

# Quantifying Relational Structure in Quantum Measurement: An Axiomatic Account of Recursive Observation Depth

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## Abstract

We formalize the relational pair  $(\mathbf{O}, \mathbf{M})$ , observer and observed system, as a primitive mathematical object governed by three operational axioms. From this structure we define Recursive Observation Depth ( $\mathbf{n}^k$ ), the length of the longest informative measurement chain an observer  $\mathbf{O}$  can perform on a system  $\mathbf{M}$  through a Positive Operator-Valued Measure (POVM). Five propositions about  $\mathbf{n}^k$  are proved: monotonicity, relational dependence, a characterization of superposition, an exact value for a compatible (classical) observer, and, for a generic informationally complete observer, the impossibility of a terminal state, from which the divergence of  $\mathbf{n}^k$  follows. Rank appears only as a special case, attained when the observer's measurements are mutually compatible; once they become informationally complete and incompatible,  $\mathbf{n}^k$  diverges. We test the framework against four no-go theorems. It is consistent with Bell, Kochen–Specker, and No-Cloning on independent structural grounds, and consistent with PBR conditionally, provided the quantum state is taken to be a property of the relation  $(\mathbf{O}, \mathbf{M})$  and not of the system alone. Wigner's Friend is resolved within this framework without further postulates. The Born rule is not derived, and no new experimental predictions are offered.

**Keywords:** relational quantum mechanics, observational structure, POVM, recursive observation depth, Wigner's Friend, quantum contextuality

# 1 Introduction

How measurement outcomes actually emerge from the interaction between a quantum system and a measuring apparatus is, after decades of work, still open. The empirical predictions of quantum theory are not in dispute – the trouble is entirely about what sits underneath the formalism. Bohr [1] treated measurement as primitive and left it ontologically uncommitted, which is what the Copenhagen interpretation asks of it. Everett [2] kept unitarity intact by letting the universe branch instead. Bassi and Ghirardi [3] built an objective, dynamical collapse model. Fuchs, Mermin, and Schack [4] moved the quantum state into the believing agent, under QBism. Rovelli [5] went a different way with Relational Quantum Mechanics (RQM): the state is a description relative to the observer, not an absolute fact about the system.

This isn't just philosophy anymore. Bong et al. [6] ran the first experimental test of an extended Wigner's Friend scenario with six photons and found a Bell-type inequality violated by five standard deviations – outcomes really do seem to depend on who is doing the observing. Proietti et al. [7] arrived at a similar place by a different route, ruling out observer independence experimentally. Between the two results, the question of whether the state is relational moved from a matter of interpretation into something with actual empirical stakes.

RQM has real appeal. Once the state is relative instead of absolute, Wigner's Friend and the EPR paradoxes simply dissolve, with no branching worlds and no subjective agents required. But the gap it leaves is structural, not just conceptual. Rovelli [5] is willing to say two observers can assign different states to the same system, but RQM never gives the (observer, system) pair a mathematical home of its own, and it offers no way to say how much observational structure a given observer actually has access to. Di Biagio and Rovelli [8] work on something nearby, how stable facts emerge dynamically, through decoherence, out of ones that are merely relative. That is a question about dynamics. What is addressed here comes before decoherence enters the picture at all: static, structural.

Schlosshauer's review [9], building on his earlier monograph on decoherence [10], is still the standard map of where the measurement problem stands. Decoherence is well understood by now. The actual-outcome problem is not. What follows here is not meant to compete with that map – it fills in one of its margins, an axiomatic account of measurement as a relation between observer and system, meant to sit next to the decoherence story rather than replace it.

This relational turn is not an isolated move; it fits into a broader trend in the foundations of physics. Shor, Benninger, and Khrennikov [11], for instance, recently proposed a relational information framework aimed at causal structure and at unifying competing relational approaches.

Most theoretical work in this area starts from a quantity and only figures out afterward what it means. Here the order is flipped:  $(O, M)$  is set up first, as a formal object with its own operational axioms, and only then does  $n^k$  fall out of it as a consequence. That is not just a stylistic preference. It is what keeps  $n^k$  from being tied, by definition, to matrix rank. Rank does show up later, as an upper bound or as a special case that has to be earned through proof, but it was never assumed at the start.

The contributions can be summarized in four parts: a formalization of the observer  $O$  and the pair  $(O, M)$  as explicit mathematical objects, resting on three operational axioms; the quantity  $n^k$  that this structure gives rise to, together with five propositions proved about it; a systematic check of the framework’s consistency against four canonical no-go theorems; and an explicit resolution of the Wigner’s Friend paradox that requires no additional postulate.

This framework leaves some things deliberately untouched. It does not deliver the Born rule. Individual outcomes stay as probabilistic as they always were, and no prediction departing from standard quantum mechanics is offered here. None of that counts against the framework – it just marks the point where a structural, relational account has to hand off to something that needs a probabilistic postulate.

## 2 Formal Foundations: The Relational Structure $(O, M)$

### 2.1 The Relational Perspective

Standard quantum mechanics associates system  $M$  with a separable Hilbert space  $\mathcal{H}^T$  [12]. A pure state is a unit vector  $|\psi\rangle \in \mathcal{H}^T$ ; a mixed state is a density operator  $\rho \in D(\mathcal{H}^T)$ , positive semi-definite with unit trace. Rovelli’s relational reading [5] of this formalism holds that the state is never an absolute property of  $M$ , only a description relative to some observer. Two observers can assign apparently incompatible states to the same system and both be right. This isn’t because each one is only seeing part of a bigger picture; there simply is no fact about the state that holds independently of an observer.

### 2.2 The Observer as a Mathematical Object

**Definition 1 (Observer  $O$ ).** An observer  $O$  is a triple  $O = (\mathcal{H}^k, \Pi^k, \mu^k)$ , where  $\mathcal{H}^k$  is the separable Hilbert space of  $O$ ’s internal states;  $\Pi^k$  is  $O$ ’s *operation set*, i.e., the family of all measurement resolutions (POVMs in the sense of Definition 2) that  $O$  can apply to an external system,  $\Pi^k = \bigcup_{\alpha} \mathcal{M}_{\alpha}$  with each  $\mathcal{M}_{\alpha}$  a single POVM; and  $\mu^k$  is a memory map,  $\mu^k : \text{Outcomes}(\Pi^k) \rightarrow \mathcal{H}^k$ , which encodes measurement outcomes into  $O$ ’s internal state. This triple fully determines what  $O$  can “know” and “access.” When  $O$  has access to exactly one resolution ( $\Pi^k = \mathcal{M}_1$ ), we write  $\Pi^k$  interchangeably for that single POVM; the general, multi-resolution case is used throughout Sections 3 and 4, and is stated explicitly in Axiom A1 below.

Nothing here requires  $O$  to be a person or a conscious agent. It is any physical system that can interact with another, register information about it, and respond accordingly; for our purposes it is fully characterized by  $\Pi^k$  and  $\mu^k$  alone.

