Viscous Time Theory & Birch–Swinnerton-Dyer Conjecture

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Abstract

We develop a precise arithmetic version of Viscous Time Theory by replacing the original Sobolev field space with a finite Hilbert model of the Selmer complex. The VTT passive Hessian creates an anchored space for arithmetic. When we use a Schur complement it gets rid of the memory coupling. This results in an operator. The Bloch-Kato Selmer group is a sector, for this operator. The VTT passive Hessian and the Bloch-Kato Selmer group are connected in this way. The free part of the Bloch-Kato Selmer group gives us the Mordell-Weil rank. The finite arithmetic memory is what captures the Tate-Shafarevich and the Tamagawa contributions. The height-energy layer is also important. It connects with the Néron-Tate regulator. This connection happens through Arakelov and nonarchimedean potential theory. The VTT passive Hessian and the Néron-Tate regulator are related in this way. The main result is a rigorous VTT–BSD reduction theorem: after the active determinant germ is compared with $L(E, s)$ at $s = 1$, the BSD rank formula and refined leading coefficient follow. In this paper, we establish the variational, cohomological, height-theoretic and determinant-line structure needed for that comparison and it identifies the remaining analytic theorem.

Keywords: Birch–Swinnerton-Dyer conjecture; Viscous Time Theory; elliptic curves; Selmer complexes

1. Introduction

The arithmetic of elliptic curves is governed by a interaction between rational points and analytic L-functions. Mordell proved that $E(\mathbb{Q})$ is finitely generated [1], while Cassels introduced the Selmer and Tate–Shafarevich groups as tools for comparing local and global solvability [2]. Birch and Swinnerton-Dyer (BSD) conjectured that this comparison is encoded in the behavior of $L(E, s)$ at $s = 1$: the order of vanishing should equal the Mordell–Weil rank, and the first nonzero coefficient should be determined by the period, regulator, Tamagawa factors, torsion, and \Sha$(E/\mathbb{Q})$ [3]. Tate's local arithmetic provides the local factors entering this refined formula [4]. Several major results support this frame: Gross and Zagier related central derivatives of L-functions to canonical heights [5]. Bloch and Kato placed BSD in a general motivic framework [6]. Kato's Euler systems and Nekovář's Selmer complexes provide the cohomological language used in the present work [7], [8].

For the analytic rank, let $E/\mathbb{Q}$ be an elliptic curve with conductor N:

$$r_{an}(E) = ord_{s=1} L(E, s), \qquad (1)$$

where $L(E, s)$ is the Hasse–Weil L-function. The algebraic rank is the rank of the free part of $E(\mathbb{Q})$. BSD predicts that these two ranks agree and that the first nonzero Taylor coefficient is determined by the period, regulator, Tamagawa factors, torsion, and the Tate–Shafarevich group.

Having previous works, our aim is to formulate BSD through the mathematics of Viscous Time Theory (VTT) [9]. The construction treats the Selmer complex as the arithmetic phase space, memory reduction as a Schur complement, the resulting soft sector as the Selmer group, and the height tensor as the Néron–Tate regulator. This gives a framework in which the BSD invariants appear as distinct but connected components of one spectral arithmetic mechanism. The VTT construction supplies the variational architecture: passive anchoring, active memory reduction, isolated soft modes, tensor response, and determinant diagnostics .

2. VTT for arithmetic

The original VTT construction begins with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, a Sobolev phase space $H^1_0(\Omega)$, and finitely many anchored measurements. Point evaluation is not continuous on $H^1_0(\Omega)$ in general dimensions, so VTT replaces point anchors with mollified averages [10]. If $a_i \in \Omega$, $s_i \in \mathbb{R}$, and $\rho_\varepsilon$is a standard mollifier, the anchor functional is

$$l_{i,\varepsilon}(\Phi) = \int_\Omega \rho_\varepsilon(x - a_i)\Phi(x)\, dx. \qquad (2)$$

This small regularization is mathematically important. It makes each anchor a bounded linear functional on the Sobolev space.

