Relativistic Geometrical Horizons(RGH) in both the original versions of Alcubierre and Natario warp drives

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Abstract

In 1994 Alcubierre developed the first warp drive work using the original $3+1$ *ADM-MTW* formalism.Seven years later in 2001 the same original $3+1$ *ADM-MTW* formalism appeared in the first part of the second warp drive work developed by Natario. But in 1997 the concept of the Relativistic Geometrical Horizon RGH was introduced by Hiscock in his work about the Alcubierre warp drive. According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00} = 0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears. The Alcubierre warp drive suffers from this pathology because in a given point of the Alcubierre warped region the $g_{00} = 0$ appears generating the Alcubierre Horizon equivalent to the Event Horizon of the Schwarzschild black hole.The Natario warp drive was formulated in polar coordinates in a $2+1$ spacetime.A dimensional reduction of the Natario warp drive from a $2+1$ to a $1+1$ spacetimes in a given point of the Natario warped region the $g_{00} = 0$ appears generating the Natario Horizon in a $1+1$ spacetime. But in the original Natario warp drive in polar coordinates in a $2+1$ spacetime the $g_{00} = 0$ never appears.Due to the presence of a second spatial dimension the $g_{00} = 0$ can be avoided.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates in [15]).(see Appendix A for the complete mathematical demonstration of the Natario calculations for the Hodge Star in [15]).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$ or better:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{1}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{2}$$

However Polar Coordinates are not real $3D$ coordinates since it uses only the two dimensional Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another alternative warp drive vector must be calculated.Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about $3D$ Spherical Coordinates in [15]).

This work introduces the concept of the Relativistic Geometrical Horizons(RGH) but in order to avoid a very long and complicated study about the RGH for the first time in this work we present only the RGH for the original Natario warp drive vector nX in Polar Coordinates with constant speeds.The RGH for the $3D$ Spherical Coordinates modified Natario warp drive vectors with constant or variable speeds will appear in future works but will appear indeed.Remember that a real spaceship must accelerate or de-accelerate.For advanced versions of the $3D$ Spherical Coordinates modified Natario warp drive vectors with constant or variable velocities see [10],[11],[12],[13] and [14] but these works do not encompass the RGH.

In order to get an idea about how many advanced versions of the $3D$ Spherical Coordinates modified Natario warp drive vectors exists see section 4 in [12].

The warp drive work that predates Natario by 7 years was written by Alcubierre in 1994.(see [2])

Alcubierre([4]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([3]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pg 67 in [4],see also eq 1 pg 3 in [2]) and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.In this work concerning the ADM formalism we adopt the Alcubierre methodology.

The ADM equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq (21.40) pg 507 in [3]).[1]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{3}$$

In the equation above α is the so-called lapse function,γ_{ij} is the $3D$ diagonalized induced metric and β^i and β^j are the so-called shift vectors.

For a complete mathematical explanation of the $3+1$ ADM formalism see Appendix A

In this work we present the Relativistic Geometrical Horizons(RGH) concept as introduced by Hiscock in 1997.(see [20])

Hiscock in 1997 applied the RGH concept to the Alcubierre warp drive(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18])) but every time in any Relativistic metric if the $g_{00} = 0$ appears we will have a RGH Relativistic Geometrical Horizon.

According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00} = 0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears.(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18]

Alcubierre used a remote frame in which a remote observer outside the bubble takes the observations but Hiscock used a ship frame in which an observer inside the bubble takes the observations.

We will remain here in the remote frame in order to avoid co-moving coordinate transformations since our presentation about RGH is being given in an introductory level.The final results are similar.

We will demonstrate the Hiscock arguments for the RGH in the Alcubierre warp drive in a way more accessible to beginners or intermediate students avoiding the co-moving coordinate transformations used by Hiscock.

Alcubierre warp drive suffers from this pathology because in a given point of the Alcubierre warped region the $g_{00} = 0$ appears generating the Alcubierre Horizon equivalent to the Event Horizon of the Schwarzschild black hole.The point of the Alcubierre warped region where the Alcubierre Horizon is located is the Alcubierre radius similar to the Schwarzschild radius.

[1] we adopt the Alcubierre notation here

The Natario warp drive in a $1+1$ spacetime like its Alcubierre counterpart becomes singular in a given point of the Natario warped region where the $g_{00} = 0$ also appears developing also the Natario Horizon as another case of a Relativistic Geometrical Horizon $g_{00} = 0$.

However in the case of the Natario warp drive in a $2+1$ polar coordinates spacetime the $g_{00} = 0$ never appears.Due to the presence of a second spatial dimension the $g_{00} = 0$ can be avoided.The Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$,the shift vectors X^r,X^θ and the Natario vector $nX = X^r e_r + X^\theta e_\theta$ all in Polar Coordinates in the $2+1$ spacetime compensates each other to neutralize the $g_{00} = 0$.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical references required for the background needed to understand the process Hiscock used to generate the Relativistic Geometrical Horizons RGH.