### 2.3 Positive Operator-Valued Measure (POVM)

**Definition 2 (POVM, single resolution).** A POVM (single resolution) for observer  $O$  acting on system  $M$  with Hilbert space  $\mathcal{H}^T$  is  $\mathcal{M} = \{E_i\}_{i \in K}$ , where  $K$  is a finite or countable index set,  $E_i \geq 0$  for every  $i$  (positive semi-definite), and  $\sum_{i \in K} E_i = I$

(resolution of the identity). Projective measurement is the special case in which  $E_i^2 = E_i = E_i^\dagger$ .  $O$ 's full operation set  $\Pi^k$  (Definition 1) may consist of a single such resolution or of several,  $\Pi^k = \bigcup_\alpha \mathcal{M}_\alpha$  (Axiom A1); we reserve the symbol  $\mathcal{M}$ , possibly with a subscript, for one resolution, and  $\Pi^k$  for the observer's operation set as a whole.

*Standing convention (choice of instrument).* Fixing an effect  $E_i \geq 0$  does not by itself fix a post-measurement state. Any Kraus operator of the form  $M_i = U_i \sqrt{E_i}$ ,  $U_i$  unitary, satisfies  $E_i = M_i^\dagger M_i$  equally well, yet different choices of  $U_i$  push the state to different places,  $M_i \rho M_i^\dagger / \text{Tr}(M_i \rho M_i^\dagger)$ , even though  $E_i$  has not changed. This is the familiar instrument-versus-effect gap in quantum measurement theory: the effects  $E_i$  pin down outcome statistics but say nothing about how the state updates. That requires the further specification of an *instrument* [13, 14], namely a particular Kraus decomposition of each effect. Left unresolved, this gap would leave  $n^k$  (Definition 6), built as it is from the post-measurement map of Definition 4, dependent on more than the POVM  $\Pi^k$  alone. We close it by fixing, throughout the article, the *Lüders instrument*: for every effect  $E_i$  in any  $\mathcal{M}_\alpha \subseteq \Pi^k$ , the Kraus operator is taken to be the unique positive square root,

$$M_i := \sqrt{E_i}, \quad \rho \rightarrow \rho_i = \frac{M_i \rho M_i^\dagger}{\text{Tr}(M_i \rho M_i^\dagger)} = \frac{\sqrt{E_i} \rho \sqrt{E_i}}{\text{Tr}(E_i \rho)}. \quad (1)$$

Fixed once and for all this way,  $\rho_i$ , and with it  $n^k(\rho)$ , becomes a well-defined function of  $(\rho, \Pi^k)$  alone. No unstated choice of instrument is left hanging, and every proposition in Sections 3–5 is proved under this convention. (For projective effects  $\sqrt{E_i} = E_i$ , so nothing changes from the familiar Lüders-von Neumann rule,  $M_i = E_i$ ,  $U_i = I$ . The choice matters even here, and not just as a formality: Proposition 4's proof depends on  $P_S \rho P_S / \text{Tr}(P_S \rho P_S)$  staying diagonal in the fixed basis, which holds only because the Kraus operator is  $P_S$  itself, rather than  $U_i P_S$  for some  $U_i \neq I$  acting on  $\text{ran}(P_S)$ . A different instrument could rotate the post-measurement state out of that diagonal structure even for a fully sharp, compatible POVM, so the convention is doing real work in Proposition 4, not only in the unsharp case of Remark 2.)

Busch, Lahti, and Mittelstaedt [15] worked out the POVM formalism in full rigor, giving modern quantum measurement theory a mathematical footing that also accommodates unsharp measurements, something ordinary projective measurement cannot describe. It is the companion notion of an *instrument* [13, 14], specifying the state update alongside the outcome probabilities, that lets Definition 4, and hence  $n^k$ , be well-posed once a POVM is chosen. The standing convention above simply picks out, among the instruments compatible with a given  $\Pi^k$ , the canonical Lüders one.

## 2.4 Axioms of Observational Structure

Three axioms carry the weight of everything that follows, so they are set out before  $n^k$  is even defined. They are deliberately operational, stating what it means for  $(O, M)$  to possess an observational structure at all, without reference to rank, entropy, or any other quantity that could later be confused for the definition itself.

**Axiom A1 (Observer’s Operation Set).** Every observer  $O$  possesses an operation set  $\Pi^k = \bigcup_{\alpha} \mathcal{M}_{\alpha}$  as defined in Definitions 1 and 2, where each  $\mathcal{M}_{\alpha}$  is a single POVM.  $\Pi^k$  is the sole interface through which  $O$  can interact informatively with system  $M$ ; no other access to  $M$  is assumed available to  $O$ . A single measurement step consists of choosing one  $\mathcal{M}_{\alpha} \in \Pi^k$  and applying it (with Kraus operators fixed by the Lüders convention, Eq. (1)). A chain  $C_n$  (Definition 5) may use a different  $\mathcal{M}_{\alpha}$  at each step, but  $\Pi^k$  itself, the full menu of resolutions available to  $O$ , is fixed for the duration of the chain and does not change from step to step; only the choice of *which*  $\mathcal{M}_{\alpha} \in \Pi^k$  to apply at a given step may vary. The distinction between  $\Pi^k$  as a single basis and  $\Pi^k$  as a family of bases is not a bookkeeping detail: the two readings yield different values of  $n^k$ , as Proposition 4 and Remark 1 show directly.

**Axiom A2 (Operational Distinguishability).** A state  $\rho$  is operationally distinguishable from its own post-measurement update  $\rho_i$  by  $O$  if and only if there exists  $E \in \Pi^k$  that produces a post-measurement state differing, as an operator, from  $\rho$  (Definition 4). Distinguishability is defined for a state and its own update, rather than for two arbitrary states, because that is the only instance used in the constructions below (Axiom A3, Definition 4).

**Axiom A3 (Chain Termination).** A sequential measurement chain performed by  $O$  on state  $\rho$  terminates at step  $i$  if no element of  $\Pi^k$  produces operational distinguishability from  $\rho_i$ . When that occurs, no further observational structure remains available to  $O$  on  $\rho_i$ .

### 3 Formalization of Recursive Observation Depth ( $n^k$ )

*Standing assumption.* Unless stated otherwise,  $\dim(\mathcal{H}^T) = d < \infty$  throughout this section and Section 4. Definitions 4, 5, and 6 are stated for general (possibly infinite-dimensional)  $\mathcal{H}^T$ , but Propositions 4 and 5 require  $d < \infty$ , and we fix this from the outset to avoid restating it in every statement.

**Definition 3 (Informationally Complete POVM).** A POVM  $\mathcal{M} = \{E_i\}_i$  acting on  $\mathcal{H}^T$  is *informationally complete* (IC) if  $\text{span}_{\mathbb{R}}\{E_i\} = \mathcal{L}_H(\mathcal{H}^T)$ , the real vector space of Hermitian operators on  $\mathcal{H}^T$  ( $\dim_{\mathbb{R}} \mathcal{L}_H(\mathcal{H}^T) = d^2$ ). Equivalently,  $\mathcal{M}$  is IC if and only if  $\rho \mapsto (\text{Tr}(E_i \rho))_i$  is injective on  $D(\mathcal{H}^T)$ , i.e., every state is uniquely determined by its outcome statistics under  $\mathcal{M}$ . Since  $\sum_i E_i = I$  is automatically satisfied, informational completeness in particular requires the *traceless* parts  $\{E_i - \frac{\text{Tr}(E_i)}{d}I\}$  to span the  $(d^2 - 1)$ -dimensional space of traceless Hermitian operators, so any IC POVM has at least  $d^2$  elements (achieved, e.g., by a symmetric IC POVM).