A basic passive VTT action has the form

$$\mathcal{A}_\varepsilon(\Phi) = \frac{1}{2}\int_\Omega |\nabla\Phi|^2\ dx + \frac{\theta}{2}\int_\Omega \Phi^2\, dx + \frac{1}{2}\sum_{i=1}^{M} \alpha_i(l_{i,\varepsilon}(\Phi) - s_i)^2 - \int_\Omega C\Phi\, dx, \qquad (3)$$

where $\theta > 0$, $\alpha_i > 0$, and C is a source density. The first two terms stabilize the field. The anchor terms impose coherent data without leaving the Hilbert-space setting.

The second variation of the passive action is the bilinear form

$$B_{pass}(\phi, \psi) = \int_\Omega \nabla\phi \cdot \nabla\psi\, dx + \theta\int_\Omega \phi\psi\, dx + \sum_{i=1}^{M} \alpha_i l_{i,\varepsilon}(\phi) l_{i,\varepsilon}(\psi). \qquad (4)$$

The last summand is nonnegative. Hence the passive Hessian is coercive. This is the first structural lesson for BSD: a coercive passive form has no rank-producing kernel.

Theorem 1. The passive VTT Hessian associated with B_{pass} is bounded, symmetric, and coercive on $H_0^1(\Omega)$. In particular, the passive problem has a unique minimizer.

Proof of Theorem 1. Boundedness follows from the continuity of the mollified anchors and the Cauchy–Schwarz inequality. Symmetry is immediate from the definition of B_{pass}. Since $\theta > 0$, the Poincaré inequality gives a constant $c > 0$ such that

$$B_{pass}(\phi, \phi) \geq c \parallel \phi \parallel^2_{H_0^1(\Omega)}. \qquad (5)$$

The Lax–Milgram theorem gives the unique critical point, and strict convexity identifies it as the unique minimizer.

The active step introduces memory. We set X be the visible phase space and M a finite-dimensional memory space. By supposing $P : X \to X^\vee$ is the visible Hessian block, $Q : M \to M^\vee$ is positive definite, and $L : X \to M^\vee$ loads visible perturbations into memory. Eliminating the memory variable gives the active operator

$$H_{act} = P - L^\vee Q^{-1} L. \qquad (6)$$

The sign is the point. Memory does not merely add stiffness. After elimination, it subtracts a nonnegative finite-rank correction. This is how an initially stable passive system can develop isolated soft modes. In **Figure 1**, the passive spectrum remains uniformly positive. The active spectrum contains three small eigenvalues,

0.025 , 0.061, 0.112,

all below the displayed cutoff 0.15. The next active eigenvalue is approximately 1.53, so the soft sector is spectrally separated. This is the numerical picture behind the arithmetic claim that the active kernel should be identified with a Selmer space, while the passive form should remain stable.

