)$	$(2)^4(3)$	$(2)^{23}$	$(2)^{23}(3)$	1
$(2)^{31}(3)$	$(2)^6$	$(2)^{31}$	$(2)^{31}$	3
$(2)(3)^2$	$(2)(3)$	$(2)(3)$	$(2)(3)$	3
$(2)^2(3)^2$	$(3)^2$	$(2)^2(3)$	$(2)^2(3)^2$	1
$(2)^3(3)^2$	$(2)^2(3)$	$(2)^3(3)$	$(2)^3(3)$	3
$(2)^5(3)^2$	$(2)(3)^2$	$(2)^5(3)$	$(2)^5(3)^2$	1
$(2)^7(3)^2$	$(2)^3(3)$	$(2)^7(3)$	$(2)^7(3)$	3
$(2)^{11}(3)^2$	$(2)^2(3)^2$	$(2)^{11}(3)$	$(2)^{11}(3)^2$	1
$(2)^{15}(3)^2$	$(2)^4(3)$	$(2)^{15}(3)$	$(2)^{15}(3)$	3
$(2)^{23}(3)^2$	$(2)^3(3)^2$	$(2)^{23}(3)$	$(2)^{23}(3)^2$	1
$(2)^{31}(3)^2$	$(2)^5(3)$	$(2)^{31}(3)$	$(2)^{31}(3)$	3
$(2)^2(3)^3$	$(2)^2(3)$	$(2)^2(3)^2$	$(2)^2(3)^2$	3
$(2)^5(3)^3$	$(2)^3(3)$	$(2)^5(3)^2$	$(2)^5(3)^2$	3
$(2)^7(3)^3$	$(2)^5$	$(2)^7(3)^2$	$(2)^7(3)^2$	3
$(2)^8(3)^3$	$(2)^2(3)^2$	$(2)^8(3)^2$	$(2)^8(3)^2$	3
$(2)^{11}(3)^3$	$(2)^4(3)$	$(2)^{11}(3)^2$	$(2)^{11}(3)^2$	3
$(2)^{15}(3)^3$	$(2)^6$	$(2)^{15}(3)^2$	$(2)^{15}(3)^2$	3
$(2)^{17}(3)^3$	$(2)^3(3)^2$	$(2)^{17}(3)^2$	$(2)^{17}(3)^2$	3
$(2)^{23}(3)^3$	$(2)^5(3)$	$(2)^{23}(3)^2$	$(2)^{23}(3)^2$	3