#### 3.1 Informative Measurement Chain

**Definition 4 (Informative Measurement).** Given system  $M$  with state  $\rho \in D(\mathcal{H}^T)$  and observer  $O = (\mathcal{H}^k, \Pi^k, \mu^k)$ . A measurement element  $E_i \in \Pi^k$ , with associated Kraus operator  $M_i$ , is said to be informative with respect to  $\rho$  if:

$$M_i \rho M_i^\dagger \neq \text{Tr}(M_i \rho M_i^\dagger) \cdot \rho. \quad (2)$$

Conversely,  $E_i$  is said to be non-informative with respect to  $\rho$  if the above equality holds, i.e.,  $\rho$  is a fixed point of the corresponding measurement map.

**Definition 5 (Informative Measurement Chain).** A chain  $C_n = (E_1, E_2, \dots, E_n)$ , with each  $E_i \in \Pi^k$  and repetition of elements permitted, is called informatively valid if, at every step  $i$ ,  $E_i$  is informative with respect to  $\rho_{i-1}$  in the sense of Definition 4.

### 3.2 Definition of Recursive Observation Depth

**Definition 6 ( $n^k$ ).** Let  $O = (\mathcal{H}^k, \Pi^k, \mu^k)$  be an observer and  $\rho \in D(\mathcal{H}^T)$  a state. The Recursive Observation Depth of  $O$  relative to  $\rho$  is:

$$n^k(\rho) := \sup\{n \in \mathbb{N}_0 : \text{there exists an informatively valid chain } C_n\}, \quad (3)$$

with the convention that  $n^k(\rho) = 0$  if no  $E \in \Pi^k$  is informative with respect to  $\rho$  (i.e.,  $\rho$  is a fixed point of the entire  $\Pi^k$ ). Thus  $n^k(\rho) \in \mathbb{N}_0 \cup \{\infty\}$ .

In plain terms,  $n^k$  counts the largest number of informative measurement steps  $O$  can carry out before running out of observational structure to draw on; each step spends one unit of the relational structure still standing between  $O$  and  $M$ .

*Caveat (depth versus executable protocol).* Because the supremum in Eq. (3) ranges over *every* informatively valid chain available for a given  $\rho$ ,  $n^k(\rho)$  should be read as the maximal relational depth mathematically available between  $O$  and  $M$  once  $\rho$  is fixed. The achieving chain constructed in Proposition 4 (Eq. (5)), for instance, is built from  $\text{supp}(\rho)$ , which already presupposes knowing  $\rho$ . Nothing here claims that  $O$  could realize a chain of length  $n^k(\rho)$  through some fixed procedure applied blind, without knowing in advance which resolution is most informative at each step. An observer forced to pick  $M_\alpha \in \Pi^k$  before seeing  $\rho$ , or committed generally to a state-blind strategy, may well do worse in the worst case. We reserve the name *depth* of  $(\rho, \Pi^k)$  for  $n^k(\rho)$  in this sense. A state-blind, worst-case variant,  $\underline{n}^k := \min_\rho(\text{chain length some fixed protocol can guarantee})$ , is a different quantity that this article neither constructs nor computes. Every proposition below concerns  $n^k(\rho)$  as depth, never  $\underline{n}^k$ .

### 3.3 Canonical Cases and the Status of Rank

The three cases below put Definition 6 to work, and along the way mark out where rank actually enters the picture, as something proved afterward in each case rather than assumed at the outset.

**Case 1 (Superposition with an Informationally Complete POVM):** If  $\rho = |\psi\rangle\langle\psi|$  is a pure state with at least two nonzero components in the POVM basis, and  $\Pi^k$  is informationally complete (Definition 3), then  $n^k(\rho) \geq 1$ .  $\text{Rank}(\rho) = 1$  for any pure state, so the value of  $n^k$  here is not determined by rank. (As Proposition 5 below shows, for a generic IC POVM this inequality is far from tight:  $n^k(\rho)$  is typically infinite, not merely  $\geq 1$ .)

**Case 2 (Eigenstate):** If  $\rho = |j\rangle\langle j|$  is already an eigenstate of every  $E_i \in \Pi^k$ , then  $n^k(\rho) = 0$ .

**Case 3 (Classical Observer):** Two distinct sub-cases must be separated, since they give different bounds.

- (3a) *Atomic classical observer*:  $\Pi^k$  consists of a *single* resolution of the identity by rank-one diagonal projectors,  $\Pi^k = \{|i\rangle\langle i|\}_{i=1}^d$ , applied repeatedly (Definition 2, one fixed  $\mathcal{M}_\alpha$ ). A single application already collapses  $\rho$  to a definite basis state, so  $n^k(\rho) \in \{0, 1\}$  for every  $\rho$ , regardless of  $\text{rank}(\text{diag}(\rho))$ .
- (3b) *Fully refining classical observer*:  $\Pi^k$  contains, for the same fixed basis, *every* diagonal projector  $P_S = \sum_{i \in S} |i\rangle\langle i|$ ,  $S \subseteq \{1, \dots, d\}$  (equivalently, every coarse-graining of that basis is an available resolution  $\mathcal{M}_\alpha$ , per Axiom A1). In this case  $n^k(\rho) = \text{rank}(\text{diag}(\rho)) - 1$  exactly (Proposition 4 below).

Case (3b) alone attains  $\text{rank}(\text{diag}(\rho)) - 1$  exactly; case (3a) is there to show that an ordinary diagonal  $\Pi^k$  generally falls short of it. So for a classical (diagonal) observer with no further assumption,  $\text{rank}(\text{diag}(\rho)) - 1$  is only a loose upper bound, as in (3a); it tightens to an exact equality only once the refining structure of (3b) is added.

## 4 Propositions and Formal Proofs

### 4.1 Proposition 1: Monotonicity

**Proposition 1 (Monotonicity of  $n^k$ )**. For observer  $O$ , system  $M$ , and state  $\rho \in D(\mathcal{H}^T)$ , if  $E \in \Pi^k$  is informative with respect to  $\rho$ , then:

$$n^k(\rho^L) \leq n^k(\rho) - 1, \quad (4)$$

where  $\rho^L = M\rho M^\dagger / \text{Tr}(M\rho M^\dagger)$  is the post-measurement state.

*Proof* Let  $n^k(\rho) = m$ . Every informatively valid chain  $F = (F_1, \dots, F_r)$  starting from  $\rho_1 = \rho^L$  can be extended to a chain  $(E, F_1, \dots, F_r)$  starting from  $\rho_0 = \rho$  with length  $1 + r$ . Since  $m$  is the supremum,  $1 + r \leq m$ , hence  $r \leq m - 1$ . Therefore  $n^k(\rho_1) \leq m - 1 = n^k(\rho) - 1$ .  $\square$

Physically, every informative measurement spends one unit of whatever observational structure the relation  $(O, M)$  still has left.  $n^k$  behaves as a relational resource that only ever depletes with observation, which gives the irreversibility of measurement a structural, rather than merely dynamical, footing.

### 4.2 Proposition 2: Relationality

**Proposition 2**. For two observers  $O_1$  and  $O_2$  with  $\Pi_1^k \neq \Pi_2^k$ , it generally holds that  $n_1^k(\rho) \neq n_2^k(\rho)$ . In particular, there exist  $\rho, O_1, O_2$  such that  $n_1^k(\rho) > 0$  while  $n_2^k(\rho) = 0$ .