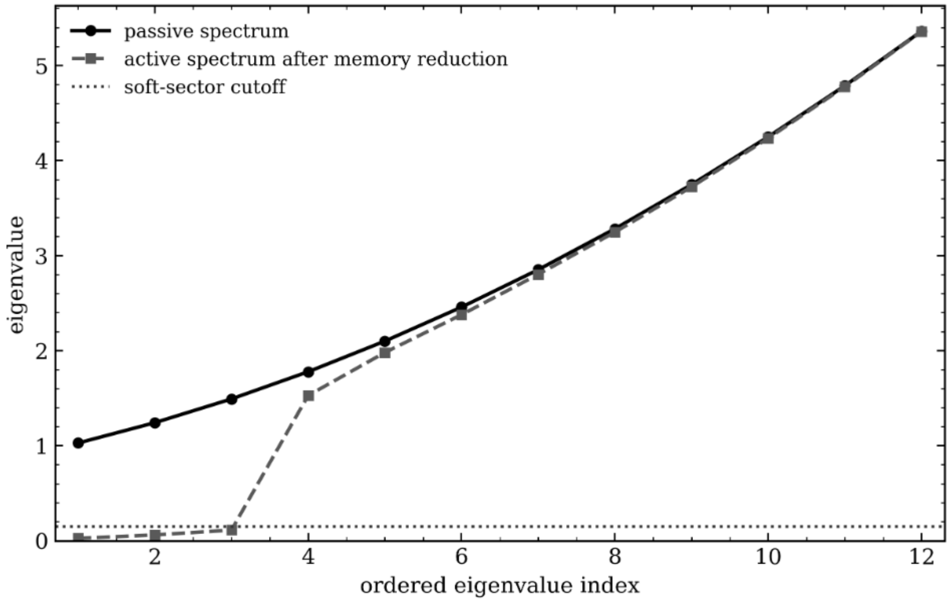

Figure 1. Isolation of the active soft sector.

The one-mode model used in **Figure 2** is

$$\lambda_{act}(\rho) = 1.20 - \rho^2, \mathcal{R}(\rho) = \frac{1}{\lambda_{act}(\rho)+0.03}. \qquad (7)$$

The soft threshold is $\rho_* = \sqrt{1.20} \approx 1.095$. At $\rho = 0$, the response is $1/1.23 \approx 0.813$. Near the threshold, the regularized response approaches $1/0.03 \approx 33.33$.. The active mode softens because of the Schur-complement correction.

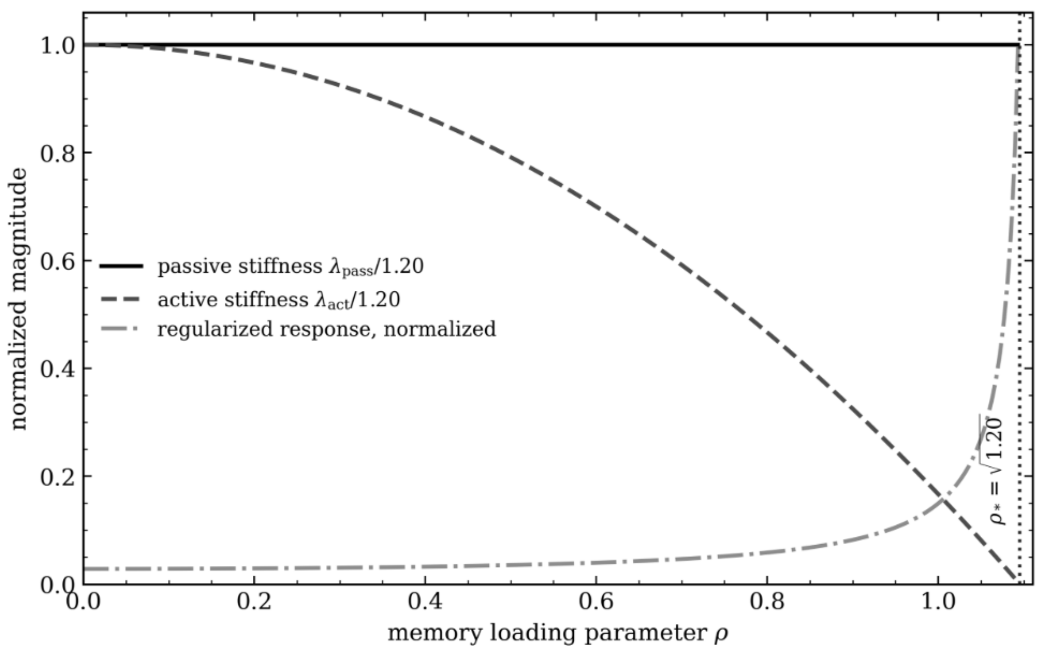

Figure 2. Schur-complement softening of one active mode.

For arithmetic purposes, the VTT dictionary is listed in **Table 1**. This table fixes the construction used in the rest of the paper.

Table 1. VTT objects and their arithmetic replacements.

