

Embedding Space Construction of Conformal Field Theory From First Principles

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July 6, 2026

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Abstract

This volume presents a systematic development of conformal field theory from first principles in quantum field theory, emphasizing complete derivations of the conformal group in arbitrary dimensions from the conformal Killing equation, proving its isomorphism to $SO(d+1, 1)$, and develop the embedding space formalism as a tool for constructing conformally covariant correlation functions constrained by Ward identities. In two dimensions, conformal symmetry enhances to an infinite-dimensional algebra: we derive the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$ from infinitesimal generators $\ell_n = -z^{n+1}\partial_z$, computing the central charge explicitly for free theories (boson $c = 1$, Majorana fermion $c = 1/2$, bc -ghosts $c = 1 - 3(2\lambda - 1)^2$) and demonstrating its physical interpretation as vacuum energy. Through radial quantization, we establish the operator-state correspondence and construct the Hilbert space via Virasoro descendants of primary states $|h, \bar{h}\rangle$, deriving the transformation law $T'(w) = (dw/dz)^{-2}[T(z) - (c/12)\{w; z\}]$ that reveals the geometric origin of the conformal anomaly. Unlike conventional treatments, we develop all mathematical machinery from standard QFT without relying on unexplained results from string theory or representation theory, exhibiting all intermediate steps explicitly to provide a detailed continuation beyond introductory QFT texts.

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Chapter 1

Introduction

1.1 About Conformal Field Theory:

Conformal symmetry is, in essence, the symmetry of **shapes without scales**. In Euclidean geometry, this is often described as "angle-preserving" symmetry. While this definition is correct in a purely spatial setting, it becomes less illuminating once we step into relativistic physics, where time is on equal footing with space. The notion of an "angle" between two events separated in time is not geometrically well-defined in the same way, so we need a more general formulation.

A more robust way to think about conformal transformations is that they are transformations which preserve the metric **up to a local rescaling**. That is, they preserve the light-cone structure of spacetime, and hence the causal relations, but may change distances by position-dependent factors:

$$g'_{\mu\nu}(x') = \Omega(x)g_{\mu\nu}(x), \quad \Omega(x) > 0.$$

Here, $\Omega(x)$ is the local scaling factor, and $\Omega(x) = 1$ corresponds to an isometry. This definition works in both Euclidean and Lorentzian settings, and it makes clear why conformal transformations generalize ordinary Poincaré symmetries: they allow stretching of spacetime while preserving angles and null directions. In flat space, an important special case is the global scaling transformation:

$$x^\mu \longrightarrow \lambda x^\mu,$$

which is not a mere relabeling of coordinates (i.e., not just a diffeomorphism), but a genuine change in the geometry. Under such a transformation, the ratios of lengths along a direction are preserved, and therefore angles are unchanged. This property underlies the term "conformal."

The study of **Conformal Field Theory (CFT)** leverages this symmetry in a very different way from conventional Quantum Field Theory (QFT). In QFT, one typically begins with a Lagrangian and derives correlation functions from the equations of motion and perturbation theory. In CFT, by contrast, the symmetry itself is so constraining that it often determines the form of correlation functions without reference to a specific Lagrangian. This leads naturally to the **conformal bootstrap** program, in which the consistency conditions of conformal symmetry, unitarity, and the operator product expansion are used to solve the theory.

Before we can use these powerful tools, it is important to distinguish conformal transformations from two related but conceptually different ideas: **Weyl rescalings** and **diffeomorphisms**. We will examine these one by one in the next section.

- A conformal field theory is a field theory, quantum or classical, that is invariant, or symmetric, under a group of transformations called the *conformal group* G .
- In a classical field theory, this means that the equations of motion are left invariant.
- In a quantum field theory, this means that, by Wigner's theorem, there is a projective unitary representation of the group G . In other words, symmetries, or transformations, that leave the transition amplitude invariant, are realized, up to a phase, by (anti)unitary operators.

This is unexpectedly nontrivial, and makes for a very rich field of study, since there is a tension between knowing the unitary representations of symmetries and demanding that the representation is locally implementable.

Chapter 2

Conformal Symmetry

2.1 General coordinate invariance (diffeomorphism)

Classical field theories can possess a variety of symmetries. One symmetry we will assume here is general coordinate invariance. Using the action principle, this symmetry can be used to show that the energy-momentum tensor is conserved.

In general, the energy-momentum tensor is defined through the variation of the action S under changes in the space-time metric:

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}.$$

By definition,

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu}$$

If the theory is invariant under general coordinate transformations, one can show that

$$(T^{\mu\nu})_{;\nu} = 0$$

where, as usual in general relativity, " ν " denotes the covariant derivative. In flat coordinates, this condition reduces to

$$\partial_\nu T^{\mu\nu} = 0$$

The concept of the field easily puts forth the idea of locality, but what constraints does conformal symmetry place on a classical field? For symmetries in a classical field theory start with the action

$$S = \int d^d x \mathcal{L}(\phi, \partial_\mu \phi), \text{ where } \phi = \{\phi_a(x)\}. \quad (2.1)$$

By writing down the action, we have assumed (1) that the equations of motion are represented by an action and (2) that the Lagrangian density depends only on the field and its first derivatives. We have effectively thrown out all non-local theories.

So, a symmetry transformation takes a spacetime location and maps it to its image under that transformation: $x \rightarrow x'$. If the transformation is *active*, then the fields transform as well

$$\phi(x) \rightarrow \phi'(x') \equiv \mathcal{F}(\phi(x)) \quad (2.2)$$

Where we note that $\mathcal{F}(\phi(x))$ depends on the previous field configuration.

For example, in an active rotation of a vector field $\mathbb{R}^{2,0}$, a nontrivial representation rotates the spacetime coordinate as well as the vector at each spacetime coordinate, the field (A trivial representation will not rotate the vector.)

After the rotation, the new field configuration at x is

$$\phi'_a(x) = \sum_b \pi(\mathcal{O})_{ab} \phi_b(\mathcal{O}^{-1}x). \quad (2.3)$$

The trivial representation of the field component b would simply be the identity $\pi(\mathcal{O})_{ab} = \delta_{ab}$, and the fundamental, nontrivial representation is written

$$\pi(\mathcal{O})_{ab} = [\mathcal{O}]_{ab}. \quad (2.4)$$

How does the action S transform under a symmetry transformation?

$$S' = \int d^d x \left| \det \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}} \right) \right| \mathcal{L} \left(\mathcal{F}(\phi(x)), \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu} \mathcal{F}(\phi(x)) \right) \quad (2.5)$$

We also know that our theory is a conformal field theory, conformally invariant, conformally symmetric if the equations of motion are invariant. This is equivalent to the Lagrangian density transforming up to a total derivative

$$\mathcal{L}' = \mathcal{L} + \text{total derivative.} \quad (2.6)$$

2.2 Weyl invariance

In addition to general coordinate invariance, many field theories possess another powerful symmetry: Weyl invariance. While diffeomorphism invariance constrains how the metric responds to arbitrary coordinate changes, Weyl invariance instead concerns how the theory behaves under local rescalings of the metric. Under a Weyl transformation, the metric changes as

$$g_{\mu\nu}(x) \rightarrow \Omega(x)g_{\mu\nu}(x)$$

or, in infinitesimal form,

$$g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) + \omega(x)g_{\mu\nu}(x)$$

The condition for the action to remain invariant under such a transformation can be expressed in terms of the energy-momentum tensor. Substituting $\delta g_{\mu\nu} = \omega(x)g_{\mu\nu}(x)$ into the earlier definition, we find

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g} T_{\mu}^{\mu} \omega(x)$$

Since this must hold for arbitrary functions $\omega(x)$, we conclude that the condition for Weyl invariance is

$$T_{\mu}^{\mu} = 0$$

Thus, just as diffeomorphism invariance implies the covariant conservation of the energy-momentum tensor, Weyl invariance implies that the energy-momentum tensor must be traceless.

2.3 Conformal invariance

2.3.1 Conformal Transformations

A conformal transformation can be defined as a coordinate transformation that acts on the metric as a Weyl transformation. Consider a general coordinate transformation

$$x \rightarrow x', \quad x^{\mu} = f^{\mu}(x')$$

The metric then transforms as

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial f^{\rho}}{\partial x'^{\mu}} \frac{\partial f^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(f(x'))$$

We now require that the transformed metric be proportional to the original one. Rotations and translations clearly satisfy this condition: they leave the metric unchanged and hence preserve all inner products

$$v \cdot w \equiv v^{\mu} g_{\mu\nu} w^{\nu}$$

They are therefore part of the conformal group. More generally, any coordinate transformation satisfying the above proportionality preserves all angles,

$$\frac{v \cdot w}{\sqrt{v^2 w^2}}$$

which is the origin of the term "conformal." Later in this chapter we will determine all such transformations explicitly.

If a field theory has a conserved and traceless energy-momentum tensor, it is invariant under both general coordinate transformations and Weyl transformations. Let the action be

$$S = \int d^d x \mathcal{L}(\partial_x, g_{\mu\nu}(x), \phi(x))$$

Here, ϕ denotes any matter field, while the metric $g_{\mu\nu}$ is written separately due to its special rôle. We have also explicitly indicated spacetime derivatives in the Lagrangian.

General coordinate invariance implies

$$S = S' \equiv \int d^d x' \mathcal{L}(\partial_{x'}, g'_{\mu\nu}(x'), \phi'(x'))$$

where $g'_{\mu\nu}$ is as given above, and the transformation of ϕ depends on its spin. For a tensor field of rank n one has

$$\phi'_{\mu_1 \dots \mu_n}(x') = \left| \frac{\partial f}{\partial x'} \right|^{\frac{\Delta}{d}} \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \dots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}} \phi_{\nu_1 \dots \nu_n}(f(x'))$$

where Δ is the scaling dimension of the field. Fields that transform according to Eq. under conformal transformations are called conformal fields, or equivalently, primary fields.

In particular, for a scalar $\phi(x)$ we have simply $\phi'(x') = \phi(f(x'))$. For the derivative of a scalar:

$$\frac{\partial}{\partial x^\mu} \phi(x) \rightarrow \frac{\partial}{\partial x^\mu} \phi'(x') = \frac{\partial f^\nu}{\partial x^\mu} \frac{\partial}{\partial f^\nu} \phi(f(x'))$$

which transforms as a vector. (However, note that n th-order ordinary derivatives do not transform as rank- n tensors; this holds only for covariant derivatives.)

If the coordinate transformation $x \rightarrow x'$ is of the above type, we can use Weyl invariance of the action to bring the metric back to its original form. This yields

$$S = S'' \equiv \int d^d x' \mathcal{L}(\partial_{x'}, g_{\mu\nu}(f(x')), \phi'(x')) = \int d^d x' \mathcal{L}(\partial_{x'}, g_{\mu\nu}(x), \phi'(x'))$$

This is the statement of conformal symmetry of the action. We should note that in some cases, an isometry of one metric can act as a conformal transformation for a different metric. In such situations, Weyl rescaling is not needed, since the transformation already preserves the metric under consideration. This happens, for example, in de Sitter (dS) and anti-de Sitter (AdS) spacetimes, where certain isometries correspond to conformal transformations of the induced boundary metric.

If we begin with a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, the background metric remains unchanged under such transformations.¹ This allows us to define conformal transformations for theories in flat space that are not coupled to gravity. We may then ignore general coordinate invariance and start with an action in which no dynamical metric appears.

In this flat-space setting, conformal invariance means that the action is unchanged when we integrate the same Lagrangian (or any scalar physical quantity) written in terms of the transformed fields $\phi'(x')$ over the new coordinates x' .

In $d = 2$ dimensions, this is not really a restriction. A general 2D metric has three independent components: $g_{11}(x)$, $g_{22}(x)$, and $g_{12}(x) = g_{21}(x)$. A general coordinate transformation provides two functions $f^1(x)$ and $f^2(x)$ that can be used to set $g_{12}(x) = 0$ and $g_{11}(x) = \pm g_{22}(x)$ (depending on the signature). The metric can then be written in the form $g(x)\eta_{\mu\nu}$, which is called conformal gauge. A Weyl transformation can remove the remaining factor $g(x)$, bringing the metric to the form $\eta_{\mu\nu}$.

In more than two dimensions, this procedure does not work in general, so restricting to flat space truly limits us to non-gravitational theories. Even in two dimensions, conformal gauge can be chosen only locally in general, meaning CFT can be applied in coordinate patches, but extra data may be needed to describe the theory globally.

2.4 Infinitesimal Conformal Transformation

Classification of Infinitesimal Conformal Translations for $d = 2$:

If $d = 2$, the spacetime metric becomes the identity

$$g_{\mu\nu} = \delta_{\mu\nu} \quad (2.7)$$

And $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$ becomes the Cauchy-Riemann equations and $\epsilon(x)$ is complex-valued, complex-differentiable, and analytic

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \text{ and } \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \quad (2.8)$$

Introduce the complex coordinates

$$z = x^1 + ix^2 \text{ and } \bar{z} = x^1 - ix^2. \quad (2.9)$$

Then we can complexify ϵ as

$$\epsilon(z) = \epsilon^1 + i\epsilon^2 \text{ and } \bar{\epsilon}(\bar{z}) = \epsilon^1 - i\epsilon^2. \quad (2.10)$$

Two-dimensional global, gotten via exponentiation of an infinitesimal transformation, conformal transformations correspond to *entire* (no singularities, invertible everywhere), *holomorphic* functions $z \rightarrow f(z)$ with holomorphic inverses $f^{-1}(z)$. The only allowable form for a conformal transformation that corresponds to an entire, holomorphic function is linear in the complex coordinates

$$f(z) = \alpha z + \beta, \text{ where } \alpha, \beta \in \mathbb{C}. \quad (2.11)$$

We may expect a larger group of symmetries with entirety and holomorphism enforced, since the space seems less constrained, but this actually constrains the space more and the group becomes smaller. So, if we were to not compactify, and add infinity as a point, as we demonstrated, the conformal space becomes linear and boring: only rotations and scaling are allowed.

To include this complex representation of the spacetime coordinates, we *extend* our manifold to the complex numbers \mathbb{C} and compactify complex space to a Riemann sphere $\mathbb{C} \cup \{\infty\}$ we get the proper space for the conformal transformations to act in

$$\mathbb{R}^{2,0} \rightarrow \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\} \rightarrow \text{Conf}(\mathbb{C} \cup \{\infty\}) \quad (2.12)$$

Where the conformal group is

$$\text{Conf}(\mathbb{C} \cup \{\infty\}) = \left\{ f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}; \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma \neq 0 \right\}. \quad (2.13)$$

This is also called the group of Moebius transformations, and is a slightly larger group of conformal transformations (symmetries), since we can map to and from infinity as a point.

A conformal field theory is a local quantum field theory that is invariant under the conformal group, a set of transformations, a change in coordinates, that leave the metric invariant up to a scale factor. In different spacetime dimensions, the conformal group takes on significantly different forms.

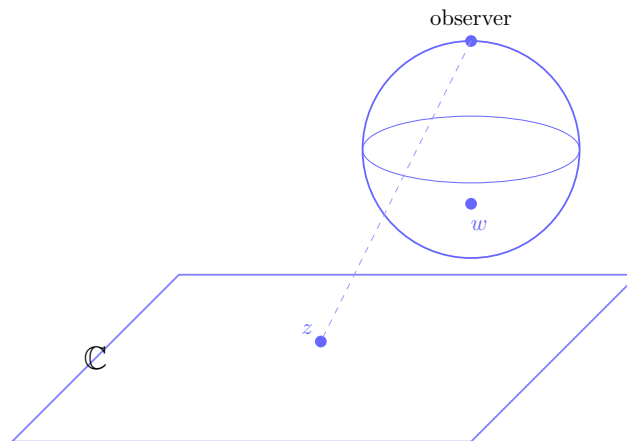


Figure 2.1:

The global conformal group in dimensions greater than two is comprised of translations, rotations, scaling, and special conformal transformations, as well as dimensions equal to two, as

long as the space is compactified. If singularities are included, functions with poles are allowed, the symmetry gets larger.

The **fundamental essence** of conformal transformations resides in their infinitesimal form, which serves as a crucial tool for investigating how fields transform under these symmetries. It plays a pivotal role in defining the generator of the conformal group and, subsequently, constraining the set of possible correlators that are compatible with conformal symmetry. Any infinitesimal transformation can be expressed as:

$$x'^{\mu} = x^{\mu} + \underbrace{\epsilon^{\mu}(x)}_{\text{infinitesimal}}$$

and subsequently,

$$x^{\mu} = x'^{\mu} - \epsilon^{\mu}(x)$$

therefore, the metric transforms like:

$$\begin{aligned} g'_{\mu\nu} &= \underbrace{\frac{\partial x^{\alpha}}{\partial x'^{\mu}}}_{\delta_{\mu}^{\alpha} - \partial_{\mu}\epsilon^{\alpha}(x)} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \\ &= \left[\delta_{\mu}^{\alpha} - \frac{\partial \epsilon^{\alpha}(x)}{\partial x'^{\mu}} \right] \left[\delta_{\nu}^{\beta} - \frac{\partial \epsilon^{\beta}(x)}{\partial x'^{\nu}} \right] g_{\alpha\beta} \\ &= \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} g_{\alpha\beta} - \delta_{\nu}^{\beta} \partial_{\mu} \epsilon^{\alpha}(x) g_{\alpha\beta} - \delta_{\mu}^{\alpha} \partial_{\nu} \epsilon^{\beta}(x) g_{\alpha\beta} + \mathcal{O}(\epsilon^2) \end{aligned}$$

$$\Omega(x)g_{\mu\nu} = g_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu}$$

In the third step, we used chain rule on $\epsilon^{\alpha}(x)$ and ignored $\mathcal{O}((\partial\epsilon)^2)$ terms. From the last line, it is reasonable to expect that:

$$\begin{aligned} g_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} &= [1 + f(x)]g_{\mu\nu} \\ \partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) &= f(x)g_{\mu\nu} \end{aligned} \tag{2.14}$$

Contracting Indices

$$\partial^{\mu}\epsilon_{\mu}(x) + \partial^{\mu}\epsilon_{\mu}(x) = f(x)\delta_{\mu}^{\mu}$$

$$2(\partial \cdot \epsilon) = \underbrace{d}_{\text{dimension of spacetime}} f(x)$$

$$f(x) = \frac{2}{d} \frac{\partial \epsilon_{\mu}(x)}{\partial x_{\mu}}$$

Substituting back in (2.14)

$$\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x) = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu} \tag{2.15}$$

Up until now, we have no made any crude assumption. However before we proceed, we will assume that the metric is Euclidean. Now, we operate on both sides by ∂^{ν} **assuming flat metric**

$$\begin{aligned} \frac{\partial}{\partial x^{\nu}} [\partial_{\mu}\epsilon_{\nu}(x) + \partial_{\nu}\epsilon_{\mu}(x)] &= \frac{\partial}{\partial x^{\nu}} \left(\frac{2}{d} \partial \cdot \epsilon(x) g_{\mu\nu} \right) \\ \partial_{\mu} \underbrace{\partial^{\nu}\epsilon_{\nu}}_{\partial \cdot \epsilon} + \underbrace{\partial^{\nu}\partial_{\nu}\epsilon_{\mu}}_{\square} &= \frac{2}{d} g_{\mu\nu} \partial^{\nu} \partial \cdot \epsilon \end{aligned}$$

⁰In perturbation theory, we often describe physics on a perturbed manifold as that of tensor fields living on a background manifold. In this case, the conformal transformation is performed on the background metric, and the change in the perturbation is again dictated by the Killing equation. We first apply the coordinate transformation to the full metric, and then perform a Weyl rescaling to restore the background metric to its original form.

$$\partial_\mu(\partial \cdot \epsilon) + \square \epsilon = \frac{2}{d} \partial_\mu(\partial \cdot \epsilon)$$

Operating by ∂_ν

$$\begin{aligned} \partial_\nu [\partial_\mu(\partial \cdot \epsilon) + \square \epsilon] &= \partial_\nu \left(\frac{2}{d} \partial_\mu(\partial \cdot \epsilon) \right) \\ \left(1 - \frac{2}{d} \right) \partial_\nu \partial_\mu(\partial \cdot \epsilon) + \square(\partial_\nu \epsilon_\mu) &= 0 \end{aligned} \quad (2.16)$$

under relabeling $\mu \leftrightarrow \nu$

$$\left(1 - \frac{2}{d} \right) \partial_\mu \partial_\nu(\partial \cdot \epsilon) + \square(\partial_\mu \epsilon_\nu) = 0 \quad (2.17)$$

adding (2.16) and (2.17)

$$2 \left(1 - \frac{2}{d} \right) \partial_\mu \partial_\nu(\partial \cdot \epsilon) + \square \underbrace{[\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)]}_{\frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}} = 0$$

$$\left(1 - \frac{2}{d} \right) \partial_\mu \partial_\nu(\partial \cdot \epsilon) + \frac{1}{d} \square(\partial \cdot \epsilon) g_{\mu\nu} = 0$$

$$[g_{\mu\nu} \square + (d-2) \partial_\mu \partial_\nu] (\partial \cdot \epsilon) = 0 \quad (2.18)$$

Contracting the indices

$$\begin{aligned} [d \square + (d-2) \square] (\partial \cdot \epsilon) &= 0 \\ 2(d-1) \square(\partial \cdot \epsilon) &= 0 \end{aligned}$$

hence,

$$\boxed{(d-1) \square(\partial \cdot \epsilon) = 0} \quad (2.19)$$

if $d = 1 \implies$ any $\epsilon^\mu(x)$ satisfies (2.19), therefore, is conformal transformation. It is interesting to note that any 1D QFT is conformal field theory, but for our purpose it's not very useful. We will be concerned with $d \neq 1$ for the rest of this notes unless stated otherwise. Consider the action of ∂_α on (2.15) and then cyclic relabeling of indices as $\alpha \rightarrow \mu \rightarrow \nu$

$$\partial_\alpha [\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)] = \frac{2}{d} \partial_\alpha g_{\mu\nu} (\partial \cdot \epsilon) \quad (2.20)$$

$$\partial_\mu [\partial_\nu \epsilon_\alpha(x) + \partial_\alpha \epsilon_\nu(x)] = \frac{2}{d} \partial_\mu g_{\nu\alpha} (\partial \cdot \epsilon) \quad (2.21)$$

$$\partial_\nu [\partial_\alpha \epsilon_\mu(x) + \partial_\mu \epsilon_\alpha(x)] = \frac{2}{d} \partial_\nu g_{\alpha\mu} (\partial \cdot \epsilon) \quad (2.22)$$

Adding the first two equation and subtracting from the last, we get [(2.20) + (2.21) - (2.22)]:

$$\begin{aligned} 2 \partial_\alpha \partial_\mu \epsilon_\nu &= \frac{2}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (\partial \cdot \epsilon) \\ \partial_\alpha \partial_\mu \epsilon_\nu &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (\partial \cdot \epsilon) \\ \partial_\alpha \partial_\mu \epsilon_\nu &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (\partial \cdot \epsilon) \end{aligned} \quad (2.23)$$

Referring to eqn 1.7.11 of “Ideas and Methods of Supersymmetry and Supergravity” by Sergio M. Kuzenko, we find that the 3rd order derivation of $\epsilon^\mu(x)$ vanishes. Therefore, the most general conformal transformation is of the type:

$$x'^\mu = x^\mu + \underbrace{a^\mu + b^\mu_\nu x^\nu + c^\mu_{\nu\alpha} x^\nu x^\alpha}_{\epsilon^\mu}$$

Where, a^μ , b^μ_ν and $c^\mu_{\nu\alpha}$ are parameters relevant to their transformation. The goal here is simple:

- First find the relevant transformations
- Then based on the transformation rule, find the generators.