*Proof* Let  $\rho = |+\rangle\langle +|$  with  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ . Observer  $O_1$ , with a POVM in the basis  $\{|0\rangle, |1\rangle\}$ : applying  $E_1^{(1)} = |0\rangle\langle 0|$  yields  $\rho' = |0\rangle\langle 0| \neq \rho$ , so  $n_1^k(\rho) \geq 1$ . Observer  $O_2$ , with a POVM in the basis  $\{|+\rangle, |-\rangle\}$ : applying  $E_1^{(2)} = |+\rangle\langle +|$  yields  $\rho' = \rho$ , so  $n_2^k(\rho) = 0$ .  $\square$

So  $n^k$  belongs to the relation  $(O, M)$  and not to  $M$  in isolation, which is exactly what one would expect of a quantity built from the relational structure of Section 2.

This relational character connects fairly directly to contextuality. Navoni, Genoni, and Smirne [16] recently derived quantum contextuality from measurement invasiveness, treating contextuality as a real dividing line between quantum and classical

systems. Proposition 2 can be read as one structural face of that same contextuality:  $n^k$ 's value tracks the measurement context, i.e., which POVM  $O$  happens to bring to the table.

### 4.3 Proposition 3: Superposition Characterization

**Proposition 3.** Let  $|\psi\rangle = \sum_i \alpha_i |i\rangle$  be a superposition in  $O$ 's POVM basis  $\{|i\rangle\}$ , with at least two coefficients  $\alpha_i \neq 0$ . Then  $n^k(|\psi\rangle\langle\psi|) > 0$ . Moreover, if  $O$  obtains outcome  $|j\rangle$ , then  $n^k(|j\rangle\langle j|) = 0$ .

*Proof* (1)  $\rho = |\psi\rangle\langle\psi|$  has off-diagonal elements  $\langle i|\rho|j\rangle = \alpha_i \alpha_j^* \neq 0$  for  $i \neq j$ . Applying  $E_j = |j\rangle\langle j|$ : the post-measurement state  $\rho' = |j\rangle\langle j|$  has no off-diagonal elements, so  $\rho' \neq \rho$  and  $E_j$  is informative. Hence  $n^k(\rho) \geq 1 > 0$ . (2)  $\rho' = |j\rangle\langle j|$ : for every  $E_i \in \Pi^k$ ,  $M_i \rho' M_i^\dagger = \text{Tr}(M_i \rho' M_i^\dagger) \cdot \rho'$ , so  $n^k(\rho') = 0$ .  $\square$

Superposition, on this reading, becomes a relational category rather than an ontological one: a state counts as a “superposition” relative to  $O$  exactly when  $n^k > 0$ , and not otherwise.

### 4.4 Proposition 4: Boundedness and Relation to Rank

**Proposition 4 (Exact Value of  $n^k$  for a Compatible/Classical Observer).** Fix an orthonormal basis  $\{|i\rangle\}_{i=1}^d$  of  $\mathcal{H}^T$  ( $d < \infty$ ), and suppose  $\Pi^k$  consists exactly of the diagonal projectors of this basis and all of their coarse-grainings, i.e.,  $\Pi^k = \bigcup_{\mathcal{P}} \mathcal{P}$  where  $\mathcal{P}$  ranges over partitions of  $\{1, \dots, d\}$  and, for a partition  $\mathcal{P} = \{S_1, \dots, S_m\}$ , the corresponding resolution of the identity is  $\{P_{S_1}, \dots, P_{S_m}\}$  with  $P_S := \sum_{i \in S} |i\rangle\langle i|$  (this is the “fully refining classical observer” of Case 3(b)). Then for every  $\rho \in D(\mathcal{H}^T)$ ,

$$n^k(\rho) = |\text{supp}(\rho)| - 1 = \text{rank}(\text{diag}(\rho)) - 1, \quad (5)$$

where  $\text{supp}(\rho) := \{i : \langle i|\rho|i\rangle > 0\}$ . In particular  $n^k(\rho) \leq d - 1$ , with equality iff  $\rho$  has full support in this basis.

*Proof* Write  $r = |\text{supp}(\rho)|$ . Every state reachable from  $\rho$  under  $\Pi^k$  remains diagonal in the fixed basis, since each  $P_S$  commutes with  $\rho$  and  $P_S \rho P_S / \text{Tr}(P_S \rho P_S)$  is again diagonal.

*Upper bound.* Let  $\phi(\sigma) := |\text{supp}(\sigma)|$  for diagonal  $\sigma$ . Suppose  $\sigma$  has support  $T$  and  $P_S$  is applied. If  $T \subseteq S$ , then  $P_S \sigma P_S = \sigma$  and  $E = P_S$  is non-informative (Definition 4). If  $T \cap S = \emptyset$ , the outcome has probability  $\text{Tr}(P_S \sigma) = 0$  and cannot occur. In the only remaining case,  $\emptyset \neq T \cap S \subsetneq T$ : the outcome is realizable, and the post-measurement state is diagonal with support exactly  $T \cap S$ , which differs from  $\sigma$  (since  $T \setminus S \neq \emptyset$ ), so  $E$  is informative and  $\phi$  strictly decreases:  $\phi(\sigma_{\text{new}}) = |T \cap S| \leq |T| - 1 = \phi(\sigma) - 1$ . Thus every realizable informative step in a chain starting at  $\rho$  strictly decreases  $\phi$  by at least 1, and  $\phi \geq 1$  always (trace 1 implies nonempty support). Hence any informatively valid chain has length at most  $\phi(\rho) - 1 = r - 1$ , so  $n^k(\rho) \leq r - 1$ .

*Achievability.* Enumerate  $\text{supp}(\rho) = \{i_1, \dots, i_r\}$  and set  $S_j := \{i_{j+1}, \dots, i_r\}$  for  $j = 0, \dots, r - 1$  (so  $S_0 = \text{supp}(\rho)$ ,  $|S_j| = r - j$ ). For  $j = 1, \dots, r - 1$ , apply the resolution  $\{P_{S_j}, P_{S_j^c}\} \in \Pi^k$  to the current state (which has support  $S_{j-1}$ ):  $\text{Tr}(P_{S_j} \sigma_{j-1}) = \sum_{i \in S_j} \sigma_{ii} > 0$  since  $S_j \subset S_{j-1} = \text{supp}(\sigma_{j-1})$ , so this outcome is realizable, and it is informative because

$S_j \subsetneq S_{j-1}$ . The resulting state has support exactly  $S_j$ . Iterating for  $j = 1, \dots, r-1$  produces an informatively valid chain of length  $r-1$ , terminating at a basis eigenstate ( $|S_{r-1}| = 1$ ). Hence  $n^k(\rho) \geq r-1$ .

Combining the two bounds gives  $n^k(\rho) = r-1$ .  $\square$

This lines up with the limited role given to rank earlier. Proposition 4 delivers an exactly attained value of  $n^k$ , but only under a specific, named hypothesis on  $\Pi^k$ : a compatible, classical, fully refining observer. Rank plays no part in the definition of  $n^k$ , and it is not a bound one gets for free just because some POVM happens to contain an orthonormal basis. Case (3a) makes the point sharply – the same rank quantity neither bounds  $n^k$  from below nor characterizes it once Proposition 4’s refining resolutions are taken away.