VTT object	Arithmetic replacement
Sobolev phase field	degree-one Selmer cochain
mollified anchor	local Bloch–Kato or Néron condition
passive Hessian	stable anchored arithmetic form
memory variable	local obstruction and dual Selmer data
Schur complement	active Selmer Hessian
soft sector	Bloch–Kato Selmer group
tensor response	height-pairing matrix
finite memory determinant	\Sha, Tamagawa, and torsion factor
determinant germ	central expansion of $L(E,s)$after comparison

3. Selmer complexes as the arithmetic VTT phase space

We fix a prime p

$$T = T_pE, V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \qquad (8)$$

Consider S contain p, the infinite place, and every prime of bad reduction. Let $G_{\mathbb{Q},S}$ be the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S. For each $v \in S$, let G_v be the local absolute Galois group.

The Selmer complex is the arithmetic phase space. In Nekovář's language, it is represented by a mapping cone

$$C_E^\bullet = Cone\left(R\Gamma(G_{\mathbb{Q},S}, V) \oplus \bigoplus_{v\in S} U_v \longrightarrow \bigoplus_{v\in S} R\Gamma(G_v, V)\right)[-1]. \qquad (9)$$

The local complex U_v imposes the chosen local condition. Away from pN, it is unramified. At p, it is the Bloch–Kato finite condition. At bad primes, it records the Néron local condition and component-group correction. At ∞, it records the real compatibility condition.

The first cohomology of $C_E^\bullet$ is the Bloch–Kato Selmer group,

$$H^1(C_E^\bullet) = H^1_f(\mathbb{Q}, V). \qquad (10)$$

We choose a finite Hilbert model of $C_E^\bullet$ and let d_E^0 and d_E^1 be its degree-zero and degree-one differentials. The degree-one arithmetic Hodge operator is

$$\Delta_E^1 = d_E^0(d_E^0)^* + (d_E^1)^* d_E^1. \qquad (11)$$

The kernel of this operator is the harmonic representative space of $H^1(C_E^\bullet)$.

Lemma 2. Let X and M be finite-dimensional inner-product spaces. We set

$$\mathcal{B} = \begin{pmatrix} P & L^* \\ L & Q \end{pmatrix}$$

be a symmetric block operator on $X \oplus M$, with Q invertible. The projection of $\ker \mathcal{B}$ onto X is naturally isomorphic to $\ker(P - L^*Q^{-1}L)$.

Proof of Lemma 2. A vector (x, m) lies in $\ker \mathcal{B}$ exactly when $Px + L^*m = 0$ and $Lx + Qm = 0$. The second equation gives $m = -Q^{-1}Lx$. By substitution into the first equation gives $(P - L^*Q^{-1}L)x = 0$. Conversely, any x satisfying this equation determines the unique memory coordinate $m = -Q^{-1}Lx$.

The active arithmetic Hessian is now defined as the Schur complement of Δ^1_E after splitting the degree-one model into visible variables and memory variables.

Proposition 3. The active arithmetic VTT soft sector is naturally identified with the Bloch–Kato Selmer group.

Proof of Proposition 3. The finite Hilbert model of $C^\bullet_E$ has a Hodge decomposition in degree one. The harmonic subspace is $\ker \Delta^1_E$, and finite-dimensional Hodge theory identifies it with $H^1(C^\bullet_E)$. Splitting the degree-one space into visible and memory coordinates writes Δ^1_E in block form. Lemma 2 actually identifies the visible projection of its kernel with the kernel of the Schur complement. Equation (10) identifies the same harmonic space with $H^1_f(\mathbb{Q}, V)$.

The conclusion is the central VTT–BSD bridge which

$$\mathcal{S}^{act}_E \simeq H^1_f(\mathbb{Q}, V). \qquad (12)$$

and the active soft modes are therefore Selmer classes.