$n = 2^u \cdot 3^v$	$\tau(n)$	$\phi(n)$	$\lambda(n)$	$n/\lambda(n)$
$(2)^{26}(3)^3$	$(2)^2(3)^3$	$(2)^{26}(3)^2$	$(2)^{26}(3)^3$	1
$(2)^{31}(3)^3$	$(2)^7$	$(2)^{31}(3)^2$	$(2)^{31}(3)^2$	3
$(2)^2(3)^5$	$(2)(3)^2$	$(2)^2(3)^4$	$(2)^2(3)^4$	3
$(2)^3(3)^5$	$(2)^3(3)$	$(2)^3(3)^4$	$(2)^3(3)^4$	3
$(2)^5(3)^5$	$(2)^2(3)^2$	$(2)^5(3)^4$	$(2)^5(3)^4$	3
$(2)^7(3)^5$	$(2)^4(3)$	$(2)^7(3)^4$	$(2)^7(3)^4$	3
$(2)^8(3)^5$	$(2)(3)^3$	$(2)^8(3)^4$	$(2)^8(3)^4$	3
$(2)^{11}(3)^5$	$(2)^3(3)^2$	$(2)^{11}(3)^4$	$(2)^{11}(3)^4$	3
$(2)^{15}(3)^5$	$(2)^5(3)$	$(2)^{15}(3)^4$	$(2)^{15}(3)^4$	3
$(2)^{17}(3)^5$	$(2)^2(3)^3$	$(2)^{17}(3)^4$	$(2)^{17}(3)^4$	3
$(2)^{23}(3)^5$	$(2)^4(3)^2$	$(2)^{23}(3)^4$	$(2)^{23}(3)^4$	3
$(2)^{26}(3)^5$	$(2)(3)^4$	$(2)^{26}(3)^4$	$(2)^{26}(3)^4$	3
$(2)^{31}(3)^5$	$(2)^6(3)$	$(2)^{31}(3)^4$	$(2)^{31}(3)^4$	3
$(2)^5(3)^7$	$(2)^4(3)$	$(2)^5(3)^6$	$(2)^5(3)^6$	3
$(2)^7(3)^7$	$(2)^6$	$(2)^7(3)^6$	$(2)^7(3)^6$	3
$(2)^8(3)^7$	$(2)^3(3)^2$	$(2)^8(3)^6$	$(2)^8(3)^6$	3
$(2)^{11}(3)^7$	$(2)^5(3)$	$(2)^{11}(3)^6$	$(2)^{11}(3)^6$	3
$(2)^{15}(3)^7$	$(2)^7$	$(2)^{15}(3)^6$	$(2)^{15}(3)^6$	3
$(2)^{17}(3)^7$	$(2)^4(3)^2$	$(2)^{17}(3)^6$	$(2)^{17}(3)^6$	3
$(2)^{23}(3)^7$	$(2)^6(3)$	$(2)^{23}(3)^6$	$(2)^{23}(3)^6$	3
$(2)^{26}(3)^7$	$(2)^3(3)^3$	$(2)^{26}(3)^6$	$(2)^{26}(3)^6$	3
$(2)^{31}(3)^7$	$(2)^8$	$(2)^{31}(3)^6$	$(2)^{31}(3)^6$	3

Table 2: The $\tau\phi$-numbers. Clearly, they are all of the form $2^u \cdot 3^v$ where $u+1$ and $v+1$ are 3-smooth. See Definition 2.1.

3.3 Table of unital $\tau\phi$-numbers

$n = 2^u \cdot 3^v$	$u+1$	$v+1$	$\tau(n)$	$\phi(n)$
1	1	1	1	1
(2)	(2)	1	(2)	1
$(2)^2(3)$	(3)	(2)	$(2)(3)$	$(2)^2$
$(2)^2(3)^2$	(3)	(3)	$(3)^2$	$(2)^2(3)$
$(2)^{26}(3)^3$	$(3)^3$	$(2)^2$	$(2)^2(3)^3$	$(2)^{26}(3)^2$
$(2)^{80}(3)^5$	$(3)^4$	$(2)(3)$	$(2)(3)^5$	$(2)^{80}(3)^4$
$(2)^{2186}(3)^7$	$(3)^7$	$(2)^3$	$(2)^3(3)^7$	$(2)^{2186}(3)^6$
$(2)^{728}(3)^8$	$(3)^6$	$(3)^2$	$(3)^8$	$(2)^{728}(3)^7$
$(2)^{59048}(3)^{11}$	$(3)^{10}$	$(2)^2(3)$	$(2)^2(3)^{11}$	$(2)^{59048}(3)^{10}$
$(2)^{14348906}(3)^{15}$	$(3)^{15}$	$(2)^4$	$(2)^4(3)^{15}$	$(2)^{14348906}(3)^{14}$
$(2)^{14348906}(3)^{17}$	$(3)^{15}$	$(2)(3)^2$	$(2)(3)^{17}$	$(2)^{14348906}(3)^{16}$

Table 3: The unital $\tau\phi$-numbers. See Definition 2.4.