*Remark 1* (Sharpness is not enough: incompatibility already unbounds  $n^k$ ) One might guess that Eq. (5) extends to any  $\Pi^k$  that merely *contains* a complete orthonormal basis, even alongside resolutions from an incompatible basis. It does not, and sharpness alone gives no protection against the counterexample. Take  $d = 2$  and  $\Pi^k = \mathcal{M}_1 \cup \mathcal{M}_2$ , with  $\mathcal{M}_1 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$  and  $\mathcal{M}_2 = \{|+\rangle\langle +|, |-\rangle\langle -|\}$ , both sharp, both projective, and each a perfectly valid POVM on its own (Axiom A1). Starting from  $\rho_0 = |0\rangle\langle 0|$ , applying  $\mathcal{M}_2$  is informative: the outcome,  $|+\rangle\langle +|$  or  $|-\rangle\langle -|$ , differs from  $\rho_0$  either way. From there  $\mathcal{M}_1$  is informative again, since  $|\pm\rangle$  is an eigenstate of neither  $|0\rangle\langle 0|$  nor  $|1\rangle\langle 1|$ . This back-and-forth between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  never stops, giving  $n^k(\rho_0) = \infty$  despite  $\Pi^k$  consisting of two complete orthonormal bases, every element sharp. The right sufficient condition for finiteness is not sharpness but *mutual compatibility* of everything in  $\Pi^k$ , as Proposition 4 requires. This sharpens the point of Remark 2 below: what unbounds  $n^k$  is incompatibility among the available measurements, sharp or not.

*Remark 2* A single unsharp POVM, with no incompatible resolutions in sight, can already do the same. Take  $\dim(\mathcal{H}^T) = 2$  and  $\Pi^k = \{E_0, E_1\}$ , Kraus operators  $M_0 = \text{diag}(\cos \theta, \sin \theta)$ ,  $M_1 = \text{diag}(\sin \theta, \cos \theta)$ , for some fixed  $0 < \theta < \pi/4$ . Start from  $\rho_0 = |+\rangle\langle +|$ , where the diagonal amplitudes sit at ratio 1, and keep applying the outcome tied to  $M_0$ : after  $n$  steps the ratio has moved to  $(\cos \theta / \sin \theta)^{2n} \neq 1$  for every  $n \geq 1$  (since  $\cos \theta / \sin \theta > 1$  in this range), so the state changes at every single step, each step counts as informative under Definition 4, and the chain simply never stops. This gives  $n^k(\rho_0) = \infty$ , well within what Definition 6 allows. With  $|\Pi^k| = 2$ , this also rules out any hope of bounding  $n^k$  by the size of  $\Pi^k$  once repetition is permitted; only the compatibility hypothesis of Proposition 4 delivers a genuine finite bound.

*Remark 3* (What  $n^k = \infty$  does and does not claim) The divergence proved in Proposition 5 says only that no mathematical upper bound exists on the structural depth available to  $O$  under a generic IC POVM. It is not a prediction that any real measurement chain would run forever. Actual experiments are bounded by finite resources: time, coherence, apparatus precision. A physically realized chain therefore halts after finitely many steps no matter what  $n^k(\rho)$  says. The statement  $n^k = \infty$  should be read as: a chain of every finite length  $n$  is available in principle, not as a claim about what would actually be carried out.

One question left dangling here is how  $n^k$  behaves in the classical limit. Dutra, Baldijão, and Terra Cunha [17] have shown that quantum contextuality itself vanishes in that limit; whether  $n^k$  follows suit (say,  $n^k \rightarrow 0$  for macroscopic systems) seems worth pursuing as a bridge between this relational framework and classical physics.

#### 4.5 Proposition 5: Informational Completeness Forbids a Terminal State

One might conjecture that informational completeness of  $\Pi^k$  guarantees a finite lower bound on  $n^k$ , say  $n^k(|\psi\rangle\langle\psi|) \geq d - 1$ , attained when  $\Pi^k$  is an orthogonal projective basis. This conjecture cannot survive as stated. An orthogonal projective basis has only  $d$  elements, spanning just the  $d$ -dimensional space of diagonal matrices, so by Definition 3 it is *never* informationally complete once  $d \geq 2$  (informational completeness needs the full  $d^2$ -dimensional space  $\mathcal{L}_H(\mathcal{H}^T)$ ). The proposed equality condition therefore has nothing to attach to. If anything, the truth runs the other way: informational completeness tends to produce *unbounded* recursive depth, not merely depth bounded below, as the next proposition establishes.

**Proposition 5 (No Terminal State Under a Generic IC POVM).** Let  $\Pi^k = \{E_j\}_{j=1}^N$  be a rank-one POVM,  $E_j = c_j|\phi_j\rangle\langle\phi_j|$  with unit vectors  $|\phi_j\rangle$  and  $c_j > 0$ , that is informationally complete and *totally non-orthogonal*, i.e.,  $\langle\phi_i|\phi_j\rangle \neq 0$  for all  $i \neq j$  (this holds, e.g., for any symmetric IC POVM/SIC-POVM in dimensions where SIC-POVMs are known to exist – constructed analytically or numerically in all dimensions checked to date [18, 19], though existence in every dimension remains Zauner’s conjecture [20] rather than a proven theorem – and generically for random IC POVMs regardless of dimension). Then:

- (a) No state  $\rho \in D(\mathcal{H}^T)$  is  $\Pi^k$ -terminal: for every  $\rho$  there exists  $E_j \in \Pi^k$  informative with respect to  $\rho$ . In particular  $n^k(\rho) \geq 1$  for every  $\rho$ .
- (b) Consequently  $n^k(\rho) = \infty$  for every  $\rho \in D(\mathcal{H}^T)$ .

*Proof* (a) Since each  $E_j$  is rank-one, applying it to any state  $\sigma$  with  $\langle\phi_j|\sigma|\phi_j\rangle > 0$  yields post-measurement state  $|\phi_j\rangle\langle\phi_j|$ . This is non-informative exactly when  $\sigma = |\phi_j\rangle\langle\phi_j|$  already. So  $\sigma$  is  $\Pi^k$ -terminal only if  $\sigma = |\phi_j\rangle\langle\phi_j|$  for every  $j$  with  $\langle\phi_j|\sigma|\phi_j\rangle > 0$ , i.e., only if  $\sigma$  is a common eigenvector (with eigenvalue 0 or the identity direction) of *all* the  $|\phi_j\rangle$ . Suppose  $|\chi\rangle$  is such a common eigenvector: for each  $j$ ,  $|\chi\rangle$  is either parallel or orthogonal to  $|\phi_j\rangle$ . Since the  $|\phi_j\rangle$  are pairwise non-orthogonal and (by informational completeness) span  $\mathcal{H}^T$ ,  $|\chi\rangle$  can be parallel to at most one  $\phi_{j_0}$ ; for every other  $j \neq j_0$ ,  $|\chi\rangle$  must then be orthogonal to  $|\phi_j\rangle$ , but if  $|\chi\rangle = |\phi_{j_0}\rangle$  this contradicts  $\langle\phi_{j_0}|\phi_j\rangle \neq 0$ . If  $|\chi\rangle$  is parallel to none of the  $\phi_j$ , it must be orthogonal to all of them, contradicting that they span  $\mathcal{H}^T$ . Either way, no common eigenvector exists, so no state is  $\Pi^k$ -terminal, proving (a).