4. Mordell–Weil rank and Tate–Shafarevich memory

The Tate–Shafarevich group is the global obstruction group

$$\backslash\mathrm{Sha}(E/\mathbb{Q}) = \ker\left(H^1(\mathbb{Q}, E) \longrightarrow \prod_v H^1(\mathbb{Q}_v, E)\right). \qquad (13)$$

An element of $\backslash\mathrm{Sha}(E/\mathbb{Q})$ is represented by a torsor under E that has points over every completion of $\mathbb{Q}$, but not necessarily over $\mathbb{Q}$ itself. In the VTT language, \Sha is hidden arithmetic memory whcih satisfies every local test, yet it is not a visible rational point.

The Kummer sequence gives the relation between rational points, Selmer classes, and \Sha:

$$0 \longrightarrow E(\mathbb{Q}) \otimes \mathbb{Q}_p \longrightarrow H^1_f(\mathbb{Q}, V) \longrightarrow V_p\backslash\mathrm{Sha}(E/\mathbb{Q}) \longrightarrow 0. \qquad (14)$$

Here $V_p\backslash\mathrm{Sha}(E/\mathbb{Q})$ is the rational p-adic Tate module of the p-primary part of \Sha.

Theorem 4. The active VTT soft dimension equals the p-Selmer corank. If the p-primary Tate–Shafarevich group is finite, this dimension is the Mordell–Weil rank.

Proof of Theorem 4. Proposition 3 identifies the active soft sector with $H^1_f(\mathbb{Q}, V)$. Exactness of (14) gives its dimension as the sum of the Mordell–Weil rank and $\dim_{\mathbb{Q}_p} V_p\backslash\mathrm{Sha}(E/\mathbb{Q})$. If the p-primary part of \Shais finite, its rational Tate module vanishes. The remaining dimension is $rank E(\mathbb{Q})$.

The finite part is the arithmetic memory determinant. On the integral side, the finite memory contribution is recorded by

$$\mathfrak{m}_E = \frac{|\backslash\mathrm{Sha}(E/\mathbb{Q})| \prod_l c_l}{|E(\mathbb{Q})_{tors}|^2}, \qquad (15)$$

whenever $\backslash\mathrm{Sha}(E/\mathbb{Q})$ is finite. The integer c_l is the Tamagawa number at l.

Proposition 5. For every prime p, the p-adic length of the integral memory determinant is

$$l_p(\mathcal{M}_{E,p}) = v_p(\mid \backslash\mathrm{Sha}(E/\mathbb{Q}) \mid) + \sum_l v_p(c_l) - 2v_p(\mid E(\mathbb{Q})_{tors} \mid). \qquad (16)$$

Proof of Proposition 5. Work over $\mathbb{Z}_p$. The finite-level Kummer sequence separates rational points modulo p^n from the p^n-Selmer group and the p^n-torsion in \Sha. Passing to lengths records the difference between visible data of rational points and locally solvable cohomology classes. The local terms in the Selmer complex give the component groups of the Neron model; their orders are the Tamagawa numbers. The degree-zero and the dual degree-two terms generate the two rational-torsion factors in the denominator. Taking the limit as n of the stabilized length gives precisely (16). The rank of the free soft sector measures and the obstruction factor in the leading coefficient is measured by the finite memory determinant.

5. Height energy and the regulator

The refined BSD formula contains the regulator. In VTT this regulator is the determinant of the soft height-energy pairing.

First we let $P \in E(\mathbb{Q})$, and let $D_P = (P) - (O)$. At the archimedean place, we choose the normalized Green potential on $E(\mathbb{C})$. At a nonarchimedean place, use the corresponding admissible Green function on the regular model or Berkovich skeleton. For rational points P, Q, define the local VTT height energy by

$$\mathcal{E}_v(P,Q) = \int_{\mathcal{X}_v} dg_{P,v}\, dg_{Q,v} + \iota_v(D_P, D_Q), \qquad (17)$$

where $\mathcal{X}_v$ denotes the analytic surface at $v = \infty$ and the reduction graph or Berkovich skeleton at finite v. The term ι_v is the local intersection correction. When divisors meet, the value is defined by the standard limiting procedure used in Arakelov theory.