3.4 Table of $\tau\phi'$-numbers

$n = p^k$	$\tau(n)$	$\phi'(n)$	$n/\lambda'(n)$	
(2)	(2)	1	(2)	2
$(2)^3$	$(2)^2$	$(2)^2$	$(2)^2$	2
$(2)^7$	$(2)^3$	$(2)^6$	$(2)^6$	2
$(2)^{15}$	$(2)^4$	$(2)^{14}$	$(2)^{14}$	2
$(2)^{31}$	$(2)^5$	$(2)^{30}$	$(2)^{30}$	2
$(2)^{63}$	$(2)^6$	$(2)^{62}$	$(2)^{62}$	2
$(2)^{127}$	$(2)^7$	$(2)^{126}$	$(2)^{126}$	2
$(2)^{255}$	$(2)^8$	$(2)^{254}$	$(2)^{254}$	2
$(2)^{511}$	$(2)^9$	$(2)^{510}$	$(2)^{510}$	2
$(2)^{1023}$	$(2)^{10}$	$(2)^{1022}$	$(2)^{1022}$	2
$(2)^{2047}$	$(2)^{11}$	$(2)^{2046}$	$(2)^{2046}$	2
$(3)^2$	(3)	(3)	(3)	3
$(3)^8$	$(3)^2$	$(3)^7$	$(3)^7$	3
$(3)^{26}$	$(3)^3$	$(3)^{25}$	$(3)^{25}$	3
$(3)^{80}$	$(3)^4$	$(3)^{79}$	$(3)^{79}$	3
$(3)^{242}$	$(3)^5$	$(3)^{241}$	$(3)^{241}$	3
$(3)^{728}$	$(3)^6$	$(3)^{727}$	$(3)^{727}$	3
$(3)^{2186}$	$(3)^7$	$(3)^{2185}$	$(3)^{2185}$	3
$(5)^4$	(5)	$(5)^3$	$(5)^3$	5
$(5)^{24}$	$(5)^2$	$(5)^{23}$	$(5)^{23}$	5
$(5)^{124}$	$(5)^3$	$(5)^{123}$	$(5)^{123}$	5
$(5)^{624}$	$(5)^4$	$(5)^{623}$	$(5)^{623}$	5
$(5)^{3124}$	$(5)^5$	$(5)^{3123}$	$(5)^{3123}$	5

Table 4: Numbers n such that both $\tau(n)$ and $\phi'(n) = n - \phi(n)$ divide n. See Remark 2.7.

4 Sequences in OEIS

The definitions of the sequences have been altered to conform the requirements of this document.

Sequence 4.1 (A000005). *$a(n) = \tau(n)$, the number of divisors of n. We set $\tau(1) = 1$.*

Sequence 4.2 (A000010). *$a(n) = \phi(n)$, where $\phi(n)$ is Euler's totient function, that is, $\phi(n)$ is the number of integers relatively prime to n and less than n. We set $\phi(1) = 1$.*

Sequence 4.3 (A002110). *Primorial numbers: $a(n) = 2 \cdot 3 \cdots p_n$, the product of the first n primes, denoted by $p_n\#$, or just $p\#$ for p a prime.*

Sequence 4.4 (A003586). *3-smooth numbers: numbers of the form $2^i \cdot 3^j$ with $i, j \geq 0$, with the standard order.*

Sequence 4.5 (A007694). *Numbers n such that n is divisible by $\phi(n)$.*

Sequence 4.6 (A009279). *$a(n) = \mathrm{lcm}(\tau(n), \phi(n))$.*

Sequence 4.7 (A033950). *Refactorable numbers: Numbers n such that n is divisible by $\tau(n)$. Also known as tau numbers.*

References

[1] Divisor function `https://en.wikipedia.org/wiki/Divisor_function`

[2] Least common multiple, Wikipedia, `https://en.wikipedia.org/wiki/Least_common_multiple`.

[3] Primorial, Wikipedia, `https://en.wikipedia.org/wiki/Primorial`

[4] Smooth number, Wikipedia, `https://en.wikipedia.org/wiki/Smooth_number`

[5] Euler's totient function, Wikipedia, `https://en.wikipedia.org/wiki/Euler%27s_totient_function`.

[6] Refactorable number, Wikipedia, `https://en.wikipedia.org/wiki/Refactorable_number`.