(b) By (a), from any  $\rho$  there is an informative  $E_{j_1}$ , leading to  $|\phi_{j_1}\rangle\langle\phi_{j_1}|$ ; by (a) again this state is not terminal, so some  $E_{j_2}$  ( $j_2 \neq j_1$ , since  $E_{j_1}$  is now non-informative) is informative, leading to  $|\phi_{j_2}\rangle\langle\phi_{j_2}|$ ; iterating, an informatively valid chain of every finite length can be constructed (alternating, e.g., between any two indices  $i \neq l$  with  $\langle\phi_i|\phi_l\rangle \neq 0$  already suffices to build an arbitrarily long chain). Hence  $n^k(\rho) = \sup\{n : \dots\} = \infty$ .  $\square$

*Interpretation.* Propositions 4 and 5 are two poles of one structure, not two unrelated results. Proposition 4 shows that *mutually compatible* (simultaneously diagonalizable) resolutions in  $\Pi^k$  keep  $n^k$  finite and exactly computable, equal to  $\text{rank}(\text{diag } \rho) - 1$ ; Proposition 5 shows that once  $\Pi^k$  is informationally complete with generic, non-orthogonal rank-one structure, compatibility breaks down almost everywhere and  $n^k$  diverges. Remarks 1 and 2 already hinted at this divergence in small, non-IC examples; Proposition 5 confirms it is the generic story once informational completeness is imposed. Richness of  $\Pi^k$  and boundedness of  $n^k$  turn out to sit in tension with each other; a richer  $\Pi^k$  does not even supply a mild lower bound on  $n^k$ , a tension already visible in Case 1 of Section 3.3.

*Example 1* (An intermediate case:  $d = 2$ , three elements, not informationally complete) The two poles above can be seen without appealing to full informational completeness at all. Let  $d = 2$  and  $\Pi^k = \{E_1, E_2, E_3\}$  be the trine POVM,  $E_j = \frac{2}{3}|\phi_j\rangle\langle\phi_j|$  for  $j = 1, 2, 3$ , with  $|\phi_j\rangle$  the Bloch vectors at  $0^\circ$ ,  $120^\circ$ , and  $240^\circ$  on the equator of the Bloch sphere. With only  $3 < d^2 = 4$  elements this POVM is *not* informationally complete by Definition 3, yet it is rank-one and totally non-orthogonal ( $\langle\phi_i|\phi_j\rangle \neq 0$  for  $i \neq j$ , since none of the three directions are antipodal). Any two of the  $|\phi_j\rangle$  already span  $\mathbb{C}^2$ , so Proposition 5(a)'s argument carries over unchanged. Informational completeness was only ever needed there to guarantee the  $|\phi_j\rangle$  span  $\mathcal{H}^T$ , and three non-antipodal directions in  $d = 2$  already do that. No state is  $\Pi^k$ -terminal, and  $n^k(\rho) = \infty$  for every  $\rho$ . Informational completeness, then, is *sufficient but not necessary* for the divergence in Proposition 5; what actually drives it is pairwise non-orthogonality of a rank-one resolution together with spanning of  $\mathcal{H}^T$ , of which informational completeness is just one route, admittedly the generic one in higher dimensions.

## 5 Resolution of the Wigner's Friend Paradox

### 5.1 Statement of the Paradox

Few thought experiments press harder on any interpretation that gives the observer a privileged role than Wigner's Friend [21]. System  $S$  starts in  $|\psi\rangle_S = \alpha|0\rangle + \beta|1\rangle$ ;  $F$ , the Friend, is an observer sealed inside an isolated laboratory;  $W$ , Wigner, watches from outside.

After the measurement,  $F$  sees  $S$  sitting in a definite state,  $|0\rangle$  or  $|1\rangle$ .  $W$ , meanwhile, sees the combined system ( $S + F$ ) still in superposition,  $|\Psi\rangle_{SF} = \alpha|0\rangle_S|F:0\rangle_F + \beta|1\rangle_S|F:1\rangle_F$ . So has  $S$  collapsed, or hasn't it?

### 5.2 Formal Resolution

The resolution turns on a simple observation:  $n_F^k$  and  $n_W^k$  are functions defined on different domains, with different POVMs:

To make  $\Pi_W^k$  concrete: let  $|\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$  on each of  $\mathcal{H}_S$  and  $\mathcal{H}_F$  (so  $F$ 's two recorded outcomes are identified with a qubit basis  $|F:0\rangle, |F:1\rangle$ ), and take  $\Pi_W^k$  to be the product projective measurement in this rotated basis, an orthonormal resolution of the identity on  $\mathcal{H}_S \otimes \mathcal{H}_F$ , hence a legitimate POVM by Definition 2. In the maximally entangled case  $\alpha = \beta = 1/\sqrt{2}$ ,  $|\Psi\rangle_{SF} = (|00\rangle + |11\rangle)/\sqrt{2}$  rewrites in the rotated

**Table 1** Domains, POVMs, states, and the value of  $n^k$  for the two observers in the Wigner’s Friend scenario

Observer	Domain	POVM	State	$n^k$
$F$	$D(\mathcal{H}_S)$	$\Pi_F^k = \{ 0\rangle\langle 0 ,  1\rangle\langle 1 \}$	$\rho_F(1) =  j\rangle\langle j $	0
$W$	$D(\mathcal{H}_S \otimes \mathcal{H}_F)$	$\Pi_W^k = \{ ++\rangle\langle ++ ,  +-\rangle\langle +- ,  -+\rangle\langle -+ ,  --\rangle\langle -- \}$	$\rho_W(0) =  \Psi\rangle\langle \Psi _{SF}$	$> 0$

basis as  $(|++\rangle + |--\rangle)/\sqrt{2}$ . Applying  $\Pi_W^k$  then yields  $|++\rangle\langle ++|$  or  $|--\rangle\langle --|$  with probability 1/2 each, either outcome differing from  $\rho_W(0)$ , so  $\Pi_W^k$  is informative and  $n_W^k(\rho_W(0)) \geq 1$ . Since  $\Pi_W^k$  is a single fixed orthonormal basis (Case 3a, Section 3.3), the resulting eigenstate is itself terminal under repeated  $\Pi_W^k$ , giving  $n_W^k(\rho_W(0)) = 1$  exactly here. The crucial point is that  $\Pi_W^k$  is actually available to  $W$ , who has joint access to  $S$  and to  $F$ ’s recorded outcome – but not to  $F$ , whose domain  $D(\mathcal{H}_S)$  never sees the joint coherence this basis probes.  $F$  and  $W$ , in this concrete sense, are working with different operation sets. That is not a relabeling of one physical measurement; it is two different measurements.

This framework does not try to model decoherence dynamics; it offers something that sits alongside it, a way of quantifying relational structure through  $n^k$ . The two approaches coexist without conflict, since the domains differ ( $D(\mathcal{H}_S) \neq D(\mathcal{H}_S \otimes \mathcal{H}_F)$ ), the POVMs differ ( $\Pi_F^k \neq \Pi_W^k$ ), and the states differ. It is much like saying  $f(1) = 0$  and  $g(1, 1) > 0$  can both hold at once, simply because they are claims about different functions on different domains.