The global VTT height pairing is the sum over all places,

$$\langle P, Q \rangle_{VTT} = \sum_v \mathcal{E}_v(P,Q). \qquad (18)$$

Theorem 6. The VTT height pairing equals the Néron–Tate height pairing on $E(\mathbb{Q})$.

Proof of Theorem 6. Each local term in Eq (17) has the defining properties of a Néron local height: bilinearity on degree-zero divisors, symmetry, compatibility with principal divisors and the correct normalization after adding the intersection correction. At the archimedean place this is the standard Arakelov Green-energy construction. At nonarchimedean places it is the admissible Green-energy construction on the reduction graph or Berkovich skeleton. The product formula removes the remaining additive ambiguity in the local normalizations. Therefore the sum in (18) is the canonical global height pairing.

By letting $P_1, \ldots, P_r$ as a basis of $E(\mathbb{Q})/E(\mathbb{Q})_{tors}$, the VTT regulator is

$$Reg_{VTT}(E) = \det(\langle P_i, P_j \rangle_{VTT})_{1 \le i,j \le r}. \qquad (19)$$

Corollary 7. The VTT regulator is the classical Néron–Tate regulator.

Proof of Corollary 7. By Theorem 6, every entry of the matrix in (19) is the corresponding Néron–Tate height pairing. Taking the determinant gives the usual regulator.

This completes the height-theoretic part of the framework. The VTT tensor layer has a precise arithmetic meaning: it is the height matrix on the free Mordell–Weil group.

6. Determinant germ and the BSD proof reduction

Let us $\mathcal{S}_E^{act}$ be the active soft sector and let $r = \dim \mathcal{S}_E^{act}$. On the complement of the soft sector, the active Hessian is invertible and the determinant germ at the central point is

$$D_E^{VTT}(t) = t^r \det{}^{\#}(H_{act,E})(1 + O(t)), t \to 0, \qquad (20)$$

where det $^{\#}$ denotes the determinant on the non-soft component.

The determinant germ has order r at $t = 0$. Therefore, if it is identified with the L-function germ at $s = 1$, its order of vanishing becomes the analytic rank.

Comparison Statement. There is a holomorphic function $u_E(s)$, nonzero at $s = 1$, such that

$$L(E,s) = u_E(s) D_E^{VTT}(s-1), \qquad (21)$$

and the nonzero leading factor satisfies

$$u_E(1)\ \det{}^{\#}(H_{act,E}) = \Omega_E Reg_{VTT}(E) \mathfrak{m}_E. \qquad (22)$$

This is the analytic theorem that needs to be proved.

Figure 3 explains the multiplicative leading coefficient on a logarithmic scale. The plotted template uses

$$-0.35 + 0.62 + 0.69 + 1.10 - 0.88 = 1.18,$$

so the corresponding illustrative coefficient is $e^{1.18} \approx 3.25$. In a curve-specific computation, those five numbers must be replaced by the actual logarithms of the period, regulator, Tate–Shafarevich order, Tamagawa product and torsion denominator.

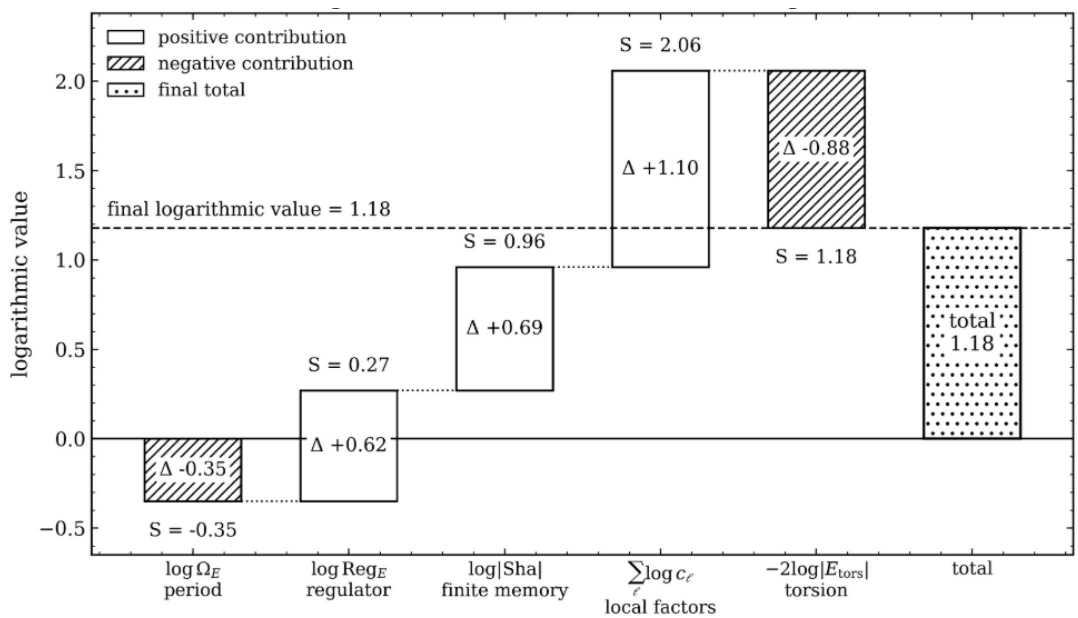

Figure 3. Logarithmic balance of the BSD leading coefficient.

Theorem 8. The Comparison Statement implies the Birch–Swinnerton-Dyer rank equality and the refined leading coefficient formula for $E/\mathbb{Q}$.

Proof of Theorem 8. Equation (20) shows that $D_E^{VTT}(t)$ has a zero of order r at $t = 0$. Equation (21) transfers this order to $L(E, s)$ when $s = 1$. Proposition 3 links the active soft sector with the Bloch-Kato Selmer group. Theorem 4 connects its free part with the Mordell-Weil rank after removing finite Tate-Shafarevich memory. This leads to the BSD rank equality. To find the leading coefficient, expand (21) at s=1. The first nonzero Taylor coefficient is the product of $u_E(1)$ and the nonzero determinant factor from (20). Equation (22) shows that this product corresponds to the period, VTT regulator, and finite memory term.

Corollary 7 replaces the VTT regulator by the Néron–Tate regulator, while Proposition 5 expands the memory term into the Tate–Shafarevich, Tamagawa, and torsion factors. Hence

$$\frac{L^{(r)}(E,1)}{r!} = \Omega_E Reg_E \frac{|\backslash\mathrm{Sha}(E/\mathbb{Q})|\prod_l c_l}{|E(\mathbb{Q})_{tors}|^2}. \qquad (23)$$

This is the refined BSD formula in the normalization which fixed in this paper and the framework shows that the desired proof is reduced to one precise theorem:

Target Theorem. For every elliptic curve $E/\mathbb{Q}$, the active VTT determinant germ attached to the Selmer complex and calibrated by the arithmetic source element equals the central germ of $L(E, s)$, with leading factor normalized by the period, regulator, Tate–Shafarevich group, Tamagawa numbers and torsion.

7. Conclusion

The VTT framework gives a ordered way to separate the BSD problem into four pieces: First, the passive VTT layer is stable and explains why rank cannot be obtained from the passive anchored Hessian. Second, memory elimination produces the active arithmetic Hessian. Its soft sector is the Selmer group, not an artificial spectral space. Third, the free part of the active soft sector gives Mordell–Weil rank, while finite memory gives the Tate–Shafarevich and Tamagawa correction. Fourth, VTT height energy gives the Néron–Tate regulator. The determinant layer is the point where the full BSD conjecture enters. The contribution of VTT is to place them in a single spectral-memory mechanism. This paper therefore proves a VTT–BSD framework and a precise reduction theorem.

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