We hope our paper is suitable to fill this proposed task.

This work was designed as a companion work to our works in [10],[11],[12],[13] and [14].

2 The Relativistic Geometrical Horizons(RGH) concept as introduced by Hiscock applied to the original Alcubierre warp drive

The warp drive metric proposed by Alcubierre using signature $(-,+,+,+)$ may be written as:(see eq 8 pg 4 in [2],eq 1 pg 3 in [20])

$$ds^2 = -dt^2 + (dx - vf(r)dt)^2 + dy^2 + dz^2 \tag{4}$$

Where $v = vs$ is the apparent velocity of the spaceship,(see pg 4 in [2],eq 2 pg 3 in [20])

$$v = vs = \frac{dx_s(t)}{dt} \tag{5}$$

$x_s(t)$ is the trajectory of the spaceship (chosen to be along the x direction), $r = rs$ is defined by:(see pg 4 in [2],eq 3 pg 3 in [20])

$$r = rs = \left[(x - x_s(t))^2 + y^2 + z^2\right]^{1/2} \tag{6}$$

And $f = f_{rs} = f(rs)$ is an arbitrary function which decreases from unity at $r = 0$ (the location of the spaceship) to zero at infinity. Alcubierre gave a particular example of such a function:(see eq 6 pg 4 in [2],eq 4 pg 3 in [20])

$$f = f_{rs}(r) = f(rs) = \frac{\tanh(\sigma(r+R)) - \tanh(\sigma(r-R))}{2\tanh(\sigma R)}, \tag{7}$$

$$f = f_{rs}(r) = f(rs) = \frac{\tanh(@(r+R)) - \tanh(@(r-R))}{2\tanh(@R)}, \tag{8}$$

The function above is the Alcubierre shape function where σ or @ and R are positive arbitrary constants.We outline the fact that σ or @ and R are positive arbitrary constants.R is the radius of the warp bubble while σ or @ are related to the thickness of the bubble.

If $\sigma >> R$ or $@ >> R$ then we can use the following useful approximation:(see eq 14 pg 8 in [16],eq 16 pg 9 in [17],eq 8 pg 6 in [18],eq 12 pg 7 in [19])

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]] \tag{9}$$

Inside the bubble $f(rs) = 1$ and outside the bubble $f(rs) = 0$ as pointed out by Alcubierre(eq 7 pg 4 in [2]) and Hiscock(pg 3 in [20] before eq 4).

The region where $f(rs)$ starts to decrease from 1 to 0 is the Alcubierre warped region $1 > f(rs) > 0$.Numerical plots of the Alcubierre warped region for a warp bubble with 100 meters of radius $R = 100$ are presented in pg 9 in [16], pg 21 in [17],pg 13 in [18],pg 11 in [19].

The Alcubierre warped region $1 > f(rs) > 0$ is the region where the derivatives of the shape function $f(rs)$ are not null defining the walls of the warp bubble.

Working with signature $(+,-,-,-)$ we get:(see eq 6 pg 6 in [18])

$$ds^2 = dt^2 - (dx - vf(r)dt)^2 - dy^2 - dz^2 \tag{10}$$

$$ds^2 = dt^2 - (dx - vf(rs)dt)^2 - dy^2 - dz^2 \tag{11}$$

Expanding the squared terms we have:(see eq 17 pg 7 in [18])(see also eq 144 in Appendix A)

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{12}$$

Reducing to a $1+1$ dimensions we have:(compare with eq 7 pg 4 in [20])

$$ds^2 = dt^2 - (dx - vf(rs)dt)^2 \tag{13}$$

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 \tag{14}$$

Alcubierre used a remote frame in which a remote observer outside the bubble takes the observations but Hiscock used a ship frame in which an observer inside the bubble takes the observations.

The Alcubierre equations in a ship frame are given by:(see eqs 22 to 34 in [18])

We will remain here in the remote frame in order to avoid co-moving coordinate transformations since our presentation about RGH is being given in an introductory level.The final results are similar.

The spacetime metric components for the Alcubierre metric in a $1+1$ dimensions are given by:(see eq 18 pg 7,eq 21 pg 8 in [18])

$$g_{00} = (1 - [vsf(rs)]^2) \tag{15}$$

$$g_{01} = vsf(rs) \tag{16}$$

$$g_{11} = 1 \tag{17}$$

According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00} = 0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears.(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18]).[2]

Unfortunately the Alcubierre warp drive suffers from this pathology because in a given point of the Alcubierre warped region $1 > f(rs) > 0$ the $g_{00} = 0$ appears generating the Alcubierre Horizon equivalent to the Event Horizon of the Schwarzschild black hole.The point of the Alcubierre warped region where the Alcubierre Horizon is located is the Alcubierre radius similar to the Schwarzschild radius.