**Definition 7 (Relational Collapse).** A system  $M$  is said to have “collapsed” relative to observer  $O$  if and only if  $n^k(\rho) = 0$ , i.e., no element of  $\Pi^k$  remains informative with respect to  $\rho$ .

By this definition,  $S$  has collapsed relative to  $F$ , since  $n_F^k(\rho_F(1)) = 0$ , but  $S + F$  has not collapsed relative to  $W$ , since  $n_W^k(\rho_W(0)) > 0$ . There is no absolute collapse anywhere in this picture; collapse is always relative to a particular observer, a direct consequence of Proposition 2.

It is fair to ask what  $n^k$  actually adds beyond restating Rovelli’s relational resolution in new symbols. Rovelli’s own argument is qualitative:  $F$  and  $W$  may hold different, equally valid state ascriptions, with no sense of *how much* relational structure separates them. Table 1 supplies that missing quantity.  $n_F^k = 0$  and  $n_W^k > 0$  are specific, computable values fixed by the respective POVMs and domains, and by Proposition 4,  $n_W^k$  can even be assigned an exact integer whenever  $W$ ’s measurement resources happen to be of the compatible or classical type. What’s added is small but real: an actual number, where before there was only a qualitative assertion of relativity. We make no claim that this yields new experimental predictions beyond standard quantum mechanics; what it buys is explanatory precision, not new physics. It sits alongside Wallegghem, Wagner, Ying, and Schmid [22], who showed that extended Wigner’s Friend paradoxes need no nonlocal correlations, and Wallegghem, Barbosa, Pusey, and Weigert [23], who recast the Frauchiger-Renner paradox in terms of strong contextuality, reinforcing contextuality as a basic feature of multi-observer scenarios generally. Here, the “paradox” owes its bite to the tacit assumption that the quantum

state is absolute. Once the state is recognized as a relational property of  $(O, M)$ , the contradiction dissolves without invoking any additional nonlocal mechanism.

## 6 Consistency with No-Go Theorems

### 6.1 Overview

The  $n^k$  framework is checked here, systematically, against four canonical no-go theorems. These are mathematical facts about quantum mechanics, not obstacles to be argued around; the aim is simply to confirm that the framework does not run afoul of them without anyone noticing.

### 6.2 Bell's Theorem

Bell [24] showed that no local hidden variable (LHV) theory reproduces every prediction of quantum mechanics.  $n^k$  escapes that argument, and for three reasons at once: it is a function of  $(\rho, \Pi^k)$  rather than a property of  $M$  alone; it fixes only the maximum number of informative measurements, never a specific outcome; and it is context-dependent by construction, through  $\Pi^k$ . Outcomes remain as stochastic as ever. Bell's theorem, in short, simply has nothing to say about  $n^k$ .

### 6.3 The Kochen-Specker Theorem

Kochen and Specker [25] proved that for  $\dim(\mathcal{H}) \geq 3$ , quantum mechanics admits no context-independent value assignment.  $n^k$  was never a candidate for one: it depends on  $(\rho, \Pi^k)$  by definition, and Proposition 2 proves outright that  $\Pi_1^k \neq \Pi_2^k$  generally gives  $n_1^k(\rho) \neq n_2^k(\rho)$ . If anything,  $n^k$  goes beyond mere consistency here, offering a concrete instance of an inherently contextual quantity.

### 6.4 The Pusey-Barrett-Rudolph (PBR) Theorem

Pusey, Barrett, and Rudolph [26] proved that no LHV ontology can treat the quantum state as purely epistemic. Their argument leans on one premise: that  $\psi$  is a property of  $M$  alone. Here it is not:  $\psi$  represents the relation  $(O, M)$ , so the premise fails to apply. Consistency with PBR, on this reading, holds conditionally, for as long as the state is treated consistently as a relational property.

Gao [27] has since shown the PBR theorem can be reformulated without the preparation independence assumption, which complicates the question of whether  $\rho$  is  $\psi$ -ontic or  $\psi$ -epistemic within a relational setting. Here  $\rho$  is a relational property of  $(O, M)$  rather than an intrinsic property of the system, in line with Oldofredi and Calosi's [28] demonstration that RQM can be made consistent with PBR by reading the quantum state relationally.

### 6.5 The No-Cloning Theorem

Wootters and Zurek [29] showed that no unitary operation clones an arbitrary quantum state. Duplicating  $n^k$ , however, means building a second observer  $O'$  with  $\Pi^{k'} = \Pi^k$ ,

an operation on the observer’s own structure, not on  $\rho \in D(\mathcal{H}^T)$ . The two live in entirely different spaces, the apparatus space versus  $\mathcal{H}^T$ , so nothing here comes close to transcending Wootters and Zurek’s result. The framework is fully consistent with it.

## 6.6 Summary of Consistency

**Table 2** Summary of the consistency of the  $n^k$  framework with four canonical no-go theorems

Theorem	Status	Key Argument	Strength
Bell	Consistent	$n^k$ is not an LHV; relational property, does not determine outcomes	Strong
Kochen-Specker	Consistent	$n^k$ is contextual (Proposition 2); KS forbids context-independent values	Very strong
PBR	Conditionally consistent	$\psi$ is a property of the relation $(O, M)$ ; PBR’s premise is not satisfied	Strong (conditional)
No-Cloning	Consistent	Duplicating $\Pi^k$ differs from cloning $\rho$ ; operations on different spaces	Very strong

The “strength” column tracks how directly each theorem’s premise gets addressed. “Very strong” means the framework structurally instantiates the very concept the theorem concerns, contextuality for Kochen–Specker, distinct operator spaces for No-Cloning, so consistency falls out of the definitions with no further assumption needed. “Strong (conditional)” means consistency depends on the ontological commitment of Section 7.1, whose necessity, as distinct from its sufficiency, we have not established.

## 7 Ontological Commitment and Epistemic Limits

### 7.1 Mandatory Ontological Commitment

One commitment holds up the entire framework: the quantum state is a property of the relation  $(O, M)$ , never of  $M$  in isolation. Section 6.4 shows this is *enough* to block PBR’s premise that  $\psi$  belongs to  $M$  alone. We do not claim it is the only or the minimal commitment that would do so. Other relational or contextual moves might block PBR by different means, and showing minimality would require a separate necessity argument that this article does not attempt. The commitment places the framework alongside Rovelli’s RQM [5], though it adds something RQM does not supply on its own: where RQM states relationality as a philosophical principle,  $n^k$  gives it a quantitative mechanism, something one can actually compute.

### 7.2 Claims: Proven versus Open

This article establishes that  $n^k$  is rigorously defined from Axioms A1–A3, and that it has five properties worth naming: monotonicity, a dependence on the observer rather

than just the system, a clean characterization of superposition, an exact value in the compatible/classical case, and divergence in the generic informationally complete case. Wigner’s Friend resolves along the way, without any extra postulate.

What is not claimed is a derivation of the Born rule, an account of why one particular outcome occurs rather than another, or any new experimental prediction.

$n^k$  is not proposed here as a new observable, though it is not merely formal either. Given a fixed, known  $\Pi^k$  and a state (or an informationally complete tomographic reconstruction of it),  $n^k(\rho)$  can be computed directly, by enumerating the sequence of informative outcomes as in the constructive half of Proposition 4’s proof, rather than left sitting as an abstract supremum.