For $\epsilon^\mu = a^\mu$:

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu \\ &= x^\mu + \delta_\nu^\mu a^\nu \\ &= x^\mu + (\partial_\nu x^\mu) a^\nu \\ &= [1 + i a^\nu (-i \partial_\nu)] x^\mu \end{aligned}$$

Thus, the generator of translation is $P_\mu - i \partial_\mu$.¹

For $\epsilon^\mu = b^\mu_\alpha x^\alpha$, we refer to (2.15)

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d} (\partial \cdot \epsilon) g_{\mu\nu}$$

$$\partial_\mu (b_{\nu\alpha} x^\alpha) + \partial_\nu (b_{\mu\alpha} x^\alpha) = \frac{2}{d} (\partial^\mu b_{\mu\alpha} x^\alpha) g_{\mu\nu}$$

$$b_{\nu\alpha} \delta_\mu^\alpha + b_{\mu\alpha} \delta_\nu^\alpha = \frac{2}{d} (b_{\mu\alpha} g^{\alpha\mu}) g_{\mu\nu}$$

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d} b^\alpha_\alpha g_{\mu\nu}$$

$$\frac{b_{\nu\mu} + b_{\mu\nu}}{2} = \frac{1}{d} b^\alpha_\alpha g_{\mu\nu}$$

now,

$$\begin{aligned} b_{\mu\nu} &= \frac{b_{\mu\nu} - b_{\nu\mu}}{2} + \frac{b_{\mu\nu} + b_{\nu\mu}}{2} \\ &= M_{\mu\nu} + \lambda g_{\mu\nu} \end{aligned}$$

If $b_{\mu\nu} = \lambda g_{\mu\nu}$ ($M_{\mu\nu} = 0$)

$$\begin{aligned} x'^\mu &= x_\mu + b^\mu_\nu x^\nu \\ &= x^\mu + \lambda g^{\mu\alpha} \underbrace{g_{\alpha\nu} x^\nu}_{x_\alpha} \\ &= x^\mu + \lambda x^\mu \\ &= x^\mu + \lambda x^\nu \delta_\nu^\mu \\ &= x^\mu + \lambda x^\nu (\partial_\nu x^\mu) \\ &= x^\mu + i \lambda x^\nu (-i \partial_\nu x^\mu) \\ &= (1 + i \lambda (-i x^\nu \partial_\nu)) x^\mu \end{aligned}$$

Thus, the generator of dilatation is $D = -i x^\mu \partial_\mu$. For $b_{\mu\nu} = M_{\mu\nu}$ ($\lambda = 0$).

$$\begin{aligned} x'^\mu &= x^\mu + M^\mu_\nu x^\nu \\ &= x^\mu + M^\alpha_\nu \delta^\mu_\alpha x^\nu \\ &= x^\mu + M^\alpha_\nu (\partial_\alpha x^\mu) x^\nu \\ &= x^\mu + M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu \\ &= x^\mu + \frac{M_{\alpha\nu} - M_{\nu\alpha}}{2} (\partial^\alpha x^\mu) x^\nu && \text{relabeling } \nu \leftrightarrow \alpha \\ &= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \frac{1}{2} M_{\nu\alpha} (\partial^\alpha x^\mu) x^\nu \\ &= x^\mu + \frac{1}{2} M_{\alpha\nu} (\partial^\alpha x^\mu) x^\nu - \frac{1}{2} M_{\alpha\nu} (\partial^\nu x^\mu) x^\alpha \\ &= x^\mu + \frac{1}{2} M_{\alpha\nu} (x^\nu \partial^\alpha - x^\alpha \partial^\nu) x^\mu \end{aligned}$$

¹if we use $[1 - a^\nu (\partial_\nu)] x^\mu$ as the definition, then $P_\mu = i \partial_\mu$ would be the generator

$$\begin{aligned}
&= x^\mu + \frac{i}{2} M_{\alpha\nu} \{-i(x^\nu \partial^\alpha - x^\alpha \partial^\nu)\} x^\mu \\
&= x^\mu + \frac{i}{2} M_{\alpha\nu} \underbrace{\{i(x^\alpha \partial^\nu - x^\nu \partial^\alpha)\}}_{L^{\alpha\nu}} x^\mu
\end{aligned}$$

Thus, the generator of rotation is $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Now, the last part $\epsilon^\mu = c^\mu_{\nu\alpha} x^\nu x^\alpha = c^\mu_{\alpha\nu} x^\nu x^\alpha$, we refer to (2.23):

$$\begin{aligned}
\partial_\alpha \partial_\mu \epsilon_\nu &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] (\partial \cdot \epsilon) \\
\partial_\alpha \partial_\mu (c_{\nu\beta} x x^\beta) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] \partial^\mu (c_{\mu\beta} x x^\beta) \\
c_{\nu\beta} \partial_\alpha (\delta_\mu x^\beta + x \delta_\mu^\beta) &= \frac{1}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] c_{\mu\beta} (g^\mu x^\beta + x g^{\beta\mu}) \\
c_{\nu\beta} (\delta_\mu \delta_\alpha^\beta + \delta_\alpha \delta_\mu^\beta) &= \frac{2}{d} [g_{\mu\nu} \partial_\alpha + g_{\nu\alpha} \partial_\mu - g_{\alpha\mu} \partial_\nu] c_{\beta} x^\beta \\
2c_{\nu\mu\alpha} &= \frac{2}{d} c_{\beta} [g_{\mu\nu} \delta_\alpha^\beta + g_{\nu\alpha} \delta_\mu^\beta - g_{\alpha\mu} \delta_\nu^\beta] \\
c_{\nu\mu\alpha} &= \frac{1}{d} \underbrace{c_{\beta}}_{b_\beta} [g_{\mu\nu} \delta_\alpha^\beta + g_{\nu\alpha} \delta_\mu^\beta - g_{\alpha\mu} \delta_\nu^\beta] \\
&= g_{\nu\mu} b_\alpha + g_{\nu\alpha} b_\mu - g_{\mu\alpha} b_\nu
\end{aligned}$$

Therefore,

$$\begin{aligned}
\epsilon_\mu &= c_{\mu\alpha\beta} x^\alpha x^\beta \\
&= (g_{\mu\alpha} b_\beta + g_{\mu\beta} b_\alpha - g_{\alpha\beta} b_\mu) x^\alpha x^\beta \\
&= x_\mu (b \cdot x) + x_\mu (b \cdot x) - b_\mu (x \cdot x) \\
&= 2x_\mu (b \cdot x) - x^2 b_\mu
\end{aligned}$$

Hence, the Special Conformal Transformation looks like:

$$\begin{aligned}
x'^\mu &= x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu = x^\mu + 2(b \cdot x) x^\nu \delta_\nu^\mu - x^2 b^\nu \delta_\nu^\mu \\
&= x^\mu + 2(b \cdot x) x^\nu \partial_\nu x^\mu - x^2 b^\nu \partial_\nu x^\mu \\
&= [1 + 2(b \cdot x) x^\nu \partial_\nu - x^2 b^\nu \partial_\nu] x^\mu \\
&= [1 + \{2b^\alpha x_\alpha x^\nu \partial_\nu - x^2 b^\alpha \partial_\alpha\}] x^\mu \\
&= \left[1 + i b^\alpha \underbrace{\{-i(2x_\alpha x^\nu \partial_\nu - x^2 \partial_\alpha)\}}_{K_\alpha} \right] x^\mu
\end{aligned}$$

Hence, the generator for Special Conformal Transformations (SCT) takes the form $K^\mu = -i(2x^\mu x \cdot \partial - x^2 \partial^\mu)$. We will now list all the **infinitesimal** transformations and their generators we found in this section.

1. Translation

$$x'^\mu = x^\mu + a^\mu \quad P_\mu = -i\partial_\mu \quad (2.24)$$

2. Rotation

$$x'^\mu = x^\mu + M^\mu_{\nu} x^\nu \quad L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \quad (2.25)$$

3. Dilatation

$$x'^\mu = (1 + \lambda)x^\mu \quad D = -ix^\mu \partial_\mu \quad (2.26)$$

4. Special Conformal Transformation

$$x'^\mu = x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu \quad K^\mu = -i(2x^\mu x \cdot \partial - x^2 \partial^\mu) \quad (2.27)$$

2.5 Finite Conformal Transformation

In the previous section, we considered the infinitesimal conformal transformation, however in this section we will consider the finite conformal transformation.

1. Translation

$$x'^{\mu} = x^{\mu} + \underbrace{a^{\mu}}_{\text{finite vector}} = e^{ia^{\nu}P_{\nu}}x^{\mu}$$

2. Dilatation

$$x'^{\mu} = \left(1 + \frac{\lambda}{N}\right)x^{\mu}$$

In order to achieve the finite dilatation, we use the infinitesimal transformation recursively by dividing the finite λ into infinitely many λ/N pieces and then transforming

$$\begin{aligned} x'^{\mu} &= \left(1 + \frac{\lambda''}{N}\right) \underbrace{\left(1 + \frac{\lambda'}{N}\right) \left(1 + \frac{\lambda}{N}\right)}_{x''^{\mu}} x^{\mu} \\ &= \lim_{N \rightarrow \infty} \left(1 + \frac{\lambda}{N}\right)^N x^{\mu} \\ &= e^{\lambda}x^{\mu} = e^{i\lambda D}x^{\mu} \end{aligned}$$

3. Rotation

$$x'^{\mu} = \left[1 + \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right]^{\mu}_{\nu} x^{\nu} = \left[e^{\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}}\right]^{\mu}_{\nu} x^{\nu} = \Lambda^{\mu}_{\nu}x^{\nu}$$

4. The special conformal transformation

infinitesimal parameter, i.e. t is small.

$$x'^{\mu} = x^{\mu} + 2x^{\mu}(b \cdot x) - x^2b^{\mu}$$

let $b^{\mu} = te^{\mu}$

$$x'^{\mu}(t) \equiv x^{\mu}(t) = x^{\mu} + 2t(e \cdot x)x^{\mu} - x^2te^{\mu}$$

To find the finite form of the transformation we have to recursively apply the above equation multiple times (Lie Algebra sense). The usual way is to integrate the infinitesimal form. The other way, and since we know that the transformations satisfy the conformal Killing equation, is to find the integral curve of the corresponding conformal Killing vector field as they are equivalent (Differential Geometry sense). Consider the t -derivative of the above.²

$$\frac{dx^{\mu}(t)}{dt} = 2(e \cdot x)x^{\mu} - x^2e^{\mu} \quad (2.28)$$

defining $y^{\mu}(t) = \frac{x^{\mu}(t)}{x^2(t)}$

quotient rule

$$\begin{aligned} \dot{y}^{\mu}(t) &= \frac{x^2\dot{x}^{\mu} - 2(\dot{x} \cdot x)x^{\mu}}{(x^2)^2} \\ &= \frac{x^2[2(e \cdot x)x^{\mu} - x^2e^{\mu}] - 2[2(e \cdot x)x^{\nu} - x^2e^{\nu}]x_{\nu}x^{\mu}}{x^4} \\ &= \frac{x^2[2(e \cdot x)x^{\mu} - x^2e^{\mu}] - 2[2(e \cdot x)x^2 - x^2(e \cdot x)]x^{\mu}}{x^4} \end{aligned}$$

²when we consider the differential equation, we are no longer thinking of it as transformation but rather flow along a trajectory parameterized by t . This part was taken from pg 16 of “Four point function in momentum spaces and topological terms in gravity”

$$= \frac{x^2[2(\cancel{e \cdot x})x^\mu - x^2 e^\mu] - 2(\cancel{e \cdot x})x^2 x^\mu}{x^4}$$

$$\dot{y}^\mu(t) = -e^\mu$$

Solving the above differential equation

$$y^\mu(t) = y^\mu(0) - te^\mu$$

$$\frac{x^\mu(t)}{x^2(t)} = \frac{x^\mu(0)}{x^2(0)} - te^\mu$$

going back to the old notation $x'^\mu \equiv x^\mu(t)$

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - te^\mu = \frac{x^\mu}{x^2} - b^\mu \quad (2.29)$$

Squaring both sides

$$\left(\frac{x'^\mu}{x'^2}\right)^2 \equiv \frac{x'^\mu x'^\mu}{x'^2 x'^2} = \left(\frac{x^\mu}{x^2} - b^\mu\right)^2$$

$$\frac{x'^2}{x'^4} = \left(\frac{x^\mu}{x^2}\right)^2 + b^2 - \frac{2(x \cdot b)}{x^2}$$

$$\frac{1}{x'^2} = \frac{1 + b^2 x^2 - 2(x \cdot b)}{x^2}$$

$$x'^2 = \frac{x^2}{1 - 2(x \cdot b) + b^2 x^2} \quad (2.30)$$

referring to (2.29)

$$x'^\mu = x'^2 \left[\frac{x^\mu}{x^2} - b^\mu \right]$$

and substituting (2.30)

$$x'^\mu = \frac{x^2}{1 - 2(x \cdot b) + b^2 x^2} \left[\frac{x^\mu}{x^2} - b^\mu \right] = \frac{x^\mu - b^\mu x^2}{1 - 2(x \cdot b) + b^2 x^2}$$

Above procedure also suggests that, finite SCT could be described as a sequence of inversion \rightarrow translation \rightarrow inversion. Where inversion is defined as:

$$I(x^\mu) = \frac{x^\mu}{x^2}$$

First we note that the inversion is a global conformal transformation and since it is undefined at origin, it does not have an infinitesimal part i.e. we can not expect inversion to be obtained by exponentiating an element from the conformal Lie algebra. It is not the connected element of conformal group and in embedding space formalism, inversion is related to parity. Another interesting point to note is that there is no parameter associated with the transformation here such as λ for dilatation or b^μ for SCT. Lastly, it is also closely related to the stereographic projection. To show this let us study the stereographic projection of sphere onto a plane.

Consider $x \in \mathbb{R}^n$, and define stereographic projection from the **north pole** of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as:

$$X^i = \frac{2x^i}{1 + |x|^2}, \quad X^{n+1} = \frac{|x|^2 - 1}{1 + |x|^2}$$

where x^i are coordinates on the projected plane and X^i are the coordinates on the sphere in embedding space. Now project this point on the sphere back to \mathbb{R}^n via stereographic projection from the **south pole**:

$$x'^i = \frac{X^i}{1 + X^{n+1}}$$

Substituting:

$$x'^i = \frac{\frac{2x^i}{1 + |x|^2}}{1 + \frac{|x|^2 - 1}{1 + |x|^2}} = \frac{2x^i}{(1 + |x|^2) + (|x|^2 - 1)} = \frac{2x^i}{2|x|^2} = \frac{x^i}{|x|^2}$$

Hence, the composition gives:

$$x^i \mapsto \frac{x^i}{|x|^2}$$

which is the **inversion** in the unit sphere. Even though this inversion does not have a killing vector associated with it, but it is reasonable to look for the killing vector associated with stereographic projection. In general, we note that these two transformation would have the following form:

$$x'^{\mu} = \Omega(x)x^{\mu}$$

If $\partial_{\mu}\partial_{\nu}(\frac{1}{\Omega}) \propto g_{\mu\nu}$. The killing vector associated with it will have the form:

$$K^A_{\mu} = \frac{1}{\Omega^2} \frac{\partial x'^A}{\partial x^{\mu}}$$

Coming back to special conformal transformation which was the topic at hand, we now look at how they scale the metric tensor. The Jacobian is given by:

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \left\{ \frac{\delta^{\mu}_{\nu} - 2b^{\mu}x_{\nu}}{\Lambda} - \frac{(x^{\mu} - b^{\mu}x^2)(-2b_{\nu} + 2b^2x_{\nu})}{\Lambda^2} \right\}$$

Transformation of the metric:

$$\begin{aligned} g_{\alpha\beta}(x) &= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g'_{\mu\nu}(x') \Big|_{x'=x'(x)} \\ &= \left\{ \frac{\delta^{\mu}_{\alpha} - 2b^{\mu}x_{\alpha}}{\Lambda} - \frac{(x^{\mu} - b^{\mu}x^2)(-2b_{\alpha} + 2b^2x_{\alpha})}{\Lambda^2} \right\} \\ &\quad \times \left\{ \frac{\delta^{\nu}_{\beta} - 2b^{\nu}x_{\beta}}{\Lambda} - \frac{(x^{\nu} - b^{\nu}x^2)(-2b_{\beta} + 2b^2x_{\beta})}{\Lambda^2} \right\} g'_{\mu\nu}(x') \\ &= \left\{ \frac{\delta^{\mu}_{\alpha} - 2b^{\mu}x_{\alpha}}{\Lambda} - \frac{(x^{\mu} - b^{\mu}x^2)(-2b_{\alpha} + 2b^2x_{\alpha})}{\Lambda^2} \right\} \\ &\quad \times \left\{ \frac{g'_{\mu\beta} - 2b_{\mu}x_{\beta}}{\Lambda} - \frac{(x_{\mu} - b_{\mu}x^2)(-2b_{\beta} + 2b^2x_{\beta})}{\Lambda^2} \right\} \\ &= \frac{g'_{\alpha\beta} - 2b_{\beta}x_{\alpha} - 2b_{\alpha}x_{\beta} + 4b^2x_{\alpha}x_{\beta}}{\Lambda^2} - \frac{(x_{\beta} - b_{\beta}x^2)(-2b_{\alpha} + 2b^2x_{\alpha})}{\Lambda^3} \\ &\quad + \frac{(2(b \cdot x)x_{\beta} - 2b^2x_{\beta}x^2)(-2b_{\alpha} + 2b^2x_{\alpha})}{\Lambda^3} - \frac{(x_{\alpha} - 2b_{\alpha}x^2)(-2b_{\beta} + 2b^2x_{\beta})}{\Lambda^3} \\ &\quad + \frac{(2(b \cdot x)x_{\alpha} - 2b^2x^2x_{\alpha})(-2b_{\beta} + 2b^2x_{\beta})}{\Lambda^3} \\ &\quad + \frac{(x^{\mu} - b^{\mu}x^2)(x_{\mu} - b_{\mu}x^2)(-2b_{\alpha} + 2b^2x_{\alpha})(-2b_{\beta} + 2b^2x_{\beta})}{\Lambda^4} \\ &= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{(-2b_{\beta}x_{\alpha} - 2b_{\alpha}x_{\beta} + 4b^2x_{\alpha}x_{\beta})(1 - 2b \cdot x + b^2x^2)}{\Lambda^3} \\ &\quad + \frac{1}{\Lambda^3} \left\{ 2b_{\alpha}x_{\beta} - 2b_{\alpha}b_{\beta}x^2 - 2b^2x_{\alpha}x_{\beta} + 2b^2x^2x_{\alpha}b_{\beta} - 4(b \cdot x)b_{\alpha}x_{\beta} \right. \\ &\quad + 4b^2x^2b_{\alpha}x_{\beta} + 4b^2(b \cdot x)x_{\alpha}x_{\beta} - 4b^4x^2x_{\alpha}x_{\beta} + 2b_{\beta}x_{\alpha} - 2b_{\beta}b_{\alpha}x^2 \\ &\quad - 2b^2x_{\alpha}x_{\beta} + 2b^2b_{\beta}x^2x_{\alpha} - 4(b \cdot x)b_{\beta}x_{\alpha} + 4b^2x^2b_{\beta}x_{\alpha} \\ &\quad \left. + 4b^2(b \cdot x)x_{\alpha}x_{\beta} - 4b^4x^2x_{\alpha}x_{\beta} \right\} \\ &\quad + \frac{\{x^2 - 2(b \cdot x)x^2 + b^2x^4\} \{4b_{\alpha}b_{\beta} - 4b^2b_{\beta}x_{\alpha} - 4b^2b_{\alpha}x_{\beta} + 4b^4x_{\alpha}x_{\beta}\}}{\Lambda^4} \end{aligned}$$

$$\begin{aligned}
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3} \\
&\quad \times \left(\cancel{-2b_\beta x_\alpha} - \cancel{2b_\alpha x_\beta} + \cancel{4b^2 x_\alpha x_\beta} + \cancel{4(b \cdot x) b_\alpha x_\beta} - \cancel{8b^2 (b \cdot x) x_\alpha x_\beta} \right. \\
&\quad - \cancel{2b^2 x^2 b_\beta x_\alpha} - \cancel{2b^2 x^2 b_\alpha x_\beta} + 4b^4 x^2 x_\alpha x_\beta + \cancel{2b_\alpha x_\beta} + \cancel{2b_\beta x_\alpha} \\
&\quad - 4b_\alpha b_\beta x^2 - \cancel{4b^2 x_\alpha x_\beta} + \cancel{2b^2 x^2 x_\alpha b_\beta} + \cancel{2b^2 x^2 x_\beta x_\alpha} - \cancel{4(b \cdot x) b_\alpha x_\beta} \\
&\quad \left. - \cancel{4(b \cdot x) b_\beta x_\alpha} + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha + \cancel{8b^2 (b \cdot x) x_\alpha x_\beta} - 8b^4 x^2 x_\alpha x_\beta \right) \\
&\quad + x^2 \frac{\Lambda}{\Lambda^4} \{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\} \\
&= \frac{g'_{\alpha\beta}}{\Lambda^2} + \frac{1}{\Lambda^3} \{-4b_\alpha b_\beta x^2 + 4b^2 x^2 b_\alpha x_\beta + 4b^2 x^2 b_\beta x_\alpha - 4b^4 x^2 x_\alpha x_\beta\} \\
&\quad + x^2 \frac{1}{\Lambda^3} \{4b_\alpha b_\beta - 4b^2 b_\beta x_\alpha - 4b^2 b_\alpha x_\beta + 4b^4 x_\alpha x_\beta\}
\end{aligned}$$

$$g_{\alpha\beta}(x) = \frac{g'_{\alpha\beta}}{\Lambda^2} \implies g'_{\alpha\beta}(x') = \Lambda^2 g_{\alpha\beta}(x)$$

This yields the final result:

$$g'_{\alpha\beta}(x') = \Lambda^2 g_{\alpha\beta}(x)$$

Jacobian of the Transformation

The following part is taken from ‘‘Conformal Field Theory Primer in $D \geq 3$ ’’ by Andrew Evans, pg 36:

$$\text{Translation: } \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = 1$$

$$\text{Rotation: } \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = 1$$

$$\text{Dilatation: } \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = \lambda^{-d}$$

$$\text{Inversion: } \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = \left(\frac{1}{\tilde{x}^2} \right)^d$$

Since the rest are easier to show, we will only focus on showing the last part:

$$\begin{aligned}
&\frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \frac{1}{\tilde{x}^2} \left[\delta_\nu^\mu - 2 \frac{\tilde{x}^\mu \tilde{x}_\nu}{\tilde{x}^2} \right] \\
\det \left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right) &= \frac{1}{d!} \epsilon_{\mu_1 \mu_2 \dots \mu_d} \epsilon^{\nu_1 \nu_2 \dots \nu_d} \frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}} \frac{\partial x^{\mu_2}}{\partial \tilde{x}^{\nu_2}} \dots \frac{\partial x^{\mu_d}}{\partial \tilde{x}^{\nu_d}} \\
&= \left(\frac{1}{\tilde{x}^2} \right)^d \frac{1}{d!} \epsilon_{\mu_1 \mu_2 \dots \mu_d} \epsilon^{\nu_1 \nu_2 \dots \nu_d} \left[\delta_{\nu_1}^{\mu_1} - 2 \frac{\tilde{x}^{\mu_1} \tilde{x}_{\nu_1}}{\tilde{x}^2} \right] \dots \left[\delta_{\nu_d}^{\mu_d} - 2 \frac{\tilde{x}^{\mu_d} \tilde{x}_{\nu_d}}{\tilde{x}^2} \right] \\
&= \left(\frac{1}{\tilde{x}^2} \right)^d \frac{1}{d!} \epsilon_{\mu_1 \mu_2 \dots \mu_d} \epsilon^{\nu_1 \nu_2 \dots \nu_d} \prod_{i=1}^d \delta_{\nu_i}^{\mu_i} - 2 \sum_{j=1}^d \frac{1}{d!} \epsilon_{\mu_1 \dots \mu_d} \epsilon^{\nu_1 \dots \nu_d} \frac{\tilde{x}^{\mu_j} \tilde{x}_{\nu_j}}{\tilde{x}^2} \prod_{\substack{i=1 \\ i \neq j}}^d \delta_{\nu_i}^{\mu_i} + 0
\end{aligned}$$

Now we use the identity

$$\epsilon_{i_1 \dots i_k i_{k+1} \dots i_n} \epsilon^{i_1 \dots i_k j_{k+1} \dots j_n} = \delta_{i_1 \dots i_k i_{k+1} \dots i_n}^{i_1 \dots i_k j_{k+1} \dots j_n} = k! \delta_{i_{k+1} \dots i_n}^{j_{k+1} \dots j_n}$$

which in our case becomes

$$\epsilon_{\mu_1 \mu_2 \dots \mu_d} \epsilon^{\nu_1 \nu_2 \dots \nu_d} \prod_{\substack{i=1 \\ i \neq j}}^d \delta_{\nu_i}^{\mu_i} = (d-1)! \delta_{\nu_j}^{\mu_j}$$

Hence

$$\begin{aligned} \det \left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right) &= \left(\frac{1}{\tilde{x}^2} \right)^d \left(\frac{d! - 2 \sum_{j=1}^d (d-1)!}{d!} \right) \\ &= \left(\frac{1}{\tilde{x}^2} \right)^d \left(1 - \frac{2d(d-1)!}{d!} \right) = - \left(\frac{1}{\tilde{x}^2} \right)^d \end{aligned}$$

How distances transform

Under translation

$$x'^\mu = x^\mu + a^\mu$$

So,

$$x'_a{}^\mu - x'_b{}^\mu = x_a^\mu + a^\mu - x_b^\mu - a^\mu = x_a^\mu - x_b^\mu$$

Thus, the distances are invariant under translation:

$$|x'_a - x'_b| = |x_a - x_b|$$

Under dilatation

$$x'^\mu = (1 + \lambda)x^\mu$$

So,

$$\begin{aligned} x'_a{}^\mu - x'_b{}^\mu &= (1 + \lambda)x_a^\mu - (1 + \lambda)x_b^\mu \\ &= (1 + \lambda)(x_a^\mu - x_b^\mu) \end{aligned}$$

We find that the distances between two point scales under dilatation, therefore the natural quantity which is invariant under both translation and dilatation is

$$\frac{|x'_a{}^\mu - x'_b{}^\mu|}{|x'_c{}^\mu - x'_d{}^\mu|} = \frac{\cancel{1+\lambda} |x_a^\mu - x_b^\mu|}{\cancel{1+\lambda} |x_c^\mu - x_d^\mu|} = \frac{|x_a^\mu - x_b^\mu|}{|x_c^\mu - x_d^\mu|}$$

Under special conformal transformation

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(x \cdot b) + b^2 x^2} = \frac{x^\mu - b^\mu x^2}{\Lambda^2(x)}$$

So,

$$x'_a{}^\mu = \frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)}, \quad x'_b{}^\mu = \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}$$

and,

$$x'_a{}^\mu - x'_b{}^\mu = \frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)}$$

squaring both sides

$$\begin{aligned} (x'_a{}^\mu - x'_b{}^\mu)^2 &= \left(\frac{x_a^\mu - b^\mu x_a^2}{\Lambda^2(x_a)} - \frac{x_b^\mu - b^\mu x_b^2}{\Lambda^2(x_b)} \right)^2 \\ &= \frac{x_a^2 + b^2(x_a^2)^2 - 2x_a^2(x_a \cdot b)}{\Lambda^4(x_a)} + \frac{x_b^2 + b^2(x_b^2)^2 - 2x_b^2(x_b \cdot b)}{\Lambda^4(x_b)} \\ &\quad - \frac{2}{\Lambda^2(x_a)\Lambda^2(x_b)} [x_a \cdot x_b - x_b^2(x_a \cdot b) - x_a^2(b \cdot x_b) + b^2 x_a^2 x_b^2] \end{aligned}$$

$$\begin{aligned}
&= x_a^2 \left[\frac{\overbrace{1 - 2(x_a \cdot b)}^{1 - \Lambda^2(x_b)} + \overbrace{2(b \cdot x_b) - b^2 x_b^2}^{1 - \Lambda^2(x_a)}}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)} \right] + x_b^2 \left[\frac{1 - 2(x_b \cdot b)}{\Lambda^4(x_b)} + \frac{2(b \cdot x_a) - b^2 x_a^2}{\Lambda^2(x_a)\Lambda^2(x_b)} \right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2 \left[\frac{1 - 2(x_a \cdot b) - \Lambda^2(x_a)}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)} \right] + x_b^2 \left[\frac{1 - 2(x_b \cdot b) - \Lambda^2(x_b)}{\Lambda^4(x_b)} + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)} \right] \\
&\quad + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)} \left] + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= x_a^2 \left[\frac{-b^2 x_a^2}{\Lambda^4(x_a)} + \frac{1}{\Lambda^2(x_a)\Lambda^2(x_b)} \right] + x_b^2 \left[\frac{-b^2 x_b^2}{\Lambda^4(x_b)} + \frac{1}{\Lambda^2(x_b)\Lambda^2(x_b)} \right] \\
&\quad + \frac{b^2(x_a^2)^2}{\Lambda^4(x_a)} + \frac{b^2(x_b^2)^2}{\Lambda^4(x_b)} - \frac{2x_a \cdot x_b}{\Lambda^2(x_a)\Lambda^2(x_b)} \\
&= \frac{(x_a - x_b)^2}{\Lambda^2(x_a)\Lambda^2(x_b)}
\end{aligned}$$

Thus, we find that the ratio of distances are not invariant under SCT.

$$\frac{|x'_a{}^\mu - x'_b{}^\mu|}{|x_a - x_b|} = \frac{1}{\Lambda(x_a)\Lambda(x_b)}$$

where $\Lambda(x_a) = \sqrt{1 - 2(x_a \cdot b) + b^2 x_a^2}$. We can however, construct another quantity which is invariant under SCT.

$$\frac{|x'_a - x'_b| |x'_d - x'_c|}{|x'_b - x'_d| |x'_c - x'_a|} = \frac{\frac{|x_a - x_b|}{\Lambda(x_a)\Lambda(x_b)} \frac{|x_d - x_c|}{\Lambda(x_d)\Lambda(x_c)}}{\frac{|x_b - x_d|}{\Lambda(x_b)\Lambda(x_d)} \frac{|x_c - x_a|}{\Lambda(x_c)\Lambda(x_a)}} = \frac{|x_a - x_b| |x_d - x_c|}{|x_b - x_d| |x_c - x_a|}$$

Such expressions are called, *anharmonic ratios* or *cross-ratios*.

2.6 Lie Algebra of Generators

$$\begin{aligned}
[P_\mu, P_\nu] &= [-i\partial_\mu, -i\partial_\nu] \\
&= -[\partial_\mu, \partial_\nu] = 0
\end{aligned}$$

Some useful identities

$$\begin{aligned}
[x_\alpha, \partial_\beta]f &= x_\alpha \partial_\beta f - \underbrace{\partial_\beta(x_\alpha f)}_{(\partial_\beta x_\alpha)f + x_\alpha \partial_\beta f} \\
&= x_\alpha \partial_\beta f - x_\alpha \partial_\beta f - (\partial_\beta x_\alpha)f \\
&= -(\partial_\beta x_\alpha)f \\
[x_\alpha, \partial_\beta] &= -\partial_\beta x_\alpha = -g_{\beta\alpha} \partial^\mu x_\alpha = g_{\beta\alpha}
\end{aligned} \tag{2.31}$$

next is,

$$\begin{aligned}
[x^2, \partial_\beta] &= [x^\alpha x_\alpha, \partial_\beta] \\
&= x^\alpha [x_\alpha, \partial_\beta] + [x^\alpha, \partial_\beta] x_\alpha \\
&= -x^\alpha g_{\beta\alpha} - \delta_\beta^\alpha x_\alpha \\
&= -x_\beta - x_\beta \\
&= -2x_\beta
\end{aligned} \tag{2.32}$$

and the last one is,

$$[x_\mu x^\nu, \partial_\beta] = x_\mu [x^\nu, \partial_\beta] + [x_\mu, \partial_\beta] x^\nu = -x_\nu \delta_\beta^\nu - x^\nu g_{\beta\alpha} \quad (2.33)$$

We will now consider, the lie algebra of different operators one by one.

$$\begin{aligned} [P_\mu, D] &= [-i\partial_\mu, -ix^\alpha \partial_\alpha] \\ &= -[\partial_\mu, x^\alpha \partial_\alpha] \\ &= -x^\alpha [\partial_\mu, \partial_\alpha] - [\partial_\mu, x^\alpha] \partial_\alpha \\ &= -\delta_\mu^\alpha \partial_\alpha = -\partial_\mu = -i(-i\partial_\mu) \\ &= -iP_\mu \end{aligned}$$

$$\begin{aligned} [P_\mu, L_{\alpha\beta}] &= [-i\partial_\mu, -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)] \\ &= -[\partial_\mu, x_\alpha \partial_\beta - x_\beta \partial_\alpha] \\ &= -[\partial_\mu, x_\alpha] \partial_\beta + [\partial_\mu, x_\beta] \partial_\alpha \\ &= g_{\alpha\mu} \partial_\beta - g_{\beta\mu} \partial_\alpha \\ &= i(g_{\alpha\mu} P_\beta - g_{\beta\mu} P_\alpha) \end{aligned}$$

$$\begin{aligned} [P_\mu, K_\nu] &= [-i\partial_\mu, -i(2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\nu)] \\ &= -[\partial_\mu, 2x_\nu x^\alpha \partial_\alpha - x^2 \partial_\nu] \\ &= -2x_\nu x^\alpha [\partial_\mu, \partial_\alpha] - 2[\partial_\mu, x_\nu x^\alpha] \partial_\alpha + x^2 [\partial_\mu, \partial_\nu] + [\partial_\mu, x^2] \partial_\nu \\ &= -2[\partial_\mu, x_\nu x^\alpha] \partial_\alpha + [\partial_\mu, x^2] \partial_\nu \\ &= -2(g_{\mu\nu} x^\alpha + \delta_\mu^\alpha x_\nu) \partial_\alpha + 2x_\mu \partial_\nu \\ &= -2g_{\mu\nu} x^\alpha \partial_\alpha - 2(x_\nu \partial_\mu - x_\mu \partial_\nu) \\ &= -2ig_{\mu\nu} D - 2iL_{\mu\nu} \\ &= -2i(g_{\mu\nu} D - L_{\mu\nu}) \end{aligned}$$

$$\begin{aligned} [D, K_\mu] &= -[x^\alpha \partial_\alpha, 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu] \\ &= -2[x^\alpha \partial_\alpha, x_\mu x^\beta \partial_\beta] + [x^\alpha \partial_\alpha, x^2 \partial_\mu] \\ &= -2\{x^\alpha [\partial_\alpha, x_\mu x^\beta] \partial_\beta + x_\mu x^\beta [x^\alpha, \partial_\beta] \partial_\alpha\} \\ &\quad + x^\alpha [\partial_\alpha, x^2] \partial_\mu + x^2 [x^\alpha, \partial_\mu] \partial_\alpha \\ &= -2\{x^\alpha (g_{\alpha\mu} x^\beta + \delta_\alpha^\beta x_\mu) \partial_\beta + x_\mu x^\beta (-\delta_\beta^\alpha) \partial_\alpha\} \\ &\quad + 2x^2 \partial_\mu - \cancel{x^\alpha x^2 \partial_\alpha \partial_\mu} + \cancel{x^2 x^\alpha \partial_\alpha \partial_\mu} - x^2 \partial_\mu \\ &= -\cancel{2x_\mu x^\beta \partial_\beta} - 2x^\beta x_\mu \partial_\beta + \cancel{2x_\mu x^\beta \partial_\beta} + x^2 \partial_\mu \\ &= -(2x^\beta x_\mu \partial_\beta - x^2 \partial_\mu) \\ &= -iK_\mu \end{aligned}$$

$$\begin{aligned} [K_\mu, L_{\alpha\beta}] &= [-i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu), i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)] \\ &= [2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu, x_\alpha \partial_\beta - x_\beta \partial_\alpha] \\ &= 2[x_\mu x^\nu \partial_\nu, x_\alpha \partial_\beta] - [x^2 \partial_\mu, x_\alpha \partial_\beta] + 2[x_\mu x^\nu \partial_\nu, x_\beta \partial_\alpha] - [x^2 \partial_\mu, x_\beta \partial_\alpha] \\ &= 2\{x_\mu x^\nu [\partial_\nu, x_\alpha] \partial_\beta + x_\alpha [x_\mu x^\nu, \partial_\beta] \partial_\nu\} \\ &\quad - x^2 [\partial_\mu, x_\alpha] \partial_\beta - x_\alpha [x^2, \partial_\beta] \partial_\mu \end{aligned}$$

$$\begin{aligned}
& - \underbrace{(\alpha \leftrightarrow \beta)}_{\text{swap}} \\
& = \cancel{2x_\mu x^\nu (g_{\nu\alpha})} \cancel{\partial_\beta} - 2x_\alpha (g_{\mu\beta} x^\nu + \cancel{\delta_\nu^\beta x_\mu}) \partial_\nu - x^2 g_{\mu\alpha} \partial_\beta + 2x_\alpha x_\beta \partial_\mu - (\alpha \leftrightarrow \beta) \\
& = -2x_\alpha g_{\mu\beta} x^\nu \partial_\nu - x^2 g_{\mu\alpha} \partial_\beta + \cancel{2x_\alpha x_\beta \partial_\mu} + 2x_\beta g_{\mu\alpha} x^\nu \partial_\nu + x^2 g_{\mu\beta} \partial_\alpha - \cancel{2x_\beta x_\alpha \partial_\mu} \\
& = -g_{\mu\beta} (2x_\alpha x^\nu \partial_\nu - x^2 \partial_\alpha) + g_{\mu\alpha} (2x_\beta x^\nu \partial_\nu - x^2 \partial_\beta) \\
& = i g_{\mu\alpha} K_\beta - i g_{\mu\beta} K_\alpha = i (g_{\mu\alpha} K_\beta - g_{\mu\beta} K_\alpha)
\end{aligned}$$

$$\begin{aligned}
[K_\mu, K_\nu] & = -[2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu, 2x_\nu x^\beta \partial_\beta - x^2 \partial_\nu] \\
& = -4[x_\mu x^\alpha \partial_\alpha, x_\nu x^\beta \partial_\beta] + 2[x_\mu x^\alpha \partial_\alpha, x^2 \partial_\nu] \\
& \quad + 2[x^2 \partial_\mu, x_\nu x^\beta \partial_\beta] - [x^2 \partial_\mu, x^2 \partial_\nu] \\
& = -4[x_\mu x^\alpha \partial_\alpha, x_\nu x^\beta \partial_\beta] + 2[x_\mu x^\alpha \partial_\alpha, x^2 \partial_\nu] + 2[x_\mu x^\alpha, x^2 \partial_\nu] \partial_\alpha + 2x_\mu x^\alpha [x^2, \partial_\nu] \partial_\alpha \\
& \quad - [x^2 \partial_\mu, x_\nu x^\beta \partial_\beta] - x_\nu x^\beta [x^2, \partial_\beta] \partial_\mu - [x^2 \partial_\mu, x^2 \partial_\nu] - x^2 [x^2, \partial_\nu] \partial_\mu
\end{aligned}$$

After expanding all commutators:

$$\begin{aligned}
& = \cancel{4x_\mu x^\alpha \partial_\alpha x_\nu x^\beta \partial_\beta} - \cancel{4x_\nu x^\beta \partial_\beta x_\mu x^\alpha \partial_\alpha} + \cancel{2x_\mu x^\alpha \partial_\alpha x^2 \partial_\nu} - \cancel{2x^2 \partial_\nu x_\mu x^\alpha \partial_\alpha} \\
& \quad + \cancel{2x^2 \partial_\mu x_\nu x^\beta \partial_\beta} - \cancel{2x_\nu x^\beta \partial_\beta x^2 \partial_\mu} - \cancel{x^2 \partial_\mu x^2 \partial_\nu} + \cancel{x^2 \partial_\nu x^2 \partial_\mu} \\
& = \cancel{4x_\mu x^2 \partial_\nu} - 2x_\mu x^2 \partial_\nu + \cancel{2x_\nu x^2 \partial_\mu} - 4x_\nu x^2 \partial_\mu - \cancel{2x^2 x_\mu \partial_\nu} + \cancel{2x^2 x_\nu \partial_\mu} \\
& = 0
\end{aligned}$$

Next, we will see that Conformal Algebra in d dimensions is isomorphic to the Lie algebra of the Lorentz group in $d + 2$ dimensions, any conformal covariant correlator in d dimensions should be obtainable from Lorentz covariant expressions in $d + 2$ dimensions via some kind of dimensional reduction procedure. This is essentially the idea behind **Embedding Formalism**. We define the following set of new operators:

$$\begin{aligned}
J_{\mu\nu} & = L_{\mu\nu} \\
J_{0,\mu} & = \frac{1}{2}(P_\mu + K_\mu) \\
J_{-1,\mu} & = \frac{1}{2}(P_\mu - K_\mu) \\
J_{-1,0} & = D
\end{aligned}$$

with the property that

$$J_{ab} = -J_{ba}$$

where

$$a, b \in \{-1, 0, 1, \dots, \underbrace{}_d\}$$

d is dimension of spacetime

These new generators, obey $SO(d + 1, 1)$ lie algebra:

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{bc} J_{ad}) \quad (2.34)$$

In this section, we will explicitly assume the form of flat metric as being euclidean, and given as:

$$g_{\mu\nu} = \eta_{\mu\nu} = \underbrace{(1, 1, \dots, 1)}_d$$

Our metric in (2.34) would be given as:

$$\eta_{ab} = (-1, \underbrace{1, 1, \dots, 1}_{\mu, \nu}) \quad \boxed{\eta_{00} = 1} \quad \boxed{\eta_{-1-1} = -1} \quad (2.35)$$

If our original metric was Minkowski, we would have had:

$$\eta_{ab} = (-1, 1, \underbrace{-1, \dots, 1}_d)$$

We will now check, if (2.34) holds true:

$$\begin{aligned} [J_{\mu\nu}, J_{0,\alpha}] &= \left[L_{\mu\nu}, \frac{1}{2}(P_\alpha + K_\alpha) \right] \\ &= \frac{1}{2}[L_{\mu\nu}, P_\alpha] + \frac{1}{2}[L_{\mu\nu}, K_\alpha] \\ &= -\frac{1}{2}[P_\alpha, L_{\mu\nu}] - \frac{1}{2}[K_\alpha, L_{\mu\nu}] \\ &= -\frac{1}{2}(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) - \frac{1}{2}(\eta_{\alpha\mu}K_\nu - \eta_{\alpha\nu}K_\mu) \\ &= -\eta_{\alpha\mu} \left[\frac{1}{2}(P_\nu + K_\nu) \right] + \eta_{\alpha\nu} \left[\frac{1}{2}(P_\mu + K_\mu) \right] \\ &= -i\eta_{\alpha\mu}J_{0,\nu} + i\eta_{\alpha\nu}J_{0,\mu} \end{aligned}$$

$$\begin{aligned} [J_{0,\mu}, J_{-1,0}] &= \left[\frac{1}{2}(P_\mu + K_\mu), D \right] \\ &= \frac{1}{2}[P_\mu, D] + \frac{1}{2}[K_\mu, D] \\ &= -\frac{1}{2}iP_\mu - \frac{1}{2}(-iK_\mu) = \frac{-i}{2}(P_\mu - K_\mu) = -iJ_{-1,\mu} \end{aligned}$$

If we assume that the metric in (2.34) is indeed given by (2.35). Then, the algebra (2.34) holds true. This shows the isomorphism between the conformal group of d -dimensional Euclidean space and the $SO(d+1, 1)$ group of $d+2$ dimensional Minkowski spacetime with $\frac{1}{2}(d+1)(d+2)$ parameters.

Conformal Generators on the Field

Finite form of conformal transformation ($x' = \Lambda x$)³

$$\Phi'_a(x') = U(\Lambda)\Phi_a(x)U^{-1}(\Lambda)$$

$$\begin{aligned} \Phi'_a(\Lambda x) &= \sum_b \pi_{ab}(\Lambda)\Phi_b(x) \\ &= \pi_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x') \\ &= \pi_{ab}(e^{i\omega_g c_g})\Phi_b(e^{-i\omega_g c_g}x') \end{aligned}$$

(2.36)

We have dropped the \sum sign and summation over repeated indices are implied. Infinitesimal form of (2.36):

generator only acting on field

$$\begin{aligned} \Phi'_a(x') &= (1 - i\omega_g T_g)_{ab}\Phi_b(\Lambda^{-1}x') \quad \text{generator which only acts on } x'^\mu \\ &= (1 - i\omega_g T_g)_{ab} \underbrace{\Phi_b[(1 - i\omega_g c_g)x'^\mu]}_{\Phi_b(x') + \{(1 - i\omega_g c_g)x'^\mu - x'^\mu\}\partial_\mu \Phi_b(x')} \\ &= (1 - i\omega_g T_g)_{ab}[\Phi_b(x') - i\omega_g c_g x'^\mu \partial_\mu \Phi_b(x')] \end{aligned}$$

³tobias osborne's lecture notes pg 18

$$\Phi'(x') = \Phi(x') - i\omega_g \underbrace{\left[T_g + c_g x'^{\mu} \frac{\partial}{\partial x'^{\mu}} \right]}_{\substack{\text{accounts for change} \\ \text{in argument of field}}} \Phi(x') + \mathcal{O}(\omega_g^2)$$

However, we will not use this approach but rather we will consider the transformations at origin and then translate it to every other point. This approach is based on studying the stabilizer subgroup of the Conformal Symmetry.⁴ So, if we study the same at origin:

$$\Phi'(0) = \Phi(x') - i\omega_g \left[T_g + c_g x'^{\mu} \frac{\partial}{\partial x'^{\mu}} \right] \Phi(x') \Big|_{x'=0} = \Phi(0) - i\omega_g T_g \Phi(0)$$

using translation operator

$$e^{ixP} \Phi'(0) e^{-ix^{\alpha} P_{\alpha}} = e^{ixP} \Phi(0) e^{-ix^{\alpha} P_{\alpha}} - e^{ixP} i\omega_g T_g \Phi(0) e^{-ix^{\alpha} P_{\alpha}}$$

$$\begin{aligned} \Phi'(x) &= \Phi(x) - e^{ixP} i\omega_g T_g e^{-ixP} \underbrace{e^{ix^{\beta} P_{\beta}} \Phi(0) e^{-ix^{\alpha} P_{\alpha}}}_{\Phi(x)} \\ &= \Phi(x) - i\omega_g \underbrace{e^{ixP} T_g e^{-ixP}}_{\text{we will find these "translated operators" later}} \Phi(x) \end{aligned}$$

For translation

$$\begin{aligned} \Phi'(x+a) &= e^{iaP} \Phi(x) e^{-ia^{\alpha} P_{\alpha}} \\ &= e^{ia[P, \cdot]} \Phi(x) \quad \text{using (2.38)} \\ &= e^{a \cdot \partial} \Phi(x) \end{aligned}$$

For rotation, at $x'^{\mu} = 0 \implies x^{\mu} = 0$

$$\Phi'_a(0) = \pi_{ab}(\Lambda) \Phi_b(\Lambda^{-1}0) = \pi_{ab}(\Lambda) \Phi_b(0)$$

Now, assuming the generator of rotation $T_g = L_{\mu\nu}$ acts like⁵

$$L_{\mu\nu} \Phi_a(0) = S_{\mu\nu} \Phi_a(0) \quad (2.37)$$

at origin. At any other point, it will behave as:

$$\begin{aligned} L_{\mu\nu} \Phi_a(x) &= e^{ix^{\beta} P_{\beta}} L_{\mu\nu} \Phi_a(0) e^{-ix^{\alpha} P_{\alpha}} \\ &= \underbrace{e^{ix^{\beta} P_{\beta}} L_{\mu\nu} e^{-ixP}}_{?} \underbrace{e^{ixP} \Phi_a(0) e^{-ix^{\alpha} P_{\alpha}}}_{\Phi_a(x)} \end{aligned}$$

by taking the derivative of second term, we obtain the following commutator:

$$\begin{aligned} \Phi_a(x) &= e^{ixP} \Phi_a(0) e^{-ix^{\alpha} P_{\alpha}} \\ \partial_{\mu} \Phi_a(x) &= (\partial_{\mu} e^{ixP}) \Phi_a(0) e^{-ix^{\alpha} P_{\alpha}} + e^{ixP} \Phi_a(0) (\partial_{\mu} e^{-ix^{\alpha} P_{\alpha}}) \\ &= iP^{\mu} e^{ixP} \Phi_a(0) e^{-ix^{\alpha} P_{\alpha}} + e^{ixP} \Phi_a(0) e^{-ix^{\alpha} P_{\alpha}} (-iP^{\mu}) \\ &= iP^{\mu} \Phi_a(x) - i\Phi_a(x) P^{\mu} \\ &= i[P^{\mu}, \Phi_a(x)] \\ \partial_{\mu} \Phi_a(x) &= i[P^{\mu}, \Phi_a(x)] \quad (2.38) \end{aligned}$$

We will now derive the form of $e^{ix^{\beta} P_{\beta}} L_{\mu\nu} e^{-ixP}$.⁶

$$\begin{aligned} e^{ix^{\beta} P_{\beta}} L_{\mu\nu} e^{-ixP} &= L_{\mu\nu} + [L_{\mu\nu}, -ix^{\alpha} P_{\alpha}] + \frac{1}{2!} [[L_{\mu\nu}, -ix^{\alpha} P_{\alpha}], -ix^{\alpha} P_{\alpha}] + \dots \\ &= L_{\mu\nu} + ix^{\alpha} \underbrace{[P_{\alpha}, L_{\mu\nu}]}_{i(g_{\alpha\mu} P_{\nu} - g_{\alpha\nu} P_{\mu})} + \dots \\ &= L_{\mu\nu} + i^2 x^{\alpha} (g_{\alpha\mu} P_{\nu} - g_{\alpha\nu} P_{\mu}) \end{aligned}$$

⁴pg 7 of "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications"

⁵pg 10, paragraph 2 of "Conformal field theory in momentum space and anomaly actions in gravity The analysis of three- and four-point functions"

⁶using BCH lemma $e^A B e^{-A} = e^{[A, \cdot]} B$

$$\begin{aligned}
&= L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \\
&= L_{\mu\nu} + \underbrace{i(x_\mu \partial_\nu - x_\nu \partial_\mu)}_{\text{we found in section 1.2}}
\end{aligned}$$

we know, at $x' = 0$ we have $L_{\mu\nu} = S_{\mu\nu}$, so for the sake of consistency we get:

$$e^{ix^\beta P_\beta} L_{\mu\nu} e^{-ix^P} = \underbrace{S_{\mu\nu}}_{\text{Spin Operator}} + \underbrace{i(x_\mu \partial_\nu - x_\nu \partial_\mu)}_{\text{transforms the argument of field}}$$

The exponential map of above can be found in any textbook on QFT which describes rotation or Lorentz transformation.⁷ If we ignore $S_{\mu\nu}$, then we can see how the last part acts on field:

$$\begin{aligned}
x'^\mu &= \left(\delta_\nu^\mu + \frac{i}{2} \omega_{\alpha\beta} L^{\alpha\beta} \right) x^\nu \\
&= x^\mu + \frac{i}{2} \omega_{\alpha\beta} i(x^\alpha \partial^\beta - x^\beta \partial^\alpha) x^\nu \\
\Phi'(x) &= \Phi(x) - \frac{i}{2} \omega_{\alpha\beta} L^{\alpha\beta} \Phi(x) \\
&= \Phi(x) - \frac{i}{2} \omega_{\alpha\beta} i(x^\alpha \partial^\beta - x^\beta \partial^\alpha) \Phi(x) \\
&= \Phi(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha g^{\beta\gamma} \partial - x^\beta g^{\alpha\gamma} \partial) \Phi(x) \\
&= \Phi(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) x \partial \Phi(x) \\
&= \Phi(x) - \frac{i}{2} \omega_{\alpha\beta} L^{\alpha\beta} x \cdot \partial \Phi(x) \\
&\approx \Phi \left(x^\mu - \frac{i}{2} \omega_{\alpha\beta} L^{\alpha\beta} x^\nu \right) \\
&\quad \Phi'(x') = \Phi(x)
\end{aligned}$$

For dilatation, at $x'^\mu = 0$, $x'^\mu = (1 + \lambda)x^\mu = 0 \implies x^\mu = 0$. We have $\omega_g = \lambda$ and $T_g = D$:

$$D\Phi_a(0) = \tilde{\Delta}\Phi_a(0) \quad (2.39)$$

corresponding commutator (by operating it on eigenstate of dilatation)

$$D|\Delta\rangle = [D, \Phi_\Delta(0)]|0\rangle + \Phi_\Delta(0)D|0\rangle$$

$$\tilde{\Delta}|\Delta\rangle = [D, \Phi_\Delta(0)]|0\rangle + 0$$

$$\tilde{\Delta}\Phi_\Delta(0)|0\rangle = [D, \Phi_\Delta(0)]|0\rangle$$

Applying the same procedure, we consider:

$$\begin{aligned}
e^{ix^\beta P_\beta} D e^{-ix^P} &= D + [D, -ix^\beta P_\beta] + \frac{1}{2!} [[D, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= D - ix^\alpha (iP_\alpha) \\
&= D + x^\alpha P_\alpha \\
&= D - ix^\alpha \partial_\alpha
\end{aligned} \quad (2.40)$$

for the sake of consistency at $x' = 0$

$$= \tilde{\Delta} - ix^\alpha \partial_\alpha$$

Now, we consider

$$D\Phi_a(x) = (\tilde{\Delta} - ix^\alpha \partial_\alpha) \Phi_a(x)$$

redefining $\tilde{\Delta} \equiv -i\Delta$, we get

$$D\Phi_a(x) = -i(\Delta + x^\alpha \partial_\alpha) \Phi_a(x)$$

⁷check eqn 1.141 and 1.150 of “QFT in curved spacetime” by Leonard Parker

Similarly,⁸

$$\begin{aligned}
[D, \Phi_a(x)] &= D e^{ix^\beta P_\beta} \Phi_a(0) e^{-ixP} - e^{ix^\beta P_\beta} \Phi_a(0) e^{-ixP} D \\
&= e^{ixP} \underbrace{e^{ix^\alpha P_\alpha} D e^{-ix^\beta P_\beta}}_{=D+x^\alpha P_\alpha} \Phi_a(0) e^{-ixP} - e^{ix^\beta P_\beta} \Phi_a(0) \underbrace{e^{-ixP} D e^{-ix^\alpha P_\alpha}}_{=D+x^\alpha P_\alpha} e^{-ixP} \\
&= e^{ix^\beta P_\beta} [D + x^\alpha P_\alpha, \Phi_a(0)] e^{-ixP} \\
&= e^{ix^\beta P_\beta} \underbrace{[D, \Phi_a(0)]}_{\tilde{\Delta}\Phi_a(0)} e^{-ixP} + e^{ix^\beta P_\beta} \underbrace{[x^\alpha P_\alpha, \Phi_a(0)]}_{=x^\alpha [P_\alpha, \Phi_a(0)]} e^{-ixP} \\
&= \tilde{\Delta}\Phi_a(x) - ix \cdot \partial\Phi_a(x) \\
&= -i(\Delta + x \cdot \partial)\Phi_a(x)
\end{aligned}$$

Finite Dilatation,⁹ we consider

$$Dx^\mu = -ix \cdot \partial x^\mu$$

$$x' = e^\lambda x = e^{i\lambda D} x = \left(1 + i\frac{\lambda}{N}D\right) \dots \left(1 + i\frac{\lambda}{N}D\right) x$$

then at origin, the field transforms (active transformation) as:

$$\begin{aligned}
\Phi'_a(0) &= \left(1 + i\frac{\lambda}{N}D\right) \dots \left(1 + i\frac{\lambda}{N}D\right) \Phi_a(0) \\
&= \left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right) \dots \left(1 + i\frac{\lambda}{N}\tilde{\Delta}_a\right) \Phi_a(0) \quad (\text{using } D\Phi(0) = \tilde{\Delta}\Phi) \\
&= e^{i\lambda\tilde{\Delta}_a} \Phi_a(0) \\
&= e^{-\lambda\Delta_a} \Phi_a(0)
\end{aligned}$$

In passive transformation

$$\begin{aligned}
\Phi'_a(0) &= \left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right) \dots \left(1 - i\frac{\lambda}{N}\tilde{\Delta}_a\right) \Phi_a(0) \\
&= e^{-i\lambda\tilde{\Delta}_a} \Phi_a(0) \\
&= e^{\lambda\Delta} \Phi_a(0)
\end{aligned}$$

For arbitrary point (ignoring the change in argument of field and thus generator c_g):

$$\begin{aligned}
\Phi'_a(x') &= \pi_{ab}(e^{i\lambda D})\Phi_b(x) \\
&= [e^{i\lambda\tilde{\Delta}}]_{ab}\Phi_b(x)
\end{aligned}$$

$$\Phi'_a(e^\lambda x) = [e^{-\lambda\Delta}]_{ab}\Phi_b(x) = e^{-\lambda\Delta}\Phi_a(x)$$

For SCT, $x'^\mu = 0 \implies x^\mu = 0$. Hence, we will consider the same equations, but in this context:

$$K_\mu \Phi_a(0) = \kappa_\mu \Phi_a(0)$$

Again, applying the same procedure,

$$\begin{aligned}
e^{ix^\beta P_\beta} K_\mu e^{-ixP} &= K_\mu + [K_\mu, -ix^\beta P_\beta] + \frac{1}{2!} [[K_\mu, -ix^\beta P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu - ix^\beta [K_\mu, P_\beta] + \frac{1}{2} [-ix^\beta [K_\mu, P_\beta], -ix^\beta P_\beta] + \dots \\
&= K_\mu + 2x^\beta (g_{\mu\beta} D - L_{\mu\beta}) + \frac{1}{2} [2x^\beta (g_{\mu\beta} D - L_{\mu\beta}), -ix^\alpha P_\alpha] \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} - ix_\mu x^\alpha [D, P_\alpha] + ix^\beta x^\alpha \underbrace{[L_{\mu\beta}, P_\alpha]}_{-i(g_{\alpha\mu} P_\beta - g_{\alpha\beta} P_\mu)}
\end{aligned}$$

⁸from pg 31 of 2309.10107, and x is not an operator here but a number

⁹look up Lectures Notes For An Introduction to Conformal Field Theory A Course Given By Dr. Tobias Osborne, pg 19

$$\begin{aligned}
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + x_\mu x^\alpha P_\alpha + x_\mu x^\beta P_\beta - x^\alpha x_\alpha P_\mu \\
&= K_\mu + 2x_\mu D - 2x^\beta L_{\mu\beta} + 2x_\mu x^\alpha P_\alpha - x^\alpha x_\alpha P_\mu
\end{aligned}$$

From the generator of dilatation and SCT, we have¹⁰

$$[D, K_\mu] = -iK_\mu \quad \text{at } x'^\mu = 0 \implies [\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu$$

and

$$[D, L_{\mu\nu}] = 0 \quad \text{at } x'^\mu = 0 \implies [\tilde{\Delta}, S_{\mu\nu}] = 0$$

For primary fields:

$$K_\mu \Phi_a(0) = 0$$

Since, for primary field $\tilde{\Delta}$ commutes with all other operators which belong to the stability subgroup. By Schur's lemma $\tilde{\Delta} \propto I$, where I is an identity operator. The SCT and momentum generator acts as ladder operator for Dilatation.

$$\begin{aligned}
[D, [P_\mu, \Phi(0)]] &= [P_\mu, [D, \Phi(0)]] + [[D, P_\mu], \Phi(0)] = -i(\Delta + 1)[P_\mu, \Phi(0)] \\
[D, [K_\mu, \Phi(0)]] &= [K_\mu, [D, \Phi(0)]] + [[D, K_\mu], \Phi(0)] = -i(\Delta - 1)[K_\mu, \Phi(0)]
\end{aligned}$$

Conformal Invariance of Scalar Field

Start from the free, massless scalar action in d dimensions:

$$S[\phi] = \frac{1}{2} \int d^d x (\partial_\mu \phi)(\partial^\mu \phi).$$

Under a scale (dilatation) transformation

$$x \mapsto x' = \lambda x, \quad \lambda > 0,$$

a primary scalar field transforms as

$$\phi(x) \mapsto \phi'(x') = \lambda^{-\Delta} \phi(x),$$

where Δ is the scaling dimension to be determined. The measure scales as¹¹

$$d^d x \mapsto d^d x' = \lambda^d d^d x,$$

and the derivative transforms as

$$\partial_\mu \phi(x) \mapsto \partial'_\mu \phi'(x') = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu (\lambda^{-\Delta} \phi(x)) = \lambda^{-1} \lambda^{-\Delta} \partial_\mu \phi(x) = \lambda^{-(\Delta+1)} \partial_\mu \phi(x).$$

Hence the kinetic density transforms as

$$(\partial\phi)^2 \mapsto \lambda^{-2(\Delta+1)} (\partial\phi)^2,$$

and the full integrand scales by

$$d^d x (\partial\phi)^2 \mapsto \lambda^{d-2(\Delta+1)} d^d x (\partial\phi)^2.$$

Classical scale invariance of the action requires the exponent of λ to vanish:

$$d - 2(\Delta + 1) = 0.$$

Solving for Δ gives

$$\boxed{\Delta = \frac{d}{2} - 1.} \tag{2.41}$$

Finite Conformal Transformation of Fields

We begin by noting that *translation* and *rotation* do not introduce any new thing that we hadn't encountered in QFT, it is only the dilatation which does. Upon exponentiating the infinitesimal

¹⁰same notes, look at eqn 65 to 70 (pg 18-19), all these commutators are for T_g

¹¹the metric and metric determinant do not change due to Weyl scaling, resulting in measure being transformation dependent

dilatation:

$$\begin{aligned}\Phi'(x') &= e^{-i\omega_g[T_g + c_g x'^\mu \frac{\partial}{\partial x'^\mu}]} \Phi(x') \\ &= e^{-i\omega_g T_g} e^{-i\omega_g c_g x' \cdot \frac{\partial}{\partial x'}} \Phi(x') \\ &= e^{-i\omega_g T_g} \Phi(e^{-i\omega_g c_g} x')\end{aligned}$$

This section is taken from “advanced mathematical methods - conformal field theory” by David Duffins.¹²

$$\Phi'_a(x') = U(\Lambda)\Phi_a(x')U^{-1}(\Lambda) = e^{-i\omega_g[T_g, \cdot]} \Phi_a e^{-i\omega_g T_g} \Phi_a(x')$$

For translation

$$\Phi(x) = e^{ixP}\Phi(0)e^{-ix^\alpha P_\alpha} = e^{x\partial}\Phi(0)$$

Or,

$$\begin{aligned}\Phi'(x') &= \Phi(x) \\ &= \Phi(x' - a) \\ &= e^{-a \frac{\partial}{\partial x'}} \Phi(x') \\ &= e^{-iaP} \Phi(x')\end{aligned}$$

For dilatation ($x' = e^\lambda x$)

$$\begin{aligned}\Phi'_a(x') &= e^{-i\lambda D} \Phi_a(x') \\ &= e^{-\lambda(\Delta + x' \cdot \partial)} \Phi_a(x') \\ &= e^{-\lambda\Delta} \underbrace{e^{-\lambda x \cdot \partial} \Phi_a(x')}_{\Phi_a[e^{-\lambda} x']} \\ &= e^{-\lambda\Delta} \Phi_a(x)\end{aligned}$$

The last part could be understood as:

$$\begin{aligned}\Phi_a \left[\left(1 - \frac{\lambda}{N}\right) x \right] &= e^{-\frac{\lambda}{N} x \cdot \partial} \Phi_a(x) \\ \Phi_a \left[\underbrace{\left(1 - \frac{\lambda}{N}\right) \dots \left(1 - \frac{\lambda}{N}\right) x}_{N \text{ terms}} \right] &= e^{-\frac{\lambda}{N} x \cdot \partial} \Phi_a \left[\underbrace{\left(1 - \frac{\lambda}{N}\right) \dots \left(1 - \frac{\lambda}{N}\right) x}_{N-1 \text{ terms}} \right] \\ \Phi_a(e^{-\lambda} x) &= e^{-\lambda x \cdot \partial} \Phi_a(x)\end{aligned}$$

or, alternatively

$$\begin{aligned}e^{-\lambda x \cdot \partial} \Phi_a(x) &= \left(1 - \frac{\lambda}{N} x \cdot \partial\right)^N \Phi_a(x) \\ &= \left(1 - \frac{\lambda}{N} x \cdot \partial\right) \dots \underbrace{\left(1 - \frac{\lambda}{N} x \cdot \partial\right)}_{\Phi_a[(1 - \frac{\lambda}{N})x]} \Phi_a(x) \\ &= \Phi_a \left[\left(1 - \frac{\lambda}{N}\right)^N x \right] = \Phi_a(e^{-\lambda} x)\end{aligned}$$

¹²Active coordinate transformation is given as: $\Phi(x') = U(\Lambda)\Phi(x)U^{-1}(\Lambda)$ whereas passive transformation is given as $\Phi(x') = U^{-1}(\Lambda)\Phi(x)U(\Lambda)$

Chapter 3

Embedding coordinates for Euclidean Space

We start in the embedding space $\mathbb{R}^{d+1,1}$ with coordinates

$$X^{-1}, X^0, \underbrace{X^1, X^2, \dots, X^d}_{X^\mu}$$

To simplify the discussion, it is convenient to introduce null coordinates, $X^M = (X^+, X^-, X^\mu)$, defined as ¹

$$\left. \begin{aligned} X^+ &= X^{-1} + X^0 \\ X^- &= X^{-1} - X^0 \end{aligned} \right\} X^{-1} = \frac{X^+ + X^-}{2}; \quad X^0 = \frac{X^+ - X^-}{2}$$

With this choice, the mostly-plus metric of the embedding space becomes

$$ds^2 = - (dX^{-1})^2 + (dX^0)^2 + \sum_{\mu=1}^d (dX^\mu)^2 = \sum_{\mu=1}^d (dX^\mu)^2 - dX^+ dX^-$$

This can be written in matrix form as

$$\eta_{MN} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & \dots \\ -1/2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}$$

The power of the embedding formalism is that the conformal generators in d -dimensional Euclidean space can be realized as Lorentz generators in $d+2$ dimensions. To make this connection explicit, we introduce new coordinates (ρ, η, x_μ) via the transformation

$$X_{-1} = \frac{\rho(1 - \eta^2 + \vec{x}^2)}{2}$$

$$X_0 = \frac{\rho(1 + \eta^2 - \vec{x}^2)}{2}$$

$$X_\mu = \rho x_\mu$$

This parametrization makes manifest an important redundancy: scaling all embedding coordinates by an overall factor λ corresponds simply to $\rho \mapsto \lambda\rho$. Thus, points related by rescalings $X'^A = \lambda X^A$ correspond to the same physical point in the conformal compactification. To proceed, we invert the transformation to express (ρ, η, x_μ) in terms of X_A :

⁰¹ the index with lowest numeric value has the same sign in both X^\pm . If we had considered, X^{d+1} rather than X^{-1} then the definition would have been something like

$$X^\pm = X^0 \pm X^{d+1}$$

$$\begin{aligned}\rho &= X_{-1} + X_0 \\ \eta &= \frac{\sqrt{\eta_{MN}X^M X^N}}{X_{-1} + X_0} \leftarrow \rho\eta = \sqrt{\eta_{MN}X^M X^N} \\ x_\mu &= \frac{X_\mu}{X_{-1} + X_0}\end{aligned}$$

Next, we compute the change of basis for derivatives. Using the chain rule, one finds

$$\begin{aligned}\frac{\partial}{\partial X_{-1}} &= \frac{\partial}{\partial \rho} - \frac{1 + \eta^2 + \vec{x}^2}{2\rho\eta} \frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho} \frac{\partial}{\partial x_\mu} \\ \frac{\partial}{\partial X_0} &= \frac{\partial}{\partial \rho} + \frac{1 - \eta^2 - \vec{x}^2}{2\rho\eta} \frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho} \frac{\partial}{\partial x_\mu} \\ \frac{\partial}{\partial X_\mu} &= \frac{x_\mu}{\rho\eta} \frac{\partial}{\partial \eta} + \frac{1}{\rho} \frac{\partial}{\partial x_\mu}\end{aligned}$$

The Lorentz generators in $d + 2$ dimensions are

$$J_{MN} = X_M \partial_N - X_N \partial_M$$

By combining them appropriately, we recover the familiar conformal generators in d dimensions.

- Translation

$$\begin{aligned}P_\mu &= J_{-1,\mu} + J_{0,\mu} = (X_{-1} + X_0) \frac{\partial}{\partial X_\mu} - X_\mu \left(\frac{\partial}{\partial X_0} - \frac{\partial}{\partial X_{-1}} \right) \\ &= \rho \left(\frac{x_\mu}{\rho\eta} \frac{\partial}{\partial \eta} + \frac{1}{\rho} \frac{\partial}{\partial x_\mu} \right) - \rho x_\mu \left(\frac{1}{\rho\eta} \frac{\partial}{\partial \eta} \right) \\ &= \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x^\mu}\end{aligned}$$

- Special Conformal Transformation

$$\begin{aligned}K_\mu &= J_{0,\mu} - J_{-1,\mu} = (X_0 - X_{-1}) \frac{\partial}{\partial X_\mu} - X_\mu \left(\frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_{-1}} \right) \\ &= \rho (\eta^2 - \vec{x}^2) \left(\frac{x_\mu}{\rho\eta} \frac{\partial}{\partial \eta} + \frac{1}{\rho} \frac{\partial}{\partial x_\mu} \right) - 2\rho x_\mu \left[\frac{\partial}{\partial \rho} - \left(\frac{\eta^2 + \vec{x}^2}{2\rho\eta} \right) - \frac{x_\mu}{\rho} \frac{\partial}{\partial x_\mu} \right] \\ &= \frac{\eta^2 - \vec{x}^2}{\eta} x_\mu \frac{\partial}{\partial \eta} + (\eta^2 - \vec{x}^2) \frac{\partial}{\partial x_\mu} - 2\rho x_\mu \frac{\partial}{\partial \rho} + x_\mu \frac{\eta^2 + \vec{x}^2}{\eta} \frac{\partial}{\partial \eta} + 2x_\mu (x \cdot \partial) \\ &= 2x_\mu (x \cdot \partial) - \vec{x}^2 \partial_\mu + \eta^2 \partial_\mu - 2\rho x_\mu \frac{\partial}{\partial \rho} + 2x_\mu \eta \frac{\partial}{\partial \eta}\end{aligned}$$

- Dilatation

$$\begin{aligned}D = J_{-10} &= X_{-1} \frac{\partial}{\partial X^0} - X_0 \frac{\partial}{\partial X^{-1}} = X_{-1} \frac{\partial}{\partial X_0} + X_0 \frac{\partial}{\partial X_{-1}} \\ &= \frac{\rho(1 - \eta^2 + \vec{x}^2)}{2} \left[\frac{\partial}{\partial \rho} + \frac{1 - \eta^2 - \vec{x}^2}{2\rho\eta} \frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho} \frac{\partial}{\partial x_\mu} \right] \\ &\quad + \frac{\rho(1 + \eta^2 - \vec{x}^2)}{2} \left[\frac{\partial}{\partial \rho} - \frac{1 + \eta^2 + \vec{x}^2}{2\rho\eta} \frac{\partial}{\partial \eta} - \frac{x_\mu}{\rho} \frac{\partial}{\partial x_\mu} \right] \\ &= \rho \frac{\partial}{\partial \rho} - x^\mu \frac{\partial}{\partial x^\mu} - \eta \frac{\partial}{\partial \eta}\end{aligned}$$

If we now restrict to the null cone $\eta = 0$ and fix ρ (using the scaling redundancy), the generators simplify to their standard flat-space form:

$$\begin{aligned} P_\mu &= \partial_\mu \\ K_\mu &= 2x_\mu(x \cdot \partial) - x^\nu x_\nu \partial_\mu \\ D &= -x^\mu \partial_\mu \end{aligned}$$

Note that conformal algebra is satisfied by both $\pm P_\mu$ and $\pm K_\mu$. The null cone corresponds to $\eta = 0$ but no condition imposed on ρ . Thus, there's a gauge redundancy: different values of ρ acting as scale factor for the coordinates correspond to the same physical point.

Next important to understand now is how the tensor fields transform under conformal transformation. We can use the embedding space to deduce their transformation law which is often more illuminating than the algebra gymnastics.

3.0.1 Tensor field under conformal transformation

The transformation rule for tensor field under conformal transformation is very complicated, therefore we will rely on Embedding formalism to find a simpler form. In this approach, physical spacetime coordinates x^μ are understood as projections from a higher-dimensional embedding space with coordinates $X^A \in \mathbb{R}^{d,2}$, where the conformal group $SO(d, 2)$ acts linearly. Tensors in physical space are then obtained by pulling back embedding space tensors via this projection. Specifically, a tensor in physical space is related to its embedding space counterpart as follows:

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x) = \frac{\partial x^{\mu_1}}{\partial X^{A_1}} \dots \frac{\partial x^{\mu_n}}{\partial X^{A_n}} \frac{\partial X^{B_1}}{\partial x^{\nu_1}} \dots \frac{\partial X^{B_m}}{\partial x^{\nu_m}} T_{B_1 \dots B_m}^{A_1 \dots A_n}(X)$$

The map from embedding to physical space is given by

$$x^\mu = \frac{X^\mu}{X^+} \equiv \frac{X^\mu}{X^0 + X^{-1}}$$

but due to the projective nature of this construction-i.e., physical points correspond to rays in the embedding space - we are free to rescale $X \sim \lambda X$, which is a manifestation of dilatation symmetry. Under this rescaling, the tensor should satisfy

$$T(\lambda X) = \lambda^{-\Delta} T(X)$$

which defines its conformal weight Δ . Using this, the projected tensor can be written as

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x) = \left(\frac{1}{X^+} \right)^\Delta \frac{\partial x^{\mu_1}}{\partial X^{A_1}} \dots \frac{\partial x^{\mu_n}}{\partial X^{A_n}} \frac{\partial X^{B_1}}{\partial x^{\nu_1}} \dots \frac{\partial X^{B_m}}{\partial x^{\nu_m}} T_{B_1 \dots B_m}^{A_1 \dots A_n} \left(\frac{X}{X^+} \right)$$

This includes both the Jacobian factors from the change of variables and the prefactor from conformal weight.

If we choose the embedding slice $X^+ = 1$, then the projection simplifies significantly. In this gauge, we define the following object:

$$e_A^\mu = X^+ \frac{\partial x^\mu}{\partial X^A}$$

and the projected tensor becomes

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x) = e_{A_1}^{\mu_1} \dots e_{A_n}^{\mu_n} e_{\nu_1}^{B_1} \dots e_{\nu_m}^{B_m} T_{B_1 \dots B_m}^{A_1 \dots A_n}(X)$$

Using the explicit form of the projection $x^\mu = X^\mu / X^+$, we compute

$$\frac{\partial x^\mu}{\partial X^A} = \frac{\delta_A^\mu (X^0 + X^{-1}) - X^\mu (\delta_A^0 + \delta_A^{-1})}{(X^0 + X^{-1})^2}$$

and setting $X^+ = 1$, we get

$$e_A^\mu = \delta_A^\mu, \quad e_A^0 = e_A^{-1} = -x^\mu$$

However, if we don't choose the slice $X^+ = 1$, then the projected tensor carries an additional scaling dependence:

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x) = \left(\frac{1}{X^+} \right)^{\Delta+n-m} \underbrace{e_{A_1}^{\mu_1} \dots e_{A_n}^{\mu_n} e_{\nu_1}^{B_1} \dots e_{\nu_m}^{B_m} T_{B_1 \dots B_m}^{A_1 \dots A_n}(X)}_{\text{depends only on physical point}}$$

This shows that tensors projected from different embedding space sections-i.e., different choices of X^+ -differ by a power of X^+ . So if two representations x and \tilde{x} correspond to the same physical point but lie on different sections (i.e., with different values of X^+), then the corresponding tensors are related as

$$T(\tilde{x}) = \left(\frac{X^+}{\tilde{X}^+} \right)^{\Delta+n-m} T(x)$$

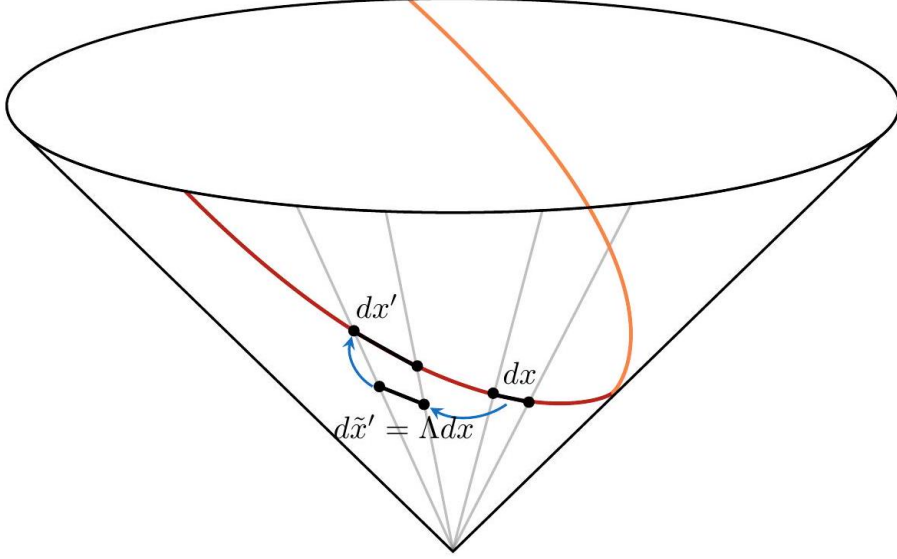


Figure 3.1: Upon Lorentz transformation, the points get mapped to different section however by utilizing the dilatation, we bring it back inside the original Euclidean section.

Next, consider how the tensor transforms under a conformal change of coordinates $x \mapsto x'$ which amounts to applying the corresponding Lorentz transformation $X \mapsto X' = \Lambda X$ in embedding space. Since the Lorentz transformation will map the tensor $T(X)$ to another tensor $T(X')$ living in different section, we will need to use above expression to map it back inside the original section labeled by X^+ . The tensor at \tilde{x}' is then:

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(\tilde{x}') = \frac{\partial x'^{\mu_1}}{\partial X'^{A_1}} \dots \frac{\partial x'^{\mu_n}}{\partial X'^{A_n}} \frac{\partial X'^{B_1}}{\partial x'^{\nu_1}} \dots \frac{\partial X'^{B_m}}{\partial x'^{\nu_m}} T_{B_1 \dots B_m}^{A_1 \dots A_n}(X')$$

Now applying the chain rule, we insert identities:

$$\frac{\partial x'^{\mu}}{\partial X'} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial X'}, \quad \frac{\partial X'}{\partial x'} = \frac{\partial X'}{\partial x^{\beta}} \cdot \frac{\partial x^{\beta}}{\partial x'}$$

This yields:

$$\underbrace{T_{\nu_1 \nu_2 \dots \nu_m}^{\mu_1 \mu_2 \dots \mu_n}(\tilde{x}')}_{\text{tensor field projected on section } X'+} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}} \right) \left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}} \right) \underbrace{\left[\frac{\partial x^{\alpha_1}}{\partial X^{A_1}} \dots \frac{\partial x^{\alpha_n}}{\partial X^{A_n}} \frac{\partial X^{B_1}}{\partial x^{\beta_1}} \dots \frac{\partial X^{B_m}}{\partial x^{\beta_m}} \right]}_{\text{tensor field projected on section } X^+} T_{B_1 B_2 \dots B_m}^{A_1 A_2 \dots A_n}(X)$$

tensor field projected on section $X'+$
 tensor field projected on section X^+

$$\begin{aligned} \left(\frac{X^+}{X'+} \right)^{\Delta+n-m} T_{\nu_1 \nu_2 \dots \nu_m}^{\mu_1 \mu_2 \dots \mu_n}(x') &= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}} \right) \left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}} \right) T_{\beta_1 \dots \beta_m}^{\alpha_1 \dots \alpha_n}(x) \\ &= \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}} \right) \left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}} \right) \left(\frac{X'+}{X^+} \right)^{\Delta+n-m} T_{\beta_1 \dots \beta_m}^{\alpha_1 \dots \alpha_n}(x) \end{aligned}$$

The factor $(X' + /X^+)^{\Delta+n-m}$ arises because x' and \tilde{x}' are physically the same point, but obtained by projecting from different embedding sections. To express this ratio in terms of the coordinate Jacobian $|\partial x'/\partial x|$, note that from the projection $x^\mu = X^\mu/X^+$ and $x^\mu = X^\mu/X'$, one finds

$$\left| \frac{\partial x'}{\partial x} \right| = \left| \frac{\partial X}{\partial x} \cdot \frac{\partial x'}{\partial X} \right| = \left| \frac{\partial X}{\partial x} \cdot \frac{\partial x'}{\partial X'} \frac{\partial X'}{\partial X} \right| = \left| \left(\frac{X^+}{X'+} \right) \frac{\partial X'}{\partial X} \right| = \left| \frac{X^+}{X'+} \right|^d |\Lambda|$$

and therefore,

$$\left(\frac{X^+}{X'+} \right)^{\Delta+n-m} = \left| \frac{\partial x'}{\partial x} \right|^{\frac{\Delta+n-m}{d}}$$

Finally, combining everything, the full transformation law for the projected tensor under a conformal coordinate transformation is:

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\frac{\Delta+n-m}{d}} \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \dots \frac{\partial x'^{\mu_n}}{\partial x^{\alpha_n}} \right) \left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}} \right) T_{\beta_1 \dots \beta_m}^{\alpha_1 \dots \alpha_n}(x)$$

This expression makes it manifest that the projected tensor transforms as a tensor under general coordinate transformations, but with an additional conformal weight $\Delta + n - m$ that reflects both the homogeneity of the embedding space tensor and the number of upper and lower indices involved in the projection.

3.0.2 Finding Correlators from Embedding Space

Needless to say, it is significantly easier to construct Lorentz covariant expressions than conformally covariant ones. Therefore, the natural question arises: once we have constructed Lorentz covariant expressions in $d+2$ dimensions, how do we descend to d dimensions without breaking covariance?

Since we have already fixed $\eta = 0$ in our derivation of the conformal generators, we now focus on the structure preserved by Lorentz transformations: the null light cone $X^2 = 0$ in embedding space. This cone, defined in $\mathbb{R}^{d+1,1}$ as the space of null rays through the origin, is given by:

$$\begin{aligned} X^2 &= -(X^0)^2 + (X^1)^2 + \dots + (X^{d+1})^2 \\ &= -X^+ X^- + \sum_{\mu=1}^d (X^\mu)^2 = 0 \end{aligned}$$

Although correlators are initially constructed as Lorentz-invariant functions over the full $d+2$ -dimensional ambient space, we now restrict them to the null cone. This constraint effectively reduces the support of such correlators to a $d+1$ -dimensional submanifold, since one of the coordinate dependencies - say, X^- - can be eliminated using the condition $X^2 = 0$ (in our case it leads to $\eta = 0$).

Next, we reinterpret embedding space as a fiber bundle over the physical d -dimensional spacetime (where the CFT is defined). Each fiber consists of null lines in the $(d+2)$ -dimensional space, and each point in the base space corresponds to an equivalence class of null vectors $X^A \sim \lambda X^A$, for any non-zero λ . This reflects the earlier observation regarding the arbitrariness of ρ : all such rescalings represent the same physical point in d dimensions.

This identification has an important consequence: As mentioned earlier, this introduces gauge redundancy in our description. To eliminate another coordinate, say $X^+ = \rho$, we fix the gauge by selecting a section of the bundle with specific choice of the slice on the embedding space, typically the Euclidean section, defined by:

$$X^+ = \rho = f(X^\mu) \equiv f(x^\mu)$$

Although Lorentz transformations may take a null vector outside this section (as they mix time-like and space-like directions), they can always be brought back by utilising the scaling equivalence $X^A \sim \lambda X^A$. This choice anchors us to physical d -dimensional spacetime, completing the descent from $d+2$ dimensions while maintaining conformal covariance inherited from Lorentz invariance in the higher-dimensional space. With these prescriptions in place we can now identify

X^μ with the Euclidean space coordinates x^μ by stripping ρ dependence.

$$X^\mu \equiv x^\mu$$

This leads to definition of X^- -based on null condition as:

$$X^- = \frac{\sum_{\mu=1}^d (X^\mu)^2}{X^+} = \frac{X^\mu X_\mu}{X^+} = \frac{x^2}{f(X^\mu)}$$

or, equivalently ²

$$\rho(-\eta^2 + \vec{x}^2) = \frac{\rho^2 \vec{x}^2}{\rho} \implies \eta = 0$$

The spacetime interval on this section is given as:

$$ds^2 = dx^2 - dX^+ dX^- \Big|_{X^+ = f(X^i), X^- = \frac{x^2}{X^+}}$$

This section satisfies two of the following conditions:

- section intersects each of the light rays at some point
- maps each point in d dimensional Euclidean space to a point on the null cone in Embedding space.

We have shown how to get generators of conformal transformation from Lorentz generators by embedding a null cone in ambient space. Let us now analyze how Lorentz Transformation acts on a generic section on that null cone. The Lorentz transformation acting as rotation on the point X^A in the null-cone will move it to another point on the null cone outside the the section $X^B = \Lambda^B_A X^A$.

However, suppose via some conformal transformation (dilatation) in d dimensional Euclidean Space, we can move X^B to X^C back into the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section.

$$\begin{aligned} ds_B^2 &= dX^M dX_M \\ &= d(\lambda(X) X^M) d(\lambda(X) X_M) \\ &= [\lambda dX^M + X^M (\nabla \lambda \cdot dX)] [\lambda dX_M + X_M (\nabla \lambda \cdot dX)] \\ &= \lambda^2 dX^M dX_M + 2\lambda \underbrace{dX^M X_M}_{=0} (\nabla \lambda \cdot dX) + \underbrace{X^M X_M}_{=0} (\nabla \lambda \cdot dX)^2 \\ &= \lambda^2 dX^M dX_M = \lambda^2 ds_C^2 \end{aligned}$$

where we used, $X^2 = 0$ and $X^\mu dX_\mu = 0$ for restricting it to null cone.

Assuming the three conditions we used for simplification applies, the Lorentz Transformation in $d+2$ dimensional spacetime is equivalent to conformal transformation in d -dimensional spacetime iff metric in d -dimensional space is Euclidean thus, dX_+ in ds^2 has to vanish. It gives us the condition for defining the Euclidean section as $X^+ = \rho = \text{constant}$ and thus, for the sake of simplicity, we take it as 1. Thus, we have two conditions which we can use to eliminate two extra degree of freedom.

In the embedding space formalism, choosing an Euclidean section corresponds to picking a specific way to embed the d -dimensional space in the $(d+2)$ -dimensional space. We define the following map between d dimensional Euclidean Space with conformal symmetry to null cone in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$(X^+, X^-, X^\mu) \equiv (1, x^2, x^\mu)$$

Here, we note that choosing a constant value for X^+ would give us a section on the cone on which the induced metric is Euclidean.

⁰² in our notation $x^\mu = \rho \vec{x}^\mu$

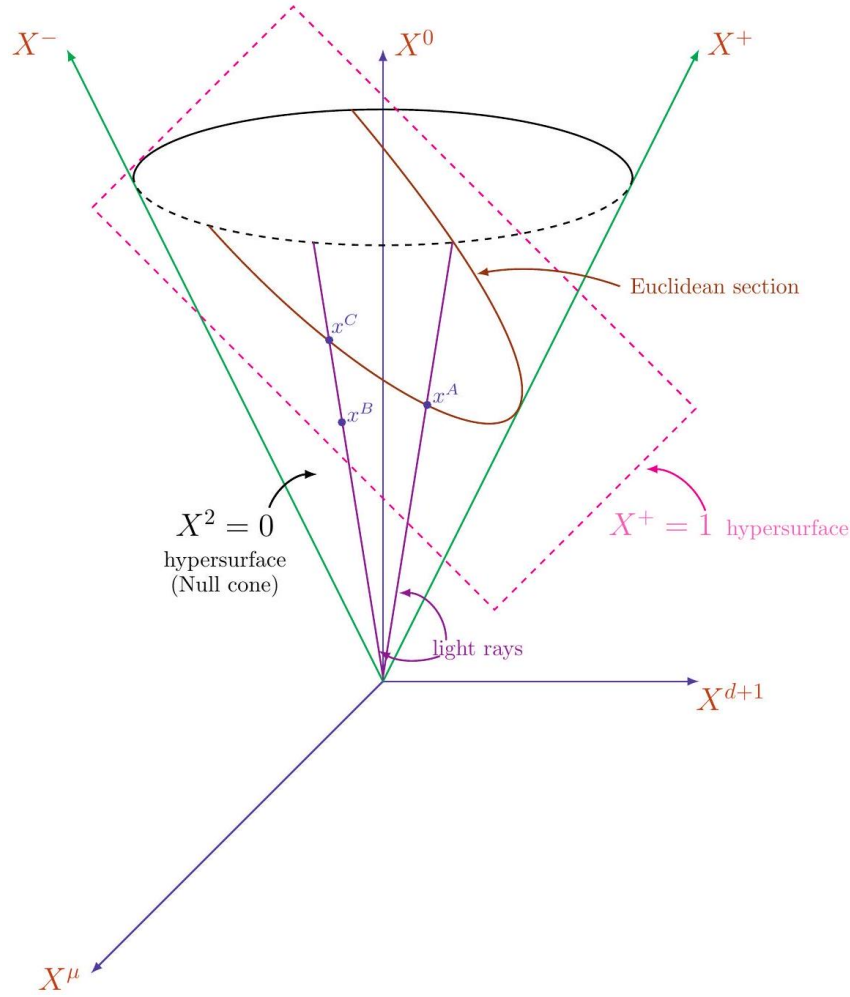


Figure 3.2: The hypersurface perpendicular to X^+ axis cutting at $X^+ = 1$ is shown as a plane and the null hypersurface is shown as the cone. The intersection of these two hypersurfaces describes the Euclidean Section. Dilatations are rotation in the $X^0 X^{d+1}$ plane and SCT or momentum generators are rotations in $X^\mu X^{d+1}$ with $X^0 X^{d+1}$ plane.

3.0.3 Tensors in Embedding Space

In this section, we will only concern ourselves with traceless and symmetric fields in \mathbb{R}^d and leave the antisymmetric tensors for future. Consider a symmetric and traceless tensor ${}^3O_{M_1 \dots M_S}$ defined on the cone $X^2 = 0$ in $\mathbb{R}^{d+1,1}$. Under the rescaling $X \rightarrow \lambda X$, the tensor transforms as

$$O_{M_1 \dots M_S}(\lambda X) = \lambda^{-\Delta} O_{M_1 \dots M_S}(X)$$

i.e. it is a homogeneous function of degree Δ . We expect $O_{M_1 \dots M_S}$ to get mapped to traceless and symmetric primary field in \mathbb{R}^d . Since, each index go from 0 to $d+1$, in $\mathbb{R}^{d+1,1}$ we find that, for $d+2$ -dimensional fields other than scalar have 2 more degree of freedom per index than d -dimensional fields. In order to remove the extra degree of freedom, we consider the transversality condition.

$$X^{M_1} O_{M_1 \dots M_S} = 0$$

We define the physical field to be:

$$\phi_{\mu\nu\lambda\dots}(x) = \left. \frac{\partial X^{M_1}}{\partial x^\mu} \frac{\partial X^{M_2}}{\partial x^\nu} \frac{\partial X^{M_3}}{\partial x^\lambda} \dots O_{M_1 \dots M_S}(X) \right|_{X=X(x)}$$

Note that, this definition implies a redundancy. Indeed, anything proportional to X^M gives zero since

$$X^2 = 0 \implies X_M \frac{\partial X^M}{\partial x^\mu} = 0$$

Therefore, $O_{M_1 \dots M_S}(X) \rightarrow O_{M_1 \dots M_S}(X) + X_{M_1} F_{M_2 \dots M_S}(X)$ gets mapped to the same physical field. This $SO(d+1, 1)$ tensor is sometimes referred to as pure gauge in the literature. It is this gauge redundancy that reduces another degree of freedom per index by making it unphysical.

3.0.4 Examples: Two point and Three point correlator

In the last section we showed how the embedding space formalism put in place could be used to deduce the conformally invariant correlator. In this section we will utilize the formalism and explicitly construct two point and three point function using the formalism developed thus far. From 1.3, we know that the ratios are only invariant under dilatation and translation. Therefore, we seek to construct an invariant out of these ratios and metric tensor which is also invariant under SCT and the exchange of indices $\mu \leftrightarrow \nu$. First we will derive the form of scalar one point correlator. A scalar primary is denoted by $\hat{\mathcal{O}}_\Delta$ (notice the absence of the Lorentz index, which indicates that it is a scalar operator). We want to enforce the invariance of correlator under conformal transformation. For a one-point function, this reduces to

$$\langle \hat{\mathcal{O}}_\Delta(x'^\mu) \rangle = \langle \hat{\hat{\mathcal{O}}}_\Delta(x'^\mu) \rangle$$

This condition must be enforced for all four conformal transformations. We will begin by enforcing translation.

$$\text{Translation: } \tilde{x}^\mu = x^\mu + a^\mu$$

It was given previously that the Jacobian for a translation is one. Therefore, our operator simply does not change under this translation

$$\hat{\hat{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \hat{\mathcal{O}}_\Delta(x^\mu)$$

enforcing this in (2.1), we are left with

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle$$

where the operators are now the same on both sides of the equation. Notice that a correlation function is just a function. Therefore, this is equivalent to saying

$$f(\tilde{x}^\mu) = f(x^\mu)$$

Since this must be true for every possible translation, this tells us that the function has the same output regardless of what the input is, which means the function must just be some constant. Therefore, by enforcing translation we can conclude that

$$\langle \hat{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle = \langle \hat{\mathcal{O}}_\Delta(x^\mu) \rangle = \text{constant} = C$$

We are not yet done. We need to make sure that all four transformations leave the one-point function invariant. Let's see what we can learn when we enforce dilatation.

$$\text{Dilatation: } \tilde{x}^\mu = \lambda x^\mu$$

Applying the Jacobian for dilatation, we see our Primary Scalar Operator transforms as

$$\hat{\hat{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right|^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \lambda^{-\Delta} \hat{\mathcal{O}}_\Delta(x^\mu)$$

We want to enforce this in (2.1) and use our results from enforcing translation invariance. This gives us

⁰³ symmetric tensors with spin s under $SO(d)$ form irreducible representations that correspond to integer spin particles (bosons). Anti-symmetric tensor fields have interpretation like they correspond to bivector of spinors etc.

$$\begin{aligned}
\langle \hat{\mathcal{O}}_{\Delta}(\tilde{x}^{\mu}) \rangle &= \langle \hat{\hat{\mathcal{O}}}_{\Delta}(\tilde{x}^{\mu}) \rangle \\
&= \langle \lambda^{-\Delta} \hat{\mathcal{O}}_{\Delta}(x^{\mu}) \rangle \\
&= \lambda^{-\Delta} \langle \hat{\mathcal{O}}_{\Delta}(x^{\mu}) \rangle
\end{aligned}$$

We found previously, by enforcing translation, that

$$\langle \hat{\mathcal{O}}_{\Delta}(\tilde{x}^{\mu}) \rangle = C$$

which means

$$C = \lambda^{-\Delta} C$$

This equation must be true for arbitrary scale factor λ . Therefore, unless $\Delta = 0$, we can conclude that $C = 0$.

For unitary CFTs, the only $\Delta = 0$ operator is the identity operator. So, with the exception of the identity, all one-point functions must vanish!

$$\langle \hat{\mathcal{O}}_{\Delta}(x^{\mu}) \rangle = 0 \text{ for } \Delta \neq 0$$

We said that we must impose all four conformal transformations, but the others are trivially satisfied at this point. So we are done with one-point correlators! Again, we'd like to highlight the fact that this is the result for ALL CFTs. You don't need to know anything else about the system, only that it has conformal symmetry.

3.1 Two-point Scalar Primary

For two-point functions, we need to enforce

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^{\mu}) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^{\mu}) \rangle = \langle \hat{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^{\mu}) \hat{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^{\mu}) \rangle$$

Again, this must be done for all four conformal transformations. As with the one-point function, we will begin by enforcing translation.

3.1.1 Translation

First, notice that $\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^{\mu}) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^{\mu}) \rangle$ is an object that takes two positions as inputs and gives back a number, so we can just write this as a function of \tilde{x}_1^{μ} and \tilde{x}_2^{μ}

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^{\mu}) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^{\mu}) \rangle = f(\tilde{x}_1^{\mu}, \tilde{x}_2^{\mu})$$

We found previously that under translations, scalar primary operators transform as

$$\hat{\hat{\mathcal{O}}}(\tilde{x}^{\mu}) = \hat{\mathcal{O}}(x^{\mu})$$

Putting this result in the right side of equation (2.3), we find that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^{\mu}) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^{\mu}) \rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^{\mu}) \hat{\mathcal{O}}_{\Delta_2}(x_2^{\mu}) \rangle$$

Notice that the only differences between the left and right side of this equation are the inputs. The function on each side is the same

$$f(\tilde{x}_1^{\mu}, \tilde{x}_2^{\mu}) = f(x_1^{\mu}, x_2^{\mu})$$

Under translation, we have $\tilde{x}^{\mu} = x^{\mu} + a^{\mu} \rightarrow x^{\mu} = \tilde{x}^{\mu} - a^{\mu}$. If we put this into the previous equation it becomes

$$f(x_1^{\mu} - a^{\mu}, x_2^{\mu} - a^{\mu}) = f(x_1^{\mu}, x_2^{\mu}), \forall a^{\mu}$$

This must be true regardless of the value of a^{μ} which means that a^{μ} must somehow cancel out. This is only satisfied if it is a function of $x_1^{\mu} - x_2^{\mu}$ so we have

$$f(x_1^\mu, x_2^\mu) = f(x_1^\mu - x_2^\mu)$$

That is, our function cannot be any function of the two positions. Rather, it can only depend on the displacement between the two positions. Therefore,

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle = f(x_1^\mu - x_2^\mu)$$

Let's now see what we can learn by enforcing rotation.

Rotation $\tilde{x}^\mu = \Lambda_\nu^\mu x^\nu$ The Jacobian for rotation is the same as for translation, 1. Therefore, scalar primary operators transform the same under rotation as they do under translation

$$\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle = \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle$$

Re-expressing this in a more familiar form, as functions, we have

$$f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) = f(x_1^\mu, x_2^\mu)$$

Next, we impose what we found by imposing translational invariance

$$f(\tilde{x}_1^\mu - \tilde{x}_2^\mu) = f(x_1^\mu - x_2^\mu)$$

Expressing the transformed coordinates in terms of our original coordinate system, we find

$$f(\Lambda_\nu^\mu (x_1^\nu - x_2^\nu)) = f(x_1^\mu - x_2^\mu)$$

This tells us that applying a rotation has no effect on the output. This means that the function must depend only on the magnitude of the separation $|x_1^\mu - x_2^\mu|$ (just a reminder, rotating a vector changes it, but rotating a scalar does nothing). So, from applying translational and rotational invariance, we can conclude that

$$\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle = f(|x_1^\mu - x_2^\mu|)$$

We will now continue by enforcing invariance under dilatation.

3.1.2 Dilatation

just a reminder, under dilatation, scalar primary operators transform as

$$\hat{\hat{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \lambda^{-\Delta} \hat{\mathcal{O}}_\Delta(x^\mu)$$

Substituting this into our two-point function condition, eqn. (2.3), we have

$$\begin{aligned} \langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle &= \langle \hat{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \\ &= \langle \lambda^{-\Delta_1} \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \lambda^{-\Delta_2} \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle \\ &= \lambda^{-\Delta_1} \lambda^{-\Delta_2} \langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \rangle \\ &= \lambda^{-(\Delta_1 + \Delta_2)} f(|x_1^\mu - x_2^\mu|) \end{aligned}$$

where we are able to pull the λ 's out of the correlator because they are just scalars. Notice also that we used what we already learned from translational and rotational invariance. This tells us that

$$f(|\tilde{x}_1^\mu - \tilde{x}_2^\mu|) = \lambda^{-(\Delta_1 + \Delta_2)} f(|x_1^\mu - x_2^\mu|)$$

We can apply the transformation to the coordinates on the left-hand-side, which gives

$$f(\lambda |x_1^\mu - x_2^\mu|) = \lambda^{-(\Delta_1 + \Delta_2)} f(|x_1^\mu - x_2^\mu|)$$

What does this mean? We can consider expanding our function in a power series

$$f(|x_1^\mu - x_2^\mu|) = \sum_n c_n |x_1^\mu - x_2^\mu|^n$$

Substituting this in above gives

$$\sum_n c_n \lambda^n |x_1^\mu - x_2^\mu|^n = \lambda^{-(\Delta_1 + \Delta_2)} \sum_n c_n |x_1^\mu - x_2^\mu|^n$$

This is only satisfied for all λ , if all $n = 0$ except $n = -(\Delta_1 + \Delta_2)$. Therefore, after enforcing translation, rotation, and dilatation symmetry we have

$$\left\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \right\rangle = C |x_1^\mu - x_2^\mu|^{-(\Delta_1 + \Delta_2)}$$

where C is some undetermined constant.

3.2 Special Conformal Transformation

Enforcing special conformal symmetry directly is a very messy business. Luckily for us, as discussed previously, a special conformal transformation is equivalent to performing an inversion, followed by a translation, followed by another inversion. Since we have already enforced translational invariance, this means it is sufficient to enforce inversion invariance, which is much easier. Recall, an inversion is given by

$$x^\mu = \frac{\tilde{x}^\mu}{\tilde{x}^2}$$

As with the other transformations, we need the Jacobian for inversion in order to see how the operators will transform. This is given by

$$\left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = \frac{1}{\tilde{x}^{2D}}$$

Therefore, under inversion, scalar primary operators transform as

$$\hat{\hat{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left(\frac{1}{\tilde{x}^{2D}} \right)^{\Delta/D} \hat{\mathcal{O}}_\Delta(x^\mu) = \frac{1}{(\tilde{x}^2)^\Delta} \hat{\mathcal{O}}_\Delta(x^\mu)$$

As usual, we will now go put this into equation (2.3) to enforce the symmetry

$$\begin{aligned} \left\langle \hat{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \right\rangle &= \left\langle \hat{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \right\rangle \\ &= \left\langle \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \right\rangle \\ &= \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \left\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \right\rangle \end{aligned}$$

Now, we can use our result from enforcing dilatation to replace $\left\langle \hat{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \right\rangle$ on the left and $\left\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \right\rangle$ on the right of this equation to get

$$\frac{C}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1 + \Delta_2}} = \frac{1}{(\tilde{x}_1^2)^{\Delta_1}} \frac{1}{(\tilde{x}_2^2)^{\Delta_2}} \frac{C}{|x_1^\mu - x_2^\mu|^{\Delta_1 + \Delta_2}}$$

With a bit of algebra, this is equivalent to

$$\frac{(\tilde{x}_1^2)^{\Delta_1} (\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1 + \Delta_2}} = \frac{1}{|x_1^\mu - x_2^\mu|^{\Delta_1 + \Delta_2}}$$

In order to put this in a more friendly form, we will use the following identity for inversions. Note: verifying this relationship requires substituting in the inversion transformation and some algebra. The reader is highly encouraged to check it.

$$\frac{\tilde{x}_1^2 \tilde{x}_2^2}{(\tilde{x}_1^\mu - \tilde{x}_2^\mu)^2} = \frac{1}{(x_1^\mu - x_2^\mu)^2}$$

Using this identity in eqn. (2.6), we find

$$\frac{(\tilde{x}_1^2)^{\Delta_1} (\tilde{x}_2^2)^{\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1 + \Delta_2}} = \left[\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2} \right]^{\frac{\Delta_1 + \Delta_2}{2}}$$

This is only satisfied if

$$\Delta_1 = \Delta_2$$

Therefore, we find that the two-point function vanishes, unless the dimensions of the two operators are the same. In summary, the two-point function for scalar primaries in ANY CFT is given by

$$\left\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \right\rangle = \frac{C \delta_{\Delta_1 \Delta_2}}{|x_1^\mu - x_2^\mu|^{\Delta_1 + \Delta_2}}$$

Note that it is standard convention to choose to normalize your operators so that $C = 1$, so you will often see this without the C constant included. We leave it here for complete generality.

3.3 From Quantum Field Theory

The Euclidean space correlator can be calculated by Fourier Transforming the momentum space propagator

$$G_E^{(0)}(x - x') = \int \frac{d^D p}{(2\pi)^D} G_E^{(0)}(p) e^{ip_\mu(x_\mu - x'_\mu)}$$

where,

$$G_E^{(0)}(p) = \frac{1}{p^2}$$

Therefore, the correlation function in real (Euclidean) space is the integral

$$G_E^{(0)}(x - x') = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip_\mu(x_\mu - x'_\mu)}}{p^2}$$

we can simplify the integrand by using Schwinger trick:

$$\frac{1}{p^2} = \frac{1}{2} \int_0^\infty d\alpha e^{-\frac{\alpha}{2} p^2}$$

Now, let's use this to re-express our correlator as double integral

$$G_E^{(0)}(x - x') = \frac{1}{2} \int_0^\infty d\alpha \int \frac{d^D p}{(2\pi)^D} e^{-\frac{\alpha}{2} p^2 + ip_\mu(x_\mu - x'_\mu)}$$

The integrand is gaussian and integral can be calculated by considering shift in variable

$$\frac{\alpha}{2} p^2 - ip_\mu(x_\mu - x'_\mu) = \frac{1}{2} \left(\sqrt{\alpha} p_\mu - i \frac{x_\mu - x'_\mu}{\sqrt{\alpha}} \right)^2 + \frac{1}{2} \left(\frac{x_\mu - x'_\mu}{\sqrt{\alpha}} \right)^2$$

and by using the Gaussian integral,

$$\int \frac{d^D p}{(2\pi)^D} e^{-\frac{1}{2} \left(\sqrt{\alpha} p_\mu - i \frac{x_\mu - x'_\mu}{\sqrt{\alpha}} \right)^2} = (2\pi\alpha)^{-D/2}$$

We find the correlation function to be,

$$G_E^{(0)}(x - x') = \frac{1}{2(2\pi)^{D/2}} \int_0^\infty d\alpha \alpha^{-D/2} e^{-\frac{|x - x'|^2}{2\alpha}} = \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{\frac{D}{2}} |x - x'|^{D-2}}$$

This is same as what we found previously by considering only symmetry constraint with $\Delta = \frac{D}{2} - 1$ as found in (1.29).

3.4 Three-point Scalar Primary:

For the three-point function, we need to enforce

$$\left\langle \hat{\mathcal{O}}_1(x_1^\mu) \hat{\mathcal{O}}_2(x_2^\mu) \hat{\mathcal{O}}_3(x_3^\mu) \right\rangle = \left\langle \hat{\hat{\mathcal{O}}}_1(x_1^\mu) \hat{\hat{\mathcal{O}}}_2(x_2^\mu) \hat{\hat{\mathcal{O}}}_3(x_3^\mu) \right\rangle$$

Enforcing the symmetries for the three-point function follows in a very similar way to the two-point function, so we will not include as much detail. The reader is encouraged to work through any excluded details on their own.

3.5 Poincaré

For translations and rotations, the same line of argumentation that was used for two-point functions can be applied. However, instead of two points at our disposal, we have three. Therefore, our function can be a function of the magnitude of the separations between any pairings of three points.

$$\left\langle \hat{\mathcal{O}}_1(x_1^\mu) \hat{\mathcal{O}}_2(x_2^\mu) \hat{\mathcal{O}}_3(x_3^\mu) \right\rangle = f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|)$$

where $|x_{12}^\mu| = |x_1^\mu - x_2^\mu|$, $|x_{23}^\mu| = |x_2^\mu - x_3^\mu|$, and $|x_{31}^\mu| = |x_3^\mu - x_1^\mu|$.

3.6 Dilatation

Enforcing dilatation invariance with our three-point function, we have

$$\begin{aligned} \left\langle \hat{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \hat{\mathcal{O}}_{\Delta_3}(\tilde{x}_3^\mu) \right\rangle &= \left\langle \hat{\hat{\mathcal{O}}}_{\Delta_1}(\tilde{x}_1^\mu) \hat{\hat{\mathcal{O}}}_{\Delta_2}(\tilde{x}_2^\mu) \hat{\hat{\mathcal{O}}}_{\Delta_3}(\tilde{x}_3^\mu) \right\rangle \\ &= \left\langle \lambda^{-\Delta_1} \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \lambda^{-\Delta_2} \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \lambda^{-\Delta_3} \hat{\mathcal{O}}_{\Delta_3}(x_3^\mu) \right\rangle \\ &= \lambda^{-\Delta_1} \lambda^{-\Delta_2} \lambda^{-\Delta_3} \left\langle \hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \hat{\mathcal{O}}_{\Delta_3}(x_3^\mu) \right\rangle \end{aligned}$$

Using our results from enforcing Poincaré invariance, this becomes

$$f(|\tilde{x}_{12}^\mu|, |\tilde{x}_{23}^\mu|, |\tilde{x}_{31}^\mu|) = \lambda^{-(\Delta_1 + \Delta_2 + \Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|)$$

Substituting in the dilatation transformation on the LHS, this is

$$f(\lambda |x_{12}^\mu|, \lambda |x_{23}^\mu|, \lambda |x_{31}^\mu|) = \lambda^{-(\Delta_1 + \Delta_2 + \Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|)$$

As with the two-point function, we can expand our function in a power series.

$$f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) = \sum_{nmp} c_{nmp} |x_{12}^\mu|^n |x_{23}^\mu|^m |x_{31}^\mu|^p$$

Substituting this in, we find that all terms must vanish, unless

$$n + m + p = -(\Delta_1 + \Delta_2 + \Delta_3)$$

Therefore, dilatation and Poincaré invariance tell us

$$\left\langle \hat{\mathcal{O}}_1(x_1^\mu) \hat{\mathcal{O}}_2(x_2^\mu) \hat{\mathcal{O}}_3(x_3^\mu) \right\rangle = \sum_{nmp=-(\Delta_1 + \Delta_2 + \Delta_3)} c_{nmp} |x_{12}^\mu|_n |x_{23}^\mu|^m |x_{31}^\mu|^p$$

3.7 Special Conformal Transformation

Again, to find the effect of imposing special conformal symmetry, we need only to impose inversion symmetry, which is much easier. Although easier, the algebra is still quite nasty and will not be shown here. Ultimately, inversion (therefore special conformal) invariance leads to the additional constraint that all terms vanish, unless

$$\begin{aligned} n &= \Delta_1 + \Delta_2 - \Delta_3 \\ m &= \Delta_1 + \Delta_3 - \Delta_2 \\ p &= \Delta_2 + \Delta_3 - \Delta_1 \end{aligned}$$

Therefore, after enforcing all of the conformal symmetries on the 3-point function of scalar primaries, we find

$$\hat{\mathcal{O}}_{\Delta_1}(x_1^\mu) \hat{\mathcal{O}}_{\Delta_2}(x_2^\mu) \hat{\mathcal{O}}_{\Delta_3}(x_3^\mu) = \frac{C_{123}}{|x_{12}^\mu|^n |x_{23}^\mu|^m |x_{31}^\mu|^p}$$

where n, m , and p are given by (2.14). We find that, as was the case with the two-point scalar primaries, the spatial dependence of 3-point scalar primaries are completely determined. We are left only with a set of constants C_{123} . It turns out that this set of constants is vitally important to defining any particular conformal field theory and they tell you how much your given operators interact. This set of constants goes by various names including the 3-point coefficients, the OPE coefficients, and the structure constants.

2.4.3 Going beyond scalars

Moving on, we next consider the two point corrector of vector field. The ansatz for such a correlator is:¹

$$\langle J_\mu(x_1) J_\nu(x_2) \rangle = C \underbrace{\frac{1}{|x_1 - x_2|^{2\Delta}}}_{\text{same as scalar case}} \left[g_{\mu\nu} + \delta \frac{(x_1 - x_2)_\mu (x_1 - x_2)_\nu}{(x_1 - x_2)^2} \right] \quad (3.1)$$

The correlation function is invariant under translation, therefore we will consider following redefinition:

$$\begin{aligned} \langle J_\mu(x_1) J_\nu(x_2) \rangle &= \langle J_\mu(x_1 - x_2) J_\nu(0) \rangle \\ &= \langle J_\mu(x_{12}) J_\nu(0) \rangle \equiv \langle J_\mu(x) J_\nu(0) \rangle \end{aligned} \quad (3.2)$$

Since SCT is just inversion \rightarrow translation \rightarrow inversion, we can use this property to our advantage. As the correlation function is already invariant under translations, it suffices to verify its invariance under inversions. If this property holds, then by extension, the correlation function will also be invariant under SCT. The inversion transformation is given as²:

$$x'_\mu = \frac{x_\mu}{x^2} \quad |x'|^2 = \frac{1}{|x|^2} \quad (3.3)$$

and

$$\frac{\partial x'_\nu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \frac{x_\nu}{x^2} = \frac{1}{x^2} \left[g_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2} \right] = x'^2 \underbrace{\left[g_{\mu\nu} - 2 \frac{x'_\mu x'_\nu}{x'^2} \right]}_{I_{\mu\nu}} \quad (3.4)$$

The vector field would transform as

$$\langle J_\mu(x'_1) J_\nu(x'_2) \rangle = \underbrace{\left| \frac{\partial x_\alpha}{\partial x'^\mu} \right|^{\Delta/d} \left| \frac{\partial x_\beta}{\partial x'^\nu} \right|^{\Delta/d}}_{\text{this was used to derive the correlation function for scalar case}} \overbrace{\frac{\partial x'_\alpha}{\partial x^\mu} \frac{\partial x'_\beta}{\partial x^\nu}}^{\text{without conformal factor}} \langle J^\alpha(x_1) J^\beta(x_2) \rangle \quad (3.5)$$

we see that

$$\left| \frac{\partial x_\alpha}{\partial x'^\mu} \right|_{x=x_1}^{\Delta/d} \left| \frac{\partial x_\beta}{\partial x'^\nu} \right|_{x=x_2}^{\Delta/d} \frac{1}{|x_{12}|^{2\Delta}} = |x'_1|^{-2\Delta} |x'_2|^{-2\Delta} \frac{1}{|x_{12}|^{2\Delta}} = \frac{1}{|x'_{12}|^{2\Delta}} \quad (3.6)$$

¹pg 24 of “CFT with boundary and defects” by Herzog

²pg 17-18 of “Quantum Gravity and Cosmology based on Conformal Field Theory” and section 4.5 of “A conformal field theory primer in $D \geq 3$ ” by Andrew Evans

where we used

$$\begin{aligned} |x'_{12}|^2 &= \left(\frac{x_1^\mu}{x_1^2} - \frac{x_2^\mu}{x_2^2} \right)^2 = \frac{|x_1|^2}{x_1^4} + \frac{|x_2|^2}{x_2^4} - 2 \frac{x_1^\mu x_{2\mu}}{x_1^2 x_2^2} \\ &= \frac{1}{x_1^2} - 2 \frac{x_1 \cdot x_2}{x_1^2 x_2^2} + \frac{1}{x_2^2} = \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} = \frac{|x_{12}|^2}{|x_1|^2 |x_2|^2} = \frac{|x_{12}|^2}{|x'_1|^{-2} |x'_2|^{-2}} \end{aligned} \quad (3.7)$$

Then, we only have to ensure that $g^{\mu\nu} + \delta \frac{x'_{12} x'_{12}}{x_{12}^2}$ is invariant under inversion.³

$$\begin{aligned} g^{\mu\nu} + \delta \frac{(x'_{12})^\mu (x'_{12})^\nu}{(x'_{12})^2} &= \left(\delta_\alpha^\mu - 2 \frac{x_1^\mu x_{1\alpha}}{|x_1|^2} \right) \left(\delta_\beta^\nu - 2 \frac{x_2^\nu x_{2\beta}}{|x_2|^2} \right) \left[g^{\alpha\beta} + \delta \frac{x_{12}^\alpha x_{12}^\beta}{x_{12}^2} \right] \Big|_{x^\mu = \frac{x'^\mu}{x'^2}} \\ &= \left(\delta_\alpha^\mu - 2 \frac{x_1^\mu x_{1\alpha}}{|x_1|^2} \right) \left(\delta_\beta^\nu - 2 \frac{x_2^\nu x_{2\beta}}{|x_2|^2} \right) \\ &\quad \times \left[g^{\alpha\beta} + \delta \frac{(x'_1 |x'_2|^2 - x'_2 |x'_1|^2)^\alpha (x'_1 |x'_2|^2 - x'_2 |x'_1|^2)^\beta}{x_{12}^2 |x'_1|^2 |x'_2|^2} \right] \\ &= \left[g^{\mu\beta} + \delta \frac{(x'_1 |x'_2|^2 - x'_2 |x'_1|^2)^\mu (x'_1 |x'_2|^2 - x'_2 |x'_1|^2)^\beta}{x_{12}^2 |x'_1|^2 |x'_2|^2} - 2 \frac{x_1^\mu x_1^\beta}{|x_1|^2} \right. \\ &\quad \left. - 2 \delta \frac{x_1^\mu (|x'_1|^2 x'_2 \cdot x'_1 - x'_2 \cdot x'_1 |x'_1|^2) (x'_1 |x'_2|^2 - x'_2 |x'_1|^2)^\beta}{x_{12}^2 |x'_1|^2 |x'_2|^2} \right] \left(\delta_\beta^\nu - 2 \frac{x_2^\nu x_{2\beta}}{|x_2|^2} \right) \\ &= \left[g^{\mu\nu} - \delta \frac{(x'_1 |x'_2|^2 + x'_2 |x'_1|^2 - 2x'_1 (x'_1 \cdot x'_2))^\mu (x'_1 |x'_2|^2 - x'_2 |x'_1|^2)^\nu}{x_{12}^2 |x'_1|^2 |x'_2|^2} - 2 \frac{x_1^\mu x_1^\nu}{|x_1|^2} \right] \\ &\quad - 2 \frac{x_2^\nu x_2^\mu}{|x_2|^2} + 2 \delta \frac{(x'_1 |x'_2|^2 + x'_2 |x'_1|^2 - 2x'_1 (x'_1 \cdot x'_2))^\mu (x'_2 \cdot x'_1 |x'_2|^2 - |x'_2|^2 |x'_1|^2) x_2^\nu}{x_{12}^2 |x'_1|^2 |x'_2|^2} \\ &\quad + 4 \frac{x_1^\mu x_2^\nu (x'_1 \cdot x'_2)}{|x_1|^2} \\ &= g^{\mu\nu} - 2 \frac{x_1^\mu x_1^\nu}{|x_1|^2} - 2 \frac{x_2^\nu x_2^\mu}{|x_2|^2} + 4 \frac{x_1^\mu x_2^\nu (x'_1 \cdot x'_2)}{|x_1|^2} \\ &\quad - \delta \frac{\{x'_1 |x'_2|^2 + x'_2 |x'_1|^2 - 2x'_1 (x'_1 \cdot x'_2)\}^\mu \{x'_1 |x'_2|^2 + x'_2 |x'_1|^2 - 2(x'_1 \cdot x'_2) x'_2\}^\nu}{x_{12}^2 |x'_1|^2 |x'_2|^2} \\ &= g^{\mu\nu} - 2 \frac{x_1^\mu x_1^\nu}{|x_1|^2} - 2 \frac{x_2^\nu x_2^\mu}{|x_2|^2} + 4 \frac{x_1^\mu x_2^\nu (x'_1 \cdot x'_2)}{|x_1|^2} \\ &\quad - \delta \frac{\{x'_2 |x'_1|^2 + |x'_1 - x'_2|^2 x'_1 - |x'_1|^2 x'_1\}^\mu \{x'_1 |x'_2|^2 + |x'_1 - x'_2|^2 x'_2 - |x'_2|^2 x'_2\}^\nu}{x_{12}^2 |x'_1|^2 |x'_2|^2} \\ &= g^{\mu\nu} - 2 \frac{x_1^\mu x_1^\nu}{|x_1|^2} - 2 \frac{x_2^\nu x_2^\mu}{|x_2|^2} + 4 \frac{x_1^\mu x_2^\nu (x'_1 \cdot x'_2)}{|x_1|^2} \\ &\quad + \delta \frac{x_{12}^\mu x_{12}^\nu}{x_{12}^2} - \delta \frac{x_{12}^\mu x_{12}^\nu}{|x_1|^2} + \delta \frac{x_{12}^\mu x_{12}^\nu - x_{12}^\mu x_2^\nu}{|x_2|^2} - \delta |x'_{12}|^2 \frac{x_1^\mu x_2^\nu}{|x_1|^2 |x_2|^2} \end{aligned}$$

³the conformal factor is there following eqn 55 of [TASI Lectures on the Conformal Bootstrap](#). The tensor operator under inversion transforms as mentioned in eqn 3.18 of [Conformal Field Theory with Boundaries and Defects](#) or eqn 1.55 and 1.60 of [EPFL Lectures on Conformal Field Theory in D ≥ 3 Dimensions](#)

$$O^\mu(x') = \left| \frac{\partial x_\alpha}{\partial x'^\beta} \right|^{\frac{\Delta+1}{d}} \frac{\partial x'_\mu}{\partial x^\nu} O^\nu(x) = \left| \frac{\partial x_\alpha}{\partial x'^\beta} \right|^{\Delta/d} I_\nu^\mu(x') O^\nu(x) \quad (3.8)$$

$$O'_\mu(x') = \left| \frac{\partial x^\alpha}{\partial x'^\beta} \right|^{\frac{\Delta-1}{d}} \frac{\partial x^\nu}{\partial x'^\mu} O_\nu(x) \quad (3.9)$$

$$\begin{aligned}
&= g^{\mu\nu} - (\delta + 2) \frac{x_1^\mu x_1^\nu}{|x_1'|^2} - (\delta + 2) \frac{x_2^\mu x_2^\nu}{|x_2'|^2} + 2(\delta + 2) \frac{x_1^\mu x_2^\nu (x_1' \cdot x_2')}{|x_1'|^2} + \delta \frac{x_1^\mu x_2^\nu}{|x_1'|^2 |x_2'|^2} \\
&\quad + \delta \frac{x_1^\nu x_2^\mu}{|x_1'|^2 |x_2'|^2} - \delta |x_2'|^2 \frac{x_1^\nu x_2^\mu}{|x_1'|^2 |x_2'|^2} - \delta |x_2'|^2 \frac{x_1^\mu x_2^\nu}{|x_1'|^2 |x_2'|^2} + \delta \frac{x_{12}^\mu x_{12}^\nu}{x_{12}^2}
\end{aligned}$$

which implies $\delta = -2$. Hence, the two point function is given as

$$\langle J_\mu(x) J_\nu(0) \rangle = \frac{C}{|x|^{2\Delta}} \left[g_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2} \right]$$

Dilations and SCTs

Dilations: Under the scale transformations, the length r_{jk} is clearly not invariant, but the ratio $\frac{r_{jk}}{r_{lm}}$ can be invariant.

SCTs: Under special conformal transformations, invariant quantities must be cross ratios of the form $\frac{r_{jk} r_{lm}}{r_{jl} r_{km}}$, since the squared length under SCTs transforms as

$$|x_1' - x_2'|^2 = \frac{|x_1 - x_2|^2}{(1 + 2b \cdot x_1 + b^2 x_1^2)(1 + 2b \cdot x_2 + b^2 x_2^2)}. \quad (3.10)$$

Two-Point Correlation Functions

Consider the classical two-point correlation function, or Green's function, of the quasi-primary fields

$$G^{(2)}(x_1, x_2) = \langle 0 | \phi_1(x_1) \phi_2(x_2) | 0 \rangle = \left| \frac{\partial x_1'}{\partial x_1} \right|^{\frac{\Delta_1}{d}} \left| \frac{\partial x_2'}{\partial x_2} \right|^{\frac{\Delta_2}{d}} \langle 0 | \hat{\phi}_1(x_1') \hat{\phi}_2(x_2') | 0 \rangle. \quad (3.11)$$

For simplicity, let us denote

$$G^{(2)}(x_1, x_2) = f(x_1, x_2) \quad (3.12)$$

Exploiting the assumed conformal symmetries of the system, use the fact that the Jacobian for a translation, as well as for a rotation, is equal to one, such that $\left| \frac{\partial x'}{\partial x} \right| = 1$, and, therefore, the Green's function can only depend on the length

$$f(x_1, x_2) = f(Rx_1 + a, Rx_2 + a), \quad (3.13)$$

requiring that

$$G^{(2)}(x_1, x_2) = f(|x_1 - x_2|) = f(r_{12}). \quad (3.14)$$

For a dilation, the Jacobian obeys a scale factor, such that $\left| \frac{\partial x'}{\partial x} \right| = \lambda^\Delta$, and the Green's function has the form

$$G^{(2)}(x_1, x_2) = \lambda^{\Delta_1 + \Delta_2} f(\lambda r_{12}). \quad (3.15)$$

To calculate the function that obeys this constraint, expand $f(r_{12}) = \sum_a f_a r_{12}^a$ in a series, noting that a can be a continuous parameter, and compare to the above to get the condition that the coefficients f_a are all zero, except for $a = -\Delta_1 - \Delta_2$.

Therefore, under translations, rotation, and dilations, we find that the Green's function is constrained to the form

$$G^{(2)}(x_1, x_2) = \frac{f_{-\Delta_1 - \Delta_2}}{r_{12}^{\Delta_1 + \Delta_2}} = \frac{c_{12}}{r_{12}^{\Delta_1 + \Delta_2}} \quad (3.16)$$

Where c_{12} is a constant determined by the normalization condition.

So far, we have come to the conclusion that the correlation function must obey a power law, and we now apply the constraints of SCTs to find that the two-point correlation function of quasi-primary fields is zero unless the two fields have the same scaling dimension $\Delta_1 = \Delta_2 = \Delta$ (**Exercise**)

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) | 0 \rangle = \frac{c_{12}}{r_{12}^{2\Delta}} \quad (3.17)$$

This means that if we solve a system and calculate the two-point correlation function of two fields with different scaling dimensions, $\Delta_1 \neq \Delta_2$, and get a nonzero result, then our system is not conformally invariant under the full global conformal group, but under a subgroup consisting of translations, rotations, and dilations.

Example: Three-Point Correlation

Following similar procedure as in the two-point case, translations, rotations, and dilations lead to conclusion that the three-point correlation function must have the form

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) | 0 \rangle = \sum_{a,b,c} \frac{C_{abc}}{r_{12}^a r_{23}^b r_{31}^c} \quad (3.18)$$

Where the summation is constrained by the field scaling dimensions to satisfy $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$.

Special conformal transformations further constrain the exponents in the power law to (**Exercise**)

$$a = \Delta_1 + \Delta_2 - \Delta_3, \quad b = -\Delta_1 + \Delta_2 + \Delta_3, \quad c = \Delta_1 - \Delta_2 + \Delta_3. \quad (3.19)$$

And the three-point correlation function has the form

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) | 0 \rangle = \frac{C_{123}}{r_{12}^{\Delta_1 + \Delta_2 - \Delta_3} r_{23}^{-\Delta_1 + \Delta_2 + \Delta_3} r_{31}^{\Delta_1 - \Delta_2 + \Delta_3}} \quad (3.20)$$

Example: Four-Point Correlation

The four-point correlation function has the form

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) | 0 \rangle = F \left(\frac{r_{12} r_{34}}{r_{13} r_{24}}, \frac{r_{12} r_{34}}{r_{23} r_{14}} \right) \prod_{j < k} r_{jk}^{-\Delta_j - \Delta_k + \frac{\Delta}{3}} \quad (3.21)$$

Where $F(\cdot, \cdot)$ is an arbitrary function of the cross products and $\Delta = \sum_j \Delta_j$.

Embedding Space Formalism

The embedding space formalism gives the same answer⁴: Considering a tensor field of $SO(d+1, 1)$ denoted as $O_{A_1 \dots A_n}(X)$, with the properties

- defined on the null-cone $X^2 = 0$,
- traceless and symmetric,
- homogeneous of degree $-\Delta$ in X , i.e., $O_{A_1 \dots A_n}(\lambda X) = \lambda^{-\Delta} O_{A_1 \dots A_n}(X)$,
- transverse $X^{A_i} O_{A_1 \dots A_n}(X) = 0$, with $i = 1, \dots, n$

It is clear that those are conditions rendering $O_{A_1 \dots A_n}(X)$ manifestly invariant under $SO(d+1, 1)$. In order to find the corresponding tensor in \mathbb{R}^d , one has to restrict $O_{A_1 \dots A_n}(X)$ to the Poincaré section and project the indices as

$$\langle O^\mu(x_1) O^\nu(x_2) \rangle = \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_1^B}{\partial x_2^\nu} \langle O_A(X_1) O_B(X_2) \rangle \quad (3.22)$$

For example, the most general form of the two-point function of two operators with spin-1 and dimension Δ can be derived as⁵:

$$\langle O^A(X_1) O^B(X_2) \rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\eta^{AB} + \alpha \frac{X_2^A X_1^B}{X_1 \cdot X_2} + \beta \frac{X_1^A X_2^B}{X_1 \cdot X_2} \right] \quad (3.23)$$

We will drop the last term as it projects to zero anyways.

$$\langle O^A(X_1) O^B(X_2) \rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\eta^{AB} + \alpha \frac{X_2^A X_1^B}{X_1 \cdot X_2} \right] \quad (3.24)$$

According to the transverse condition

⁴Section 2.4 of “Conformal field theory in momentum space and anomaly actions in gravity: The analysis of three- and four-point functions” or section 5.2.2 of “Conformal Field Theory” by Lorian Bonora

⁵Here the terms in bracket are chosen such that they are invariant under the replacement $x \rightarrow \lambda x$. We are not using the transformation law for any of them. Under which, even the metric will change to $\eta_{AB} \rightarrow \lambda^{-2} \eta_{AB}$.

$$X_{A1} \langle O^A(X_1) O^B(X_2) \rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} [X_1^B + \alpha X_1^B] = 0 \implies \alpha = -1 \quad (3.25)$$

We now use the projection to find the correlation function in \mathbb{R}^d :

$$\langle O_\mu(x_1) O_\nu(x_2) \rangle = \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \langle O_A(X_1) O_B(X_2) \rangle \quad (3.26)$$

$$= \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\eta_{AB} - \frac{X_{A2} X_{B1}}{X_1 \cdot X_2} \right] \quad (3.27)$$

$$= \frac{C_{12}}{(X_1 \cdot X_2)^\Delta} \left[\frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \eta_{AB} - \frac{\partial X_1^A}{\partial x_1^\mu} \frac{\partial X_2^B}{\partial x_2^\nu} \frac{X_{A2} X_{B1}}{X_1 \cdot X_2} \right] \quad (3.28)$$

$$= \frac{C_{12}}{(x_1 - x_2)^{2\Delta}} \left[g_{\mu\nu} - 2 \frac{(x_1 - x_2)_\mu (x_1 - x_2)_\nu}{(x_1 - x_2)^2} \right] \quad (3.29)$$

where we used $X^A = (X^a, X^+, X^-) = (x^a, 1, x^2)$, $X_B = (x_a, -\frac{1}{2}x^2, -\frac{1}{2})$ and $\eta_{ab} = I_{d \times d}$ with $\eta_{+-} = \eta_{-+} = -1/2$

$$\frac{\partial X_1^A}{\partial x_1^\mu} X_{A2} = \eta_{ab} \frac{\partial x_1^a}{\partial x_1^\mu} x_2^b - \frac{1}{2} \frac{\partial 1}{\partial x_1^\mu} x_2^2 - \frac{1}{2} \frac{\partial x_1^2}{\partial x_1^\mu} \cdot 1 \quad (3.30)$$

$$= \eta_{ab} \delta_\mu^a x_2^b - x_{1\mu} = (x_2 - x_1)_\mu = -(x_1 - x_2)_\mu \quad (3.31)$$

$$\frac{\partial X_2^B}{\partial x_2^\nu} X_{B1} = \eta_{ab} \frac{\partial x_2^a}{\partial x_2^\nu} x_1^b - \frac{1}{2} \frac{\partial 1}{\partial x_2^\nu} x_1^2 - \frac{1}{2} \frac{\partial x_2^2}{\partial x_2^\nu} \cdot 1 = (x_1 - x_2)_\nu \quad (3.32)$$

$$\frac{\partial X_2^B}{\partial x_2^\nu} X_{B1} = \eta_{ab} \frac{\partial x_2^a}{\partial x_2^\nu} x_1^b - \frac{1}{2} \frac{\partial 1}{\partial x_2^\nu} x_1^2 - \frac{1}{2} \frac{\partial x_2^2}{\partial x_2^\nu} \cdot 1 = (x_1 - x_2)_\nu \quad (3.33)$$

$$(X_1 - X_2)^A (X_1 - X_2)_A = (x_1 - x_2)^a (x_1 - x_2)_a - \frac{1}{2}(1-1)(x_1^2 - x_2^2) - (x_1^2 - x_2^2) \left(\frac{1}{2} - \frac{1}{2} \right) \quad (3.34)$$

$$\implies X_1 \cdot X_2 = -\frac{1}{2}(x_1 - x_2)^2 \quad (3.35)$$

Three-Point Correlator via Embedding Space

Next, we bootstrap the three-point correlator.⁶ On the null cone we will have

$$\langle \phi_1(X_1) \phi_2(X_2) J_M(X_3) \rangle = \frac{W_M}{(-2X_1 \cdot X_2)^{\alpha_{123}} (-2X_1 \cdot X_3)^{\alpha_{132}} (-2X_2 \cdot X_3)^{\alpha_{231}}} \quad (3.36)$$

where the powers α_{123} , α_{132} , α_{231} of the scalar factor are determined by the dilatation as in case of scalar operators and the tensor structure W_M equals to

$$W_M = \frac{(-2X_2 \cdot X_3) X_{1M} - (-2X_1 \cdot X_3) X_{2M} - (-2X_1 \cdot X_2) X_{3M}}{(-2X_1 \cdot X_2)^{\frac{1}{2}} (-2X_1 \cdot X_3)^{\frac{1}{2}} (-2X_2 \cdot X_3)^{\frac{1}{2}}} \quad (3.37)$$

Let us comment a few things on the tensor structure. The relative sign is, as before, fixed by transversality.

$$(X_1)^M W_M = 0 \quad (3.38)$$

$$(X_2)^M W_M = 0 \quad (3.39)$$

⁶See also the discussion in the previous section on three-point functions from the direct approach.

$$(X_3)^M W_M = 0 \quad (3.40)$$

We drop the term proportional to X_{3M} , since it would project to zero anyway. The scaling behavior of the correlation function under dilatation is completely determined in the scalar part so the tensor structure has scaling 0 in all variables ($X \rightarrow \lambda X \implies W_\mu \rightarrow \lambda^0 W_\mu$). Finally, it is immediate to check that the tensor structure is transverse, i.e. $(X_3)_M W_M = 0$. Projecting to physical space as:

$$\langle \phi_1(x_1) \phi_2(x_2) J_\mu(x_3) \rangle = \frac{\partial X_3^M}{\partial x_3^\mu} \langle \phi_1(X_1) \phi_2(X_2) J_M(X_3) \rangle \quad (3.41)$$

we find, as explicitly computed before,

$$\frac{\partial X_3^M}{\partial x_3^\mu} X_{iM} = (x_i - x_3)_\mu, \quad i = 1, 2 \quad (3.42)$$

$$-2X_i \cdot X_j = (x_i - x_j)^2, \quad i = 1, 2, 3 \ (i < j), \quad (3.43)$$

so that we end up with the tensor structure

$$W_\mu = \frac{|x_2 - x_3|^2 (x_1 - x_3)_\mu - |x_1 - x_3|^2 (x_2 - x_3)_\mu}{|x_1 - x_2| |x_1 - x_3| |x_2 - x_3|} = \frac{|x_{23}|^2 (x_{13})_\mu - |x_{13}|^2 (x_{23})_\mu}{|x_{12}| |x_{13}| |x_{23}|} \quad (3.44)$$

Therefore, the three-point function of two scalars and one spin-1 operator in physical space is given by

$$\langle \phi_1(x_1) \phi_2(x_2) J_\mu(x_3) \rangle = \frac{\frac{|x_{23}|^2 (x_{13})_\mu - |x_{13}|^2 (x_{23})_\mu}{|x_{12}| |x_{13}| |x_{23}|}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}} \quad (3.45)$$

$$= \frac{|x_{23}|^2 (x_{13})_\mu - |x_{13}|^2 (x_{23})_\mu}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3 + 1} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2 + 1} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1 + 1}} \quad (3.46)$$

The three-point function of higher-spin operators $J_{\mu_1 \dots \mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that W_μ is the only indexed object for three points that is conformal invariant.

Conformal Theory in (2+0) Dimensions

Thus far, we have been studying the structure of conformal theories in arbitrary spacetime dimension d . In (2+0) dimensions, conformal theories will take on a new definition.

Recall that we defined the complex fields $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, and made the assumption that we can *analytically* continue to arbitrary, independent z and \bar{z} , such that

$$\Phi(x_1, x_2) = \Phi(z, \bar{z}) \quad (3.47)$$

And work out the consequences from there. We now extend our definition of quasi-primary fields, by analogy to the general, arbitrary dimension case. We will now refer to these fields as *primary fields* of type, or conformal weight, (h, \bar{h}) with the proposed form

$$\Phi(z, \bar{z}) = \left(\frac{\partial f}{\partial z} \right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}} \right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})) \quad (3.48)$$

Note here that there are indeed enough examples of primary fields of conformal field theories to make these assumptions interesting to study.

Expand the primary field infinitesimally, as it is the more convenient method opposed to working with the full global transformation, by sending $z \rightarrow z + \epsilon(z)$, and complex conjugate, to order ϵ

$$\delta_{\epsilon, \bar{\epsilon}} \Phi(z, \bar{z}) = \left((h \partial_z \epsilon(z) + \epsilon(z) \partial_z) + (\bar{h} \partial_{\bar{z}} \bar{\epsilon}(\bar{z}) + \bar{\epsilon}(\bar{z}) \partial_{\bar{z}}) \right) \Phi(z, \bar{z}). \quad (3.49)$$

And write the two-point correlation function as

$$G^{(2)}(z, \bar{z}) = \langle 0 | \Phi_1(z_1, \bar{z}_1) \Phi_2(z_2, \bar{z}_2) | 0 \rangle \quad (3.50)$$

Where $z \equiv (z_1, z_2)$ and $\bar{z} \equiv (\bar{z}_1, \bar{z}_2)$.

Applying the infinitesimal equation for $\delta_{\epsilon, \bar{\epsilon}}\Phi(z, \bar{z})$ from above, and setting equal to zero by conformal invariance to order ϵ , we get

$$\delta_{\epsilon, \bar{\epsilon}}G^{(2)}(z, \bar{z}) = \langle 0|\delta_{\epsilon, \bar{\epsilon}}\Phi_1, \Phi_2|0\rangle + \langle 0|\Phi_1, \delta_{\epsilon, \bar{\epsilon}}\Phi_2|0\rangle = 0. \quad (3.51)$$

From here, as before in the arbitrary dimension case, we can work out the constraints of each type of conformal transformation on the correlation function.

Translation:

$$\epsilon(z) = \epsilon \implies G^{(2)}(z, \bar{z}) \propto z_{12} = z_1 - z_2 \text{ and } \bar{z}_{12} = \bar{z}_1 - \bar{z}_2. \quad (3.52)$$

Rotation & Dilation:

$$\epsilon(z) = z \implies G^{(2)}(z, \bar{z}) = \frac{c_{12}}{z_{12}^{h_1+h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}. \quad (3.53)$$

SCT:

$$\epsilon(z) = z^2 \implies G^{(2)}(z, \bar{z}) = \frac{c_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}}. \quad (3.54)$$

Now, for example, suppose we have a bosonic field which applies the constraint $h - \bar{h} = 0$ to the conformal weights. Setting $h + \bar{h} = \Delta$, the analytically continued two-point correlation function for the bosonic field becomes

$$G^{(2)}(z, \bar{z}) = \frac{c_{12}}{|z_{12}|^{2\Delta}} \quad (3.55)$$

Example: Three-Point Correlation in 2D

Similarly, the three-point correlation function has the form

$$G^{(3)}(z, \bar{z}) = c_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{-h_1+h_2+h_3} z_{31}^{h_1-h_2+h_3}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{-\bar{h}_1+\bar{h}_2+\bar{h}_3} \bar{z}_{31}^{\bar{h}_1-\bar{h}_2+\bar{h}_3}} \quad (3.56)$$

In summary, we have introduced what we consider quantum field theories and have analyzed the consequences of conformal invariance of these quantum theories. We have also analyzed subclasses of theories that obey additional constraints (e.g., $\Delta = h + \bar{h}$ is real).

3.8 Fermions in Embedding Space

Following is taken from section 3.2 of Lectures on Conformal Field Theories by Hugh Osborn. To discuss spinor fields in the embedding formalism requires extending the usual d -dimensional gamma matrices to $d + 2$ dimensions. For $d = 2n$, we define ¹⁰

$$\begin{aligned} a_0^\pm &= \frac{1}{2} (\pm\gamma^0 + \gamma^1) \\ a_1^\pm &= \frac{1}{2} (\gamma^2 \pm i\gamma^3) \\ a_2^\pm &= \frac{1}{2} (\gamma^4 \pm i\gamma^5) \\ &\vdots \\ a_{\frac{d-2}{2}}^\pm &= \frac{1}{2} (\gamma^{d-2} \pm i\gamma^{d-1}) \end{aligned}$$

where gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

One can show that:

$$\begin{aligned} \{a_i^-, a_j^-\} &= \{a_i^+, a_j^+\} = 0 \\ \{a_i^-, a_j^+\} &= \delta_{ij} \quad i, j = 0, 1, 2 \dots \frac{d-2}{2} \end{aligned}$$

⁶⁹ pg 30 of Masters Thesis on "Spinning Correlators at Finite Temperature" of Oscar Arandes Tejerina

¹⁰ we have abused notation for the sake of avoiding cluttering of indices and \pm

In the literature $d - 2/2$ is defined as another variable labeled by k , but for the sake for clarity we will keep it explicit. This is the algebra of raising and lowering operators for $d/2$ independent two-level systems. We ask how many basis vectors are there (including lowest weight state) which could be formed by operating $d/2$ raising a_i^+ on lowest weight state: ¹¹

$$\sum_{r=0}^{d/2} \binom{d/2}{r} = 2^{d/2}$$

It implies that in d -dimensions, we have $2^{d/2} \times 2^{d/2}$ dimensional matrix representation for γ -matrices. We will use the highest weight representation to determine a_j^\pm and then use them to construct γ_μ . From (2.16), we quickly observe that

$$(a_i^-)^2 = 0 = (a_i^+)^2$$

It implies that we can only act a_i or a_i^\dagger once on a state, the second time it acts the state is annihilation. We will build off our intuition from harmonic oscillator (fermionic) and assume that there is a lowest weight state $|\xi\rangle$ such that

$$a_i^- |\xi\rangle = 0 \quad \text{for all } i$$

Similarly, acting on it once by each a_i^\dagger for all i , we can construct states in the representation. The states can be labeled $s = (s_0, s_1, \dots, s_{\frac{d-2}{2}})$, where each of the $s_a = \pm \frac{1}{2}$:

$$|\xi^{(s)}\rangle = (a_{\frac{d-2}{2}}^+)^{s_{\frac{d-2}{2}} + \frac{1}{2}} \dots (a_0^+)^{s_0 + \frac{1}{2}} |\xi\rangle$$

The lowest weight state $|\xi\rangle$ corresponds to all $s_a = -\frac{1}{2}$. Taking the $|\xi^s\rangle$ as a basis, we derive the matrix elements of γ_μ from the definitions and the anti-commutation relation. Starting with $d = 2$, we have a single two-level system:

$$|\xi^{(\frac{1}{2})}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\xi^{(-\frac{1}{2})}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we can construct the raising and lowering operator connecting these two matrices as:

$$a_0^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad a_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we find:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $d = 4$, we have 2 independent fermionic oscillator:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we construct, the following a_i^+ and a_i^- operators for $i = 0, 1$.

$$a_0^+ = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad a_0^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$a_1^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad a_1^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

From (2.17) we see that ¹²

⁶¹¹ we would like to remind ourselves that number of linearly independent basis is defined as the dimension of space.

$$a_0^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad a_0^+ a_1^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, we conclude that the gamma matrices are gives as:

$$\gamma^0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

The above choice of gamma matrices satisfy the clifford algebra, however the chosen basis is not familiar from QFT textbooks. Given a representation γ^μ in d dimensions, we can construct a representation Γ^μ in $d + 2$ dimensions using the prescription,

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^\mu \otimes \sigma^3, \quad \mu = 0, \dots, d-3,$$

$$\Gamma^{d-2} = \mathbb{I} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^1, \quad \Gamma^{d-1} = \mathbb{I} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbb{I} \otimes \sigma^2$$

where the σ^i obey

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

The 2×2 matrices that we add act on the index $s_{d-2/2}$, which newly appears in going from $d = 2k$ to $2k + 2$ dimensions. In odd dimensions the first $d - 1$ gamma matrices can be constructed as above, and $\Gamma_d = \pm \Gamma_1 \Gamma_2 \dots \Gamma_{d-1}$ completes the gamma matrix algebra. There are two independent representations of the gamma matrix algebra in odd dimensions, differing in the sign of Γ_d . These representations are exchanged by parity, and both representations appear in a parity-conserving theory.

We now move onto calculating the correlation function involving spinors in embedding space formalism. To define spinor fields on null cone in embedding space requires that the number of component in \mathbb{R}^d is half the number of components in $\mathbb{R}^{d+1,1}$.

$$\psi(x) \rightarrow \Psi(X), \quad \bar{\psi}(x) \rightarrow \bar{\Psi}(X)$$

which satisfies the following homogeneity condition:

$$\Psi(\lambda X) = \lambda^{-\Delta+\frac{1}{2}} \Psi(X), \quad \bar{\Psi}(\lambda X) = \lambda^{-\Delta+\frac{1}{2}} \bar{\Psi}(X)$$

The degrees of freedom of $\Psi; \bar{\Psi}$ are reduced to those for $\psi; \bar{\psi}$ by imposing the transversality condition like before:

$$\bar{\Gamma}_A X^A \Psi(X) = 0 \quad \bar{\Psi}(X) \Gamma_A X^A = 0$$

It introduces the gauge invariance and thus the degrees of freedom are now halved by imposing the equivalence relations

$$\Psi'(X) \sim \Psi' + \bar{\Gamma}_A X^A \zeta(X) \quad \bar{\Psi}'(X) \sim \bar{\Psi}' + \bar{\zeta}(X) \Gamma_A X^A$$

for arbitrary spinor $\zeta(X); \bar{\zeta}(X)$ of appropriate homogeneity. From standard QFT, we are familiar that

$$V_A = \bar{\Psi} \Gamma_A \Psi'$$

transforms like a vector and we also have

⁶¹² a_0^\pm acts like raising and lowering operator in the same oscillator while a_1^\pm changes the oscillator.

$$\Psi(X) = \Gamma_B X^B \Psi'(X)$$

$$\bar{\Psi}(X) = \bar{\Psi}'(X) \bar{\Gamma}_B X^B$$

Now compute the contraction:

$$\begin{aligned} V_A &= \bar{\Psi}(X) \Gamma_A \Psi'(X) = (\bar{\Psi}'(X) \bar{\Gamma}_B X^B) \Gamma_A \Psi'(X) \\ &= \bar{\Psi}'(X) \bar{\Gamma}_B X^B \Gamma_A \Psi'(X) \end{aligned}$$

Use the Clifford algebra identity:

$$\bar{\Gamma}_A \Gamma_B = -\bar{\Gamma}_B \Gamma_A + 2\eta_{AB}$$

$$\Gamma_A \bar{\Gamma}_B = -\Gamma_B \bar{\Gamma}_A + 2\eta_{AB}$$

from (2.18), and above we rewrite:

$$V_A = \bar{\Psi}' \bar{\Gamma}_B X^B \Gamma_A \Psi' = -\bar{\Psi}' \bar{\Gamma}_A \Gamma_B X^B \Psi' + 2X_A \bar{\Psi}' \Psi'$$

The second term, $2X_A \bar{\Psi}' \Psi'$, is proportional to X_A and is hence pure gauge under the equivalence relation:

$$V_A \sim V_A + X_A f(X)$$

so it can be discarded in physical quantities. Therefore, we obtain:

$$\bar{\Psi} \Gamma_A \Psi' + X_A f(X) \sim -\bar{\Psi}' \bar{\Gamma}_A \Gamma_B X^B \Psi' + 2X_A \bar{\Psi}' \Psi'$$

$$\bar{\Psi}(X) \Gamma_A \Psi'(X) \sim -\bar{\Psi}'(X) \bar{\Gamma}_A \Psi(X)$$

whereas,

$$\bar{\Psi} \Psi$$

transforms like a scalar. However, the above is only under (2.18) in odd dimensions. So it does not correspond to a scalar on the projective null cone in even dimensions.

Chapter 4

2D CFT

Conformal invariance takes a new meaning in two dimensions. As already apparent in previous chapter, the case $d = 2$ requires special attention. Indeed, there exists in two dimensions an infinite variety of coordinate transformations that, although not everywhere well-defined, are locally conformal: they are holomorphic mappings from the complex plane (or part of it) onto itself. Among this infinite set of mappings one must distinguish the 6-parameter global conformal group, made of one-to-one mappings of the complex plane into itself. The analysis of the previous chapter still holds when considering these transformations only. However, a local field theory should be sensitive to local symmetries, even if the related transformations are not globally defined. It is local conformal invariance that enables exact solutions of two-dimensional conformal field theories.

4.1 Conformal Group in Two Dimensions

We begin by considering the flat two-dimensional Euclidean space with coordinates x^0 and x^1 . A conformal transformation in this space takes the form

$$x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$$

where the conformal Killing equation imposes the condition

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \partial^\rho \epsilon_\rho$$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = g_{\mu\nu} (\partial_0 \epsilon_0 + \partial_1 \epsilon_1)$$

Evaluating these relations component by component, we obtain:

- For $\mu = \nu = 0$

$$2\partial_0 \epsilon_0 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_0 \epsilon_0 = \partial_1 \epsilon_1$$

- For $\mu = \nu = 1$

$$2\partial_1 \epsilon_1 = \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \implies \partial_1 \epsilon_1 = \partial_0 \epsilon_0$$

- For $\mu = 0, \nu = 1$

$$\partial_0 \epsilon_1 + \partial_1 \epsilon_0 = 0 \implies \partial_0 \epsilon_1 = -\partial_1 \epsilon_0$$

Thus only two independent constraints remain:

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1$$

$$\partial_0 \epsilon_1 = -\partial_1 \epsilon_0$$

To make further progress, it is convenient to introduce complex coordinates $(z_1, z_2) \equiv (z, \bar{z})$ by embedding the \mathbb{R}^2 plane in \mathbb{C}^2 . This embedding allows us to treat the two real directions as components of a single complex variable on the hypersurface $z_1^* = z_2$, which greatly simplifies the form of the constraints.

$$\left. \begin{aligned} z &= x^0 + ix^1 \\ \bar{z} &= x^0 - ix^1 \end{aligned} \right\} \begin{aligned} x^0 &= \frac{z + \bar{z}}{2} \\ x^1 &= \frac{z - \bar{z}}{2i} \end{aligned}$$

Now let us see how the transformation acts in these coordinates. Under $x^\mu \rightarrow x^\mu + \epsilon^\mu$, we find

$$z \rightarrow x^0 + \epsilon^0 + i(x^1 + \epsilon^1) = (x^0 + ix^1) + (\epsilon^0 + i\epsilon^1) = z + \epsilon(z, \bar{z})$$

and similarly,

$$\bar{z} \rightarrow \bar{z} + \bar{\epsilon}(z, \bar{z})$$

where we define $\bar{\epsilon} = \epsilon^0 - i\epsilon^1$. To rewrite the constraints, we must also translate derivatives into complex coordinates. Using the chain rule,

$$\begin{aligned} \partial_0 &= \frac{\partial}{\partial x^0} = \frac{\partial z}{\partial x^0} \frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^0} \frac{\partial}{\partial \bar{z}} = \partial + \bar{\partial} \\ \partial_1 &= \frac{\partial}{\partial x^1} = \frac{\partial z}{\partial x^1} \frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x^1} \frac{\partial}{\partial \bar{z}} = i(\partial - \bar{\partial}) \end{aligned}$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$ are the Wirtinger derivatives:

$$\begin{aligned} \partial &= \frac{\partial}{\partial z} = \frac{\partial x^0}{\partial z} \frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial z} \frac{\partial}{\partial x^1} = \frac{1}{2} \left(\frac{\partial}{\partial x^0} - i \frac{\partial}{\partial x^1} \right) \\ \bar{\partial} &= \frac{\partial}{\partial \bar{z}} = \frac{\partial x^0}{\partial \bar{z}} \frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial \bar{z}} \frac{\partial}{\partial x^1} = \frac{1}{2} \left(\frac{\partial}{\partial x^0} + i \frac{\partial}{\partial x^1} \right) \end{aligned}$$

We also note that

$$\begin{aligned} \epsilon_0 &= \frac{\epsilon + \bar{\epsilon}}{2} \\ \epsilon_1 &= \frac{\epsilon - \bar{\epsilon}}{2i} \end{aligned}$$

Substituting into the first constraint $\partial_0 \epsilon_0 = \partial_1 \epsilon_1$, we obtain.

$$\begin{aligned} \partial_0 \epsilon_0 &= \partial_1 \epsilon_1 \\ (\partial + \bar{\partial}) \left(\frac{\epsilon + \bar{\epsilon}}{2} \right) &= i(\partial - \bar{\partial}) \left(\frac{\epsilon - \bar{\epsilon}}{2i} \right) \\ \partial \epsilon + \partial \bar{\epsilon} + \bar{\partial} \epsilon + \bar{\partial} \bar{\epsilon} &= \partial \epsilon - \partial \bar{\epsilon} - \bar{\partial} \epsilon + \bar{\partial} \bar{\epsilon} \\ \bar{\partial} \epsilon &= -\partial \bar{\epsilon} \end{aligned} \tag{3.3}$$

From the second constraint $\partial_0 \epsilon_1 = -\partial_1 \epsilon_0$, we similarly find

$$\begin{aligned} \partial_0 \epsilon_1 &= -\partial_1 \epsilon_0 \\ (\partial + \bar{\partial}) \left(\frac{\epsilon - \bar{\epsilon}}{2i} \right) &= -i(\partial - \bar{\partial}) \left(\frac{\epsilon + \bar{\epsilon}}{2} \right) \\ \partial \epsilon - \partial \bar{\epsilon} + \bar{\partial} \epsilon - \bar{\partial} \bar{\epsilon} &= \partial \epsilon + \partial \bar{\epsilon} - \bar{\partial} \epsilon - \bar{\partial} \bar{\epsilon} \\ \bar{\partial} \epsilon &= \partial \bar{\epsilon} \end{aligned} \tag{3.4}$$

Combining both (3.3) and (3.4), we conclude

$$\partial\bar{\epsilon} = 0 \implies \frac{\partial\bar{\epsilon}}{\partial z} = 0$$

$$\bar{\partial}\epsilon = 0 \implies \frac{\partial\epsilon}{\partial\bar{z}} = 0$$

This shows that $\epsilon(z)$ is holomorphic while $\bar{\epsilon}(\bar{z})$ is antiholomorphic. Therefore, the conformal transformation factorizes neatly as

$$z \rightarrow z + \epsilon(z)$$

$$\bar{z} \rightarrow \bar{z} + \bar{\epsilon}(\bar{z})$$

or equivalently

$$z \rightarrow f(z)$$

with

$$\frac{\partial f}{\partial\bar{z}} = 0$$

i.e. $f(z)$ is analytic or holomorphic. Since $\epsilon(z)$ is holomorphic, it admits a Laurent expansion.

$$\begin{aligned} z' &= z + \epsilon(z) \\ &= z + \sum_{n \in \mathbb{Z}} \epsilon_n z^n \\ &= z + \sum_{n \in \mathbb{Z}} \epsilon_n z^n \partial z \\ &= \left(1 + \sum_{n \in \mathbb{Z}} \epsilon_n z^n \partial \right) z \end{aligned}$$

It is now easy to read off the generators of conformal transformation in 2 d from above. We define,

$$r_n = z^n \partial$$

Similarly,

$$\begin{aligned} \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) \\ &= \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \\ &= \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial} \bar{z} \\ &= \left(1 + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^n \bar{\partial} \right) \bar{z} \end{aligned}$$

and hence

$$\bar{r}_n = \bar{z}^n \bar{\partial}$$

To determine the structure of the conformal algebra, we now compute the commutator between two such generators. A straightforward calculation shows that the r_n satisfy the following commutation relation:

$$\begin{aligned}
[r_m, r_n] f(z, \bar{z}) &= [z^m \partial, z^n \partial] f(z, \bar{z}) \\
&= z^m \partial (z^n \partial f) - z^n \partial (z^m \partial f) \\
&= z^{m+n} \partial^2 f + z^m (\partial z^n) \partial f - z^{m+n} \partial^2 f - z^n (\partial z^m) \partial f \\
&= (n - m) z^{m+n-1} \partial f \\
&= (n - m) r_{m+n-1} f
\end{aligned}$$

Thus,

$$[r_m, r_n] = (n - m) r_{m+n-1}$$

Following the same steps, we can also find

$$[\bar{r}_m, \bar{r}_n] = (n - m) \bar{r}_{m+n-1}$$

and

$$[r_m, \bar{r}_n] = 0$$

It is conventional to redefine the generators as $r_n = -l_{n-1}$. With this shift, the algebra becomes

$$[r_m, r_n] = (n - m) r_{m+n-1}$$

$$[-l_{m-1}, -l_{n-1}] = -(n - m) l_{m+n-2}$$

and then shifting $m \rightarrow m + 1$ and $n \rightarrow n + 1$

$$[l_m, l_n] = (m - n) l_{m+n}$$

similarly,

$$[\bar{l}_m, \bar{l}_n] = (m - n) \bar{l}_{m+n}$$

$$[l_m, \bar{l}_n] = 0$$

where

$$l_n = -r_{n+1} = -z^{n+1} \partial$$

$$\bar{l}_n = -\bar{r}_{n+1} = -\bar{z}^{n+1} \bar{\partial}$$

This is recognized as Witt algebra. They generate the conformal transformation in 2d Euclidean plane. To find the subset of generators l_n which generate global conformal transformation, we note that z^n has two singularity, one at $z = 0$ for $n < 0$ and another at $z = \infty$ for $n > 0$. Therefore, we impose the restriction that the representations of Witt algebra does not blow up in the limit $z \rightarrow 0$ or $z \rightarrow \infty$. We start with $z \rightarrow 0$ limit, $l_n = -z^{n+1} \partial$ needs to converge, implying:

$$n + 1 \geq 0 \implies n \geq -1$$

Since $z \rightarrow \infty$ does not appear as a pole, we consider the conformal mapping $w = \frac{1}{z}$ which brings the infinity at origin. Then,

$$\begin{aligned}
\partial &= \frac{\partial}{\partial z} = \frac{\partial w}{\partial z} \frac{\partial}{\partial w} \\
&= -\frac{1}{z^2} \frac{\partial}{\partial w} = -w^2 \frac{\partial}{\partial w}
\end{aligned}$$

Then,

$$\begin{aligned}
l_n &= -z^{n+1} \partial = w^{-n-1} w^2 \partial_w \\
&= w^{-n+1} \partial_w
\end{aligned}$$

Then,

$$1 - n \geq 0 \implies n \leq 1$$

Therefore, l_{-1}, l_0 and l_1 generate global conformal transformation on the 2d Euclidean space.

4.2 Global Conformal Transformation

Last section was devoted to finding the generators of global conformal transformation. In this part, we look at the infinitesimal transformation generated by l_{-1} , l_0 and l_1 and then find their finite counterpart. For $n = -1$, the generators associated with the infinitesimal transformation $z \rightarrow z + \epsilon_{-1}$ looks like:

$$l_{-1} = z^{-1+1}\partial = \partial$$

$$\bar{l}_{-1} = \bar{z}^{-1+1}\bar{\partial} = \bar{\partial}$$

So, this generates translation. For $n = 0$, the generators associated with the infinitesimal transformation $z \rightarrow z + \epsilon_0 z$ looks like:

$$l_0 = -z^{0+1}\partial = -z\partial$$

$$\bar{l}_0 = -\bar{z}\bar{\partial}$$

To interpret how the works in complex plane, we will use the polar coordinates $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$ where $r = \sqrt{z\bar{z}}$ and $\theta = \frac{1}{2i} \ln\left(\frac{z}{\bar{z}}\right)$

$$\begin{aligned} \partial &= \frac{\partial}{\partial z} = \frac{\partial r}{\partial z} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} \\ &= \frac{\bar{z}}{2\sqrt{z\bar{z}}} \frac{\partial}{\partial r} + \frac{1}{2i} \frac{\bar{z}}{z} \frac{\partial(z/\bar{z})}{\partial z} \frac{\partial}{\partial \theta} \\ &= \frac{\bar{z}}{2r} \frac{\partial}{\partial r} + \frac{1}{2iz} \frac{\partial}{\partial \theta} \\ &= \frac{e^{-i\theta}}{2} \frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir} \frac{\partial}{\