[2] we will discuss briefly in this work the the Event Horizon of the Schwarzschild black hole in order to compare with the Alcubierre Horizon

We will demonstrate the Hiscock arguments for the RGH in the Alcubierre warp drive in a way more accessible to beginners or intermediate students avoiding the co-moving coordinate transformations used by Hiscock.

Examining again the g_{00} in the Alcubierre metric:

$$g_{00} = (1 - [vsf(rs)]^2) \tag{18}$$

Inside the warp bubble $f(rs) = 1$ and $g_{00} = 1-vs^2$ while outside the warp bubble $f(rs) = 0$ and $g_{00} = 1$.

Assuming a continuous decreasingly behavior of $f(rs)$ from 1 to zero implying in a decreasingly $vsf(rs)$ from vs inside the bubble to zero outside the bubble then in a given point of the Alcubierre warped region $1 > f(rs) > 0$ and in consequence $vsf(rs) = 1$ because $vs > vsf(rs) > 0$,$vs > 1 > 0$ implying in a $g_{00} = 0$.The Alcubierre metric becomes singular.(see eqs 19 and 20 pg 7 in [18]).This singularity is the Alcubierre Horizon similar to the Event Horizon of a Schwarzschild black hole.

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = \frac{1}{vs} \tag{19}$$

A better explanation:

- 1)-Inside the warp bubble:
  $g_{00} = (1 - [vsf(rs)]^2)$,$f(rs) = 1$,$vsf(rs) = vs$,$g_{00} = (1 - vs^2)$
- 2)-Outside the warp bubble:
  $g_{00} = (1 - [vsf(rs)]^2)$,$f(rs) = 0$,$vsf(rs) = 0$,,$g_{00} = 1$
- 2)-In the warp bubble walls(Alcubierre warped region):
  $g_{00} = (1 - [vsf(rs)]^2)$,$[1 > f(rs) > 0]$,$[vs > vsf(rs) > 0]$,$[vs > 1 > 0]$
  $[g_{00} = (1 - vs^2)] > [g_{00} = (1 - [vsf(rs)]^2)] > [g_{00} = 1]$

We are working with units $G = c = 1$ then a velocity two times light speed means a $vs = 2$,four times light speed means a $vs = 4$ and 1000 times light speed means a $vs = 1000$.

Inside the bubble $f(rs) = 1$,outside the bubble $f(rs) = 0$ and in the warp bubble walls(Alcubierre warped region) $[1 > f(rs) > 0]$.

When $vs = 2$,$f(rs) = \frac{1}{vs} = \frac{1}{2} = 0,5$ exactly in the middle of the Alcubierre warped region,when $vs = 4$,$f(rs) = \frac{1}{vs} = \frac{1}{4} = 0,25$ more close to the end of the Alcubierre warped region and when $vs = 1000$,$f(rs) = \frac{1}{vs} = \frac{1}{1000} = 0,0001$ even more close to the end of the Alcubierre warped region.

When $vs = 2$,$f(rs) = \frac{1}{vs} = \frac{1}{2} = 0,5$,$vsf(rs) = 2\frac{1}{2} = 1$ Alcubierre Horizon exactly in the middle of the Alcubierre warped region,when $vs = 4$,$f(rs) = \frac{1}{vs} = \frac{1}{4} = 0,25$,$vsf(rs) = 4\frac{1}{4} = 1$ Alcubierre Horizon more close to the end of the Alcubierre warped region and when $vs = 1000$,$f(rs) = \frac{1}{vs} = \frac{1}{1000} = 0,0001$,$vsf(rs) = 1000\frac{1}{1000} = 1$ Alcubierre Horizon even more close to the end of the Alcubierre warped region.

Our approximated Alcubierre shape function is:

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]] \quad (20)$$

We recall this function numerical plots of the Alcubierre warped region for a warp bubble with 100 meters of radius $R = 100$ presented in pg 9 in [16], pg 21 in [17],pg 13 in [18],pg 11 in [19].

In the Alcubierre Horizon

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)]] = \frac{1}{vs} \quad (21)$$

The point rs in the Alcubierre warped region where the Alcubierre Horizon occurs is the Alcubierre radius similar to the Schwarzschild radius and is given by:(see Appendix G in [17])

$$rs = R + \frac{1}{@}arctanh(1 - \frac{2}{vs}) \quad (22)$$

Although we described the RGH Alcubierre Horizon $vsf(rs) = 1$ from the Hiscock point of view in which $g_{00} = 0$,the Alcubierre Horizon also appears in the study of Horizons of null-like geodesics when $ds^2 = 0$.(see pgs 9,10,11 and Appendix D in [17]),(see pg 16 in [18]),(see pg 17 in [19]).(see pg 34 in [5],pg 268 in [6])

However while in the study of Horizons of null-like geodesics when $ds^2 = 0$,the Horizon occurs if and only if an observer in the center of the bubble send a photon to the front of the bubble and if such this observer remains "quiet" we do not need to worry ourselves about Horizons,unfortunately the RGH Alcubierre Horizon $vsf(rs) = 1$ from the Hiscock point of view $g_{00} = 0$ always exists inside the metric wether we like it or not.

Independently from "quiet" or "photon sender" observers inside the bubble "throwing powerful laser beams to the front of the bubble (or not)" in the system of units $G = c = 1$ and assuming a continuous growth of the speed vs from 0 to 1000 times light speed $vs = 1000$ then in a given moment of time $vs = 1$ because $0 <= vs <= 1000$ and $0 < 1 < 1000$ and when $vs = 1$(light speed) even inside the bubble where $f(rs) = 1$ we have:

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = 1 \dashrightarrow vs = 1 \dashrightarrow g_{00} = (1 - [1 \times 1]^2) = 0 \quad (23)$$

The Alcubierre Horizon similar to the Event Horizon of a Schwarzschild black hole according to Hiscock appears every time a spaceship reaches the speed of light $vs = 1$ and the Alcubierre metric becomes singular.$g_{00} = 0$

We are now ready to compare the Alcubierre warp drive with the Schwarzschild black hole.

The Schwarzschild metric is described by the following equation:(see eqs 5.1 and 5.2 pg 193 in [7])(see also eq 6.1.44 pg 124 in [8])(see also eq 11.13 pg 229 in [9])

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{24}$$

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \tag{25}$$

The equations in [7] are being given with signature $(-,+,+,+)$ in the system of units $c = 1$.

The term that interest to ourselves is the $g_{00} = g_{tt}$ given by:(see eq 5.22 pg 196 in [7])

$$g_{00} = g_{tt} = -(1 - \frac{2GM}{r}) \tag{26}$$

Passing to the signature $(+,-,-,-)$ we get:

$$ds^2 = (1 - \frac{2GM}{r})dt^2 - (1 - \frac{2GM}{r})^{-1}dr^2 - r^2 d\Omega^2 \tag{27}$$

$$g_{00} = g_{tt} = (1 - \frac{2GM}{r}) \tag{28}$$

Rewriting the term $g_{00} = g_{tt}$ in normal SI units MKS we have:

$$g_{00} = g_{tt} = (1 - \frac{2GM}{c^2 r}) \tag{29}$$

Hiscock applied the point of view of $g_{00} = 0$ to the Alcubierre warp drive.The same point of view $g_{00} = 0$ applied to the Schwarzschild metric gives:(see eq 6.1.45 pg 124 in [8])

$$g_{00} = (1 - \frac{2GM}{c^2 r}) = 0 \dashrightarrow 1 = \frac{2GM}{c^2 r} \dashrightarrow r = \frac{2GM}{c^2} \tag{30}$$

The Schwarzschild metric becomes singular $g_{00} = 0$ when $r = \frac{2GM}{c^2}$.The term r is the Schwarzschild radius.For our Sun a star with about 1 million kilometers in diameter the corresponding Schwarzschild radius is approximately 3 kilometers which means to say that if we could shrink the volume of the Sun to a small size of about 3 kilometers the Sun would become a black hole.The Sun will never be a black hole because it do not possesses enough mass but stars with more than 3 solar masses cannot withstand their structures against the gravitational pull when the nuclear furnace ceases the production of energy and then these stars starts to shrink in catastrophic gravitational collapse to very small volumes compatible with their Schwarzschild radius becoming black holes in the process generating Event Horizons in the point of the Schwarzschild radius.(see pgs 124,125 in [8])(see pg 231 in [9]).For details about black holes see [7],[8] and [9].

The Schwarzschild metric:

$$ds^2 = (1 - \frac{2GM}{r})dt^2 - (1 - \frac{2GM}{r})^{-1}dr^2 - r^2 d\Omega^2 \tag{31}$$

The Schwarzschild Event Horizon:$g_{00} = 0$

$$g_{00} = (1 - \frac{2GM}{c^2 r}) = 0 \dashrightarrow 1 = \frac{2GM}{c^2 r} \dashrightarrow r = \frac{2GM}{c^2} \tag{32}$$

The Alcubierre metric:

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{33}$$

The Alcubierre Horizon:$g_{00} = 0$

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = \frac{1}{vs} \tag{34}$$

Both Horizons are RGH Relativistic Geometrical Horizons because in both cases the $g_{00} = 0$ appears.

Hiscock in 1997 applied the RGH concept to the Alcubierre warp drive(see eq 10 pg 4 in [20],pg 5 before eq 15 in [20])(see eqs 18,19 and 20 pg 7 in [18])(see eqs 33 to 37 pg 9 in [18])) but every time in any Relativistic metric if the $g_{00} = 0$ appears we will have a RGH Relativistic Geometrical Horizon.

Although we described the RGH Alcubierre Horizon $vsf(rs) = 1$ from the Hiscock point of view in which $g_{00} = 0$,the Alcubierre Horizon also appears in the study of Horizons of null-like geodesics when $ds^2 = 0$.(see pgs 9,10,11 and Appendix D in [17]),(see pg 16 in [18]),(see pg 17 in [19]).(see pg 34 in [5],pg 268 in [6])

In the case of null-like geodesics $ds^2 = 0$ a photon stops and cannot reach the outermost layers of the warp bubble which remains "causally disconnected" from the "photon sender" observer in the center of the bubble.

The solution that allows contact with the bubble walls was presented in pg 83 in [22].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [22] to be sent to the large warp bubble keeping it in causal contact.This idea seems to be endorsed by pg 34 in [5],pg 268 in [6] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

But unfortunately the Alcubierre Horizon $g_{00} = 0$ cannot be circumvented.

3 The Relativistic Geometrical Horizons(RGH) concept applied to the original Natario warp drive-Simplest case

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates in [15]).(see Appendix A for the complete mathematical demonstration of the Natario calculations for the Hodge Star in [15]).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$ or better:

$$nX = -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{35}$$

or

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{36}$$

We prefer the second equation above.

The equation of the Natario warp drive vector nX in polar coordinates with a constant speed is given by(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{37}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [15])

$$X^r = 2v_s f(r)\cos\theta \tag{38}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{39}$$

Considering a valid $f(r) = f(rs)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large $r = rs$(outside the warp bubble) and $f(r) = 0$ for small $r = rs$(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1]).The Natario warp drive was computed in ship frame coordinates.See Appendices G and H in [15].

The generic warp drive equation is given by:(see Appendix A)

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{40}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{41}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{42}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ for the original Natario warp drive are given by:

$$X_i = \gamma_{ii} X^i \tag{43}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r) \cos\theta = X^r \tag{44}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s (2f(r) + (r)f'(r)) \sin\theta \tag{45}$$

Then the equation of the Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{46}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{47}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{48}$$

In order to simplify our analysis we consider first motion in the $x - axis$ only (like Natario did) or the equatorial plane $x - y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1])

The Natario warp drive equation in the $1+1$ spacetime is then given by:

$$ds^2 = (1 - X_r X^r)dt^2 + 2X_r dr dt - dr^2 \tag{49}$$

In order to avoid confusion between the shape functions of Natario and Alcubierre, $f(r) = f(rs)$ is the Alcubierre shape function while the Natario shape functions are represented by the letters $n(r) = n(rs)$ and $N(r) = N(rs)$,$v_s = vs$,$r = rs$.Then we have:

$$ds^2 = (1 - X_r X^r)dt^2 + 2X_r dr dt - dr^2 \tag{50}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s N(r) = X^r \tag{51}$$

$$ds^2 = (1 - [2vsN(rs)][2vsN(rs)])dt^2 + 2[2vsN(rs)]dr dt - dr^2 \tag{52}$$

$$ds^2 = (1 - [2vsN(rs)]^2)dt^2 + 4[vsN(rs)]drdt - dr^2 \tag{53}$$

$$ds^2 = (1 - 4[vsN(rs)]^2)dt^2 + 4[vsN(rs)]drdt - dr^2 \tag{54}$$

Compare with the Alcubierre metric:

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 \tag{55}$$

$$ds^2 = (1 - X_r^2)dt^2 + 2X_r drdt - dr^2 \tag{56}$$

The spacetime metric tensor component g_{00} is given by:

$$g_{00} = (1 - X_r^2) \dashrightarrow X_r = 2v_s N(r) \tag{57}$$

A Relativistic Geometrical Horizon RGH occurs when $g_{00} = 0$ and this means the following result:

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2v_s N(r) = 1 \tag{58}$$

We are ready to introduce now the Natario shape functions $n(r) = n(rs)$ and $N(r) = N(rs)$.The first Natario shape function $n(rs)$ is given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)] \tag{59}$$

According with Natario(pg 5 in [1]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive. Another Natario shape function can be defined by:

$$N(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{60}$$

This shape function gives the result of $N(rs) = 0$ inside the warp bubble and $N(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < N(rs) < \frac{1}{2}$ in the Natario warped region. Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $N(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $N(rs) = \frac{1}{2}$(exterior of the Natario bubble).

The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.We prefer the second Natario shape function because it behaves much better in the negative energy density requirements to sustain a warp bubble.See section 3 in [16].See section 2 in [24]

Considering a valid $N(rs)$ as a Natario shape function being $N(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $N(rs) = 0$ for small rs(inside the warp bubble) while being $0 < N(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]) and assuming a continuous behavior for $N(rs)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2v_sN(rs)$ from 0 to vs we can clearly see that inside the bubble $2v_sN(rs) = 0$ because $N(rs) = 0$ and outside the bubble $2v_sN(rs) = vs$ because $N(rs) = \frac{1}{2}$ and assuming also continuous values from 0 to vs with $vs > 1$[3] then somewhere in the Natario warped region where $0 < N(rs) < \frac{1}{2}$ we have the situation in which $2v_sN(rs) = 1$ because 1 lies in the continuous interval from 0 to vs and in consequence:

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2v_sN(r) = 1 \dashrightarrow g_{00} = 0!!! \tag{61}$$

The Natario warp drive like its Alcubierre counterpart becomes singular $g_{00} = 0$ developing also the Natario Horizon as another case of a Relativistic Geometrical Horizon $g_{00} = 0$.

This situation resembles the descriptions in pg 9 in [10] and pg 11 in [23].In the case of null-like geodesics $ds^2 = 0$ a photon stops and cannot reach the outermost layers of the warp bubble which remains "causally disconnected" from the "photon sender" observer in the center of the bubble.If the "photon sender" remains "quiet" and do not send photons to the front of the bubble then the null-like geodesics $ds^2 = 0$ do not occurs but the Natario Horizon $g_{00} = 0$ unfortunately cannot be avoided.

What happens when a Natario warp drive reaches the speed of light(vs=1)?Look to the region outside the bubble where $N(rs) = \frac{1}{2}$

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2N(r) = 1 \dashrightarrow X_r = 2\frac{1}{2} = 1 \dashrightarrow g_{00} = 0!!! \tag{62}$$

The Natario warp drive metric in the $1 + 1$ spacetime becomes singular $g_{00} = 0$

[3]Remember that we are working with Geometrized Units in which $G = c = 1$

4 The Relativistic Geometrical Horizons(RGH) concept applied to the original Natario warp drive-Complete case

The Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:(see Appendix A)

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{63}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{64}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{65}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [15])

$$X^r = 2v_s f(r) \cos\theta \tag{66}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta \tag{67}$$

Remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ for the original Natario warp drive are given by:

$$X_i = \gamma_{ii} X^i \tag{68}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = 2v_s f(r) \cos\theta = X^r \tag{69}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2 v_s(2f(r) + (r)f'(r)) \sin\theta \tag{70}$$

The g_{00} for the complete Natario warp drive in polar coordinates is given by:

$$g_{00} = (1 - X_r X^r - X_\theta X^\theta) \tag{71}$$

$$g_{00} = (1 - [X_r X^r + X_\theta X^\theta]) \tag{72}$$

$$X_r X^r = [2v_s f(r) \cos\theta]^2 \tag{73}$$

$$X_\theta X^\theta = (r^2)[v_s(2f(r) + (r)f'(r)) \sin\theta]^2 \tag{74}$$

Now we have motion in two dimensions r and θ and this affects the Relativistic Geometrical Horizon $g_{00} = 0$

$$g_{00} = (1 - [[2v_s f(r) \cos\theta]^2 + (r^2)[v_s(2f(r) + (r)f'(r)) \sin\theta]^2]) \tag{75}$$

$$g_{00} = (1 - [[2v_s f(r)\cos\theta]^2 + (r^2)[v_s(2f(r) + (r)f'(r))\sin\theta]^2]) \tag{76}$$

The Relativistic Geometrical Horizon $g_{00} = 0$ never occurs in the complete case of the Natario warp drive in polar coordinates in the $2+1$ spacetime due to the terms r^2 and $(r)f'(r)$.Also the terms $\cos\theta$ and $\sin\theta$ complements each other except when $\theta = 0$,$\sin\theta = 0$ and $\cos\theta = 1$.When $\theta = 90$,$\sin\theta = 1$,$\cos\theta = 0$ but the term $(r)f'(r)$ avoids the Natario Horizon.

Due to the presence of 2 dimensions the RGH can be avoided.This resembles a situation described in section 6 and specially eqs 76 and 77 pg 16 all in [10] for null-like geodesics $ds^2 = 0$ Horizons.

In the previous presented cases of the Schwarzschild metric,the Alcubierre warp drive metric and the Natario warp drive metric in the $1+1$ spacetime the Relativistic Geometrical Horizon $g_{00} = 0$ appears and all these metrics becomes singular.

$$g_{00} = (1 - \frac{2GM}{c^2 r}) = 0 \dashrightarrow 1 = \frac{2GM}{c^2 r} \dashrightarrow r = \frac{2GM}{c^2} \tag{77}$$

In the case of the Alcubierre warp drive the RGH Alcubierre Horizon $g_{00} = 0$ appears even inside the bubble when the velocity reaches the speed of light $vs = 1$.

$$g_{00} = (1 - [vsf(rs)]^2) = 0 \dashrightarrow vsf(rs) = 1 \dashrightarrow f(rs) = 1 \dashrightarrow vs = 1 \dashrightarrow g_{00} = (1 - [1 \times 1]^2) = 0 \tag{78}$$

But in the case of the Natario warp drive in the $1+1$ spacetime the RGH Natario Horizon $g_{00} = 0$ appears outside the bubble when the velocity reaches the speed of light $vs = 1$.

$$g_{00} = (1 - X_r^2) = 0 \dashrightarrow X_r^2 = 1 \dashrightarrow X_r = 2vsN(r) = 1 \dashrightarrow vs = 1 \dashrightarrow N(r) = \frac{1}{2} \dashrightarrow X_r = 2\frac{1}{2} = 1 \dashrightarrow g_{00} = 0!!! \tag{79}$$

Metrics of the form given below appears many times in General Relativity.Applyimg the Einstein Summing Convention we have:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} g_{\mu\nu} dx^\mu dx^\nu \tag{80}$$

$$ds^2 = g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + ... + g_{33} dx^3 dx^3 \tag{81}$$

General Relativity metrics are physically consistent only when $g_{00} \neq 0$.A given spacetime metric cannot have RGH Relativistic Geometrical Horizons $g_{00} = 0$.

The Natario warp drive in $2+1$ polar coordinates is physically consistent because it can avoid the RGH Relativistic Geometrical Horizons $g_{00} = 0$.

5 Conclusion

In 1994 Alcubierre developed the first warp drive work using the original $3+1$ ADM-MTW formalism.[2].

Seven years later in 2001 the same original $3+1$ ADM-MTW formalism appeared in the first part of the second warp drive work developed by Natario.[1].

But in 1997 the concept of the Relativistic Geometrical Horizon RGH was introduced by Hiscock in his work about the Alcubierre warp drive.[20].

According to Hiscock if the time component of a given spacetime metric tensor becomes null $g_{00} = 0$ a Horizon similar to the Event Horizon of the Schwarzschild black hole appears.

Unfortunately the Alcubierre warp drive suffers from this pathology because in a given point of the Alcubierre warped region the $g_{00} = 0$ appears generating the Alcubierre Horizon equivalent to the Event Horizon of the Schwarzschild black hole.The point of the Alcubierre warped region where the Alcubierre Horizon is located is the Alcubierre radius similar to the Schwarzschild radius.

The Natario warp drive was formulated in polar coordinates in a $2+1$ spacetime.A dimensional reduction of the Natario warp drive from a $2+1$ to a $1+1$ spacetimes in a given point of the Natario warped region the $g_{00} = 0$ appears generating the Natario Horizon in a $1+1$ spacetime.

But in the original Natario warp drive in polar coordinates in a $2+1$ spacetime the $g_{00} = 0$ never appears.Due to the presence of a second spatial dimension the $g_{00} = 0$ can be avoided.The Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$,the shift vectors X^r,X^θ and the Natario vector $nX = X^r e_r + X^\theta e_\theta$ all in Polar Coordinates in the $2+1$ spacetime compensates each other to neutralize the $g_{00} = 0$.

In this work we presented the original Alcubierre warp drive and the original Natario warp drive with the Natario case in Polar Coordinates in the $2+1$ spacetime.Both warp drives were presented with constant speeds.

However Polar Coordinates are not real $3D$ coordinates since it uses only the two dimensional Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Remember that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must uses all the $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.Also a real warp drive must accelerate and de-accelerate in order to be accepted as a physical valid model.

Future works using more advanced versions of the Natario vectors in $3D$ spherical coordinates with constant or variable speeds as depicted in [10],[11],[12],[13] and [14] specially in section 4 in [12] will appear.

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel due to its negative energy density distribution.(see Appendices K,M and N all in [17]).Remember that negative energy density have repulsive gravitational behavior deflecting hazardous objects(asteroids comets photons etc) that may be in front of the ship.(see pg 116 in [21]).

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pg 98 in [8] and pg 377 in [9]).

For the experimental verification of the acceleration of the Universe see for example the bottom of pg 355 and top of pg 356 eq 8.155 in [7].

6 Appendix A:mathematical demonstration of the original Alcubierre and Natario warp drive equations for a constant speed vs in the original $3+1$ ADM-MTW Formalism

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg 64 in [4])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg 65) in [4].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg 65 in [4])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg 66 in [4])
(see also fig 21.2 pg 506 in [3] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg 507 in [3])[4]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[4] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs 507 and 508 in [3]
with the eqs (2.2.5) and (2.2.6) pg 67 in [4] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{82}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{83}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix} \tag{84}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{85}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{86}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{87}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{88}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{89}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i \beta^i dt^2 = \beta_i \beta^i dt^2 \tag{90}$$

$$(dx^i)^2 = dx^i dx^i \tag{91}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i \beta^i dt^2 \tag{92}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i \beta^i dt^2 \tag{93}$$

$$ds^2 = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{94}$$

Note that the expression above is exactly the eq (2.2.4) pg 67 in [4].It also appears as eq 1 pg 3 in [2].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{95}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{96}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{97}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{98}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{99}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{100}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{101}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{102}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{103}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{104}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{105}$$

The generic warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [1])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{106}$$

The generic Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{107}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{108}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{109}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{110}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{111}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{112}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_iX^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have the generic warp drive equation:

$$ds^2 = (1 - X_iX^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{113}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_iX^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{114}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^iX^i \end{pmatrix} \tag{115}$$

Looking to the equation of the original Natario vector nX in Polar Coordinates(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{116}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])(see also Appendices A and B in [15] for details)

$$X^r = 2v_s f(r)\cos\theta \tag{117}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{118}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ are given by:

$$X_i = \gamma_{ii}X^i \tag{119}$$

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = 2v_s f(r)\cos\theta = X^r \tag{120}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2X^\theta = -r^2v_s(2f(r) + (r)f'(r))\sin\theta \tag{121}$$

The generic equations of the warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{122}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{123}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{124}$$

Then the equation of the Natario warp drive spacetime in Polar Coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{125}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{126}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{127}$$

The Alcubierre warp drive can also be defined in a Natario-like vectorial form $nA = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.In the case of the Alcubierre warp drive (pg 3 in [1]) the Coordinates System is the Cartezian one with the original Alcubierre vector given by:

$$nA = X^i e_i = X^x e_x + X^y e_y + X^z e_z \tag{128}$$

$X^i = X^x = vsf(rs)$,$X^y = X^z = 0$ is the Alcubierre shift vector and $e_i = e_x$ is the Canonical Basis of the x-axis where the motion occurs.The Alcubierre vector is then given by:

$$nA = vsf(rs)e_x \tag{129}$$

Back again to the generic equations of the warp drive in the $3+1$ ADM formalism with signature $(+,-,-,-)$ given by:

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{130}$$

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{131}$$

In Cartezian Coordinates all the terms of the induced metric components $\gamma_{ii} = 1$ and therefore $X_i = \gamma_{ii} X^i = X_i = X^i$.Inserting the Alcubierre shift vector $X^i = X^x = vsf(rs)$,$X^y = X^z = 0$ we get:

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 = dt^2 - (dx - vsf(rs)dt)^2 - dy^2 - dz^2 \tag{132}$$

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{133}$$

$$ds^2 = dt^2 - (dx - vsf(rs)dt)^2 - dy^2 - dz^2 \tag{134}$$

$$ds^2 = (1 - [vsf(rs)]^2)dt^2 + 2vsf(rs)dxdt - dx^2 - dy^2 - dz^2 \tag{135}$$

And we recover the original equation of the Alcubierre warp drive with signature $(+, -, -, -)$.Compare with eq 8 pg 4 in [2] and eq 7 pg 4 in [20].

Note that in the Alcubierre case the motion occurs only over the x-axis implying in an Alcubierre shift vector defined by $X^i = X^x = vsf(rs)$,$X^y = X^z = 0$,$X^y e_y + X^z e_z = 0$.

The equation of the original Natario warp drive vector nX in polar coordinates with a constant speed is given by(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{136}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details all in [12])

$$X^r = 2v_s f(r)\cos\theta \tag{137}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{138}$$

The equation of the original Alcubierre warp drive vector nA in cartezian coordinates with a constant speed is given by:

$$nA = X^i e_i = X^x e_x + X^y e_y + X^z e_z \tag{139}$$

$$nA = vsf(rs)e_x \tag{140}$$

With the contravariant shift vector components X^i and given by $X^i = X^x = vsf(rs)$,$X^y = X^z = 0$.

The Natario original warp drive vector have 2 non-null contravariant shift vector components in 2 corresponding canonical basis while the Alcubierre original warp drive vector have only one non-null contravariant shift vector component in one corresponding canonical basis.

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