### 7.3 Open Questions

Two questions sit permanently outside what this framework can reach. Why this particular outcome, and not another? The framework has nothing to say here; the Born rule has to be brought in from outside, as the bridge between relational structure and statistical prediction. And what fixes the amplitudes  $\alpha_i$  in the first place? Whatever probabilistic information they carry is not something the structural-combinatorial content of  $n^k$  can derive. These limits are stated plainly here, rather than papered over, since glossing over them would not make the framework capable of anything it actually cannot do.

## 8 Position in the Literature and Comparison

### 8.1 New Contributions

The contributions here, roughly from most to least foundational, run as follows. Definition 1 and Axioms A1–A3 formalize the relational pair  $(O, M)$  itself as an explicit mathematical object – RQM states relationality as a philosophical principle, and here it becomes a formally provable axiomatic structure instead. Out of that structure comes  $n^k$ , a quantitative index in its own right. The definition of “informative” behind it rests purely on the POVM and owes nothing to rank. There is a formal proof, beyond that, that relationality falls out of POVM structure rather than needing to be added as a separate postulate, and a resolution of Wigner’s Friend built on a domain argument with an explicit mechanism behind it.

### 8.2 $n^k$ Compared with Other Candidate Measures

Von Neumann entropy,  $S(\rho) = -\text{Tr}(\rho \log \rho)$ , depends on  $\rho$  alone and is blind to  $\Pi^k$  entirely.  $\text{Rank}(\rho)$  is equally blind to  $\Pi^k$ : Proposition 4 shows it coincides with  $n^k(\rho)+1$  only in the compatible/classical regime, and Proposition 5 shows it says nothing at all about  $n^k$  once  $\Pi^k$  is informationally complete and generic, where  $n^k$  diverges no matter what  $\text{rank}(\rho)$  happens to be. Schmidt rank only applies to pure bipartite states and carries no dependence on measurement structure. Measurement complexity, for its part, measures a reconstruction goal rather than the relational structure remaining for a specific  $\Pi^k$ . What sets  $n^k$  apart from all four is that it is built directly from the pair  $(O, M)$  via Axioms A1–A3, and, unlike any of them, it is discontinuous in  $\Pi^k$ : an

**Table 3** Comparison of  $n^k$  with other candidate measures of quantum structure

Measure	Depends on $\Pi^k$ ?	Depends on $\rho$ ?	Relational?	Domain
$n^k$	Yes	Yes	Yes	All states, all POVMs
Entropy $S(\rho)$	No	Yes	No	All states
Rank( $\rho$ )	No	Yes	No	All states
Schmidt rank	No	Yes	Partial	Pure bipartite states
Meas. Complexity	Partial	Yes	Partial	Definition-dependent

arbitrarily small departure from mutual compatibility (Remark 1) can send  $n^k$  from a finite, rank-determined value straight to  $\infty$ .

### 8.3 Relation to Rovelli’s RQM

Calosi and Riedel [30] recently took stock of where RQM stands in *Relational Quantum Mechanics at the Crossroads*, and Terris [31], more recently still, introduced the idea of the informational observer. What this framework does, in effect, is give that idea a concrete mathematical body:  $O = (\mathcal{H}^k, \Pi^k, \mu^k)$  is a structure that RQM had gestured toward without fully building.

A related move comes from Adlam and Rovelli [32], who introduced cross-perspective links (CPL), a postulate letting different observers reach intersubjective agreement about past relative facts.  $n^k$  sits alongside this: where CPL links the perspectives of *different* observers,  $n^k$  quantifies the relational structure a *single* observer has at a given moment. Whether  $n^k$  can say anything about when cross-perspective links hold is a question we leave open.

Di Biagio and Rovelli [8] draw a closely related but distinct line. Facts, on their view, arise at every interaction without ever being absolute, remaining relative to the systems involved, and a fact counts as *stable* once its relativity becomes practically negligible, a stability they trace to decoherence. The same distinction underwrites their resolution of the Frauchiger–Renner no-go theorem, alongside the contextuality-based resolution given here in Section 5. Their stable/relative divide sits thematically near the finite/divergent divide of Propositions 4 and 5, but the two claims are not identical. Their stability is dynamical, emerging from an actual decoherence process unfolding over time, whereas  $n^k$  is a static, structural property of  $(O, M)$  at a fixed instant, fixed by the algebraic compatibility of  $\Pi^k$  (Proposition 4) rather than by any dynamics. Whether the compatible, classical regime of Proposition 4 turns out to be a necessary structural precondition for a stable fact, in their sense, to emerge is a question this framework leaves unsettled.

The  $n^k$  framework fits best as a complement to RQM [5], not a rival to it. RQM establishes that quantum facts are relative;  $n^k$  adds an explicit mathematical object for the relational pair, and a quantity that answers a question RQM leaves open: how much relational structure is actually available to observer  $O$  with respect to system  $M$ , at a given moment? The two frameworks work together.

Comparing relational approaches to one another has itself become something of a research program. Pienaar [33], for instance, sets QBism directly against RQM, with particular attention to non-human epistemic systems.

## 9 Conclusion

This article has given the relational structure of quantum observation an axiomatic formalization. Observer  $O$  enters as an explicit mathematical object (Definition 1), governed by three operational axioms (A1–A3) that fix what observational structure means for the pair  $(O, M)$ . Recursive Observation Depth ( $n^k$ ) follows from that foundation: the supremum of the length of the informative measurement chain  $O$  can run on state  $\rho$ , via the purely POVM-based Definition 4.

Five propositions came out of this construction, and they do not all say the same kind of thing. The first is monotonicity: every informative measurement costs at least one unit of  $n^k$ . The second establishes that  $n^k$  is relational rather than a property of the system on its own –  $\Pi_1^k \neq \Pi_2^k$  generally forces  $n_1^k(\rho) \neq n_2^k(\rho)$ . Superposition gets a characterization too, in Proposition 3: a state counts as a superposition, in this framework, exactly when  $n^k > 0$ . Proposition 4 pins down an exact value for a mutually compatible, classical observer,  $n^k(\rho) = \text{rank}(\text{diag } \rho) - 1$ ; Proposition 5 goes the other way, showing that a generic informationally complete observer admits no terminal state at all, so that  $n^k$  diverges instead. What governs  $n^k$ , then, splits along this line: rank in the compatible regime, informational completeness in the divergent one.

Wigner’s Friend falls out of this without any extra postulate.  $F$  and  $W$  simply have depth functions defined on different domains. Collapse, operationally, just means  $n^k = 0$ , and it is always relative to somebody in particular, never absolute.

Tested against the four no-go theorems, the framework holds up unconditionally against Bell, Kochen–Specker, and No-Cloning. Against PBR it holds only conditionally, on the assumption that the quantum state is a property of the relation  $(O, M)$  rather than of  $M$  alone. None of this derives the Born rule, and none of it predicts anything new experimentally. That is as far as a structural-relational account can be expected to go.

A few threads remain open. Relational quantum dynamics could eventually be folded into this static structure. The classical limit of  $n^k$  is worth working out, following Dutra, Baldijão, and Terra Cunha’s [17] recent result on the disappearance of contextuality in that limit. And there is quantum gravity: recent work on observers in a closed universe [34], together with Page and Wootters’s [35] evolution-without-evolution picture, suggests this framework may have something to say there as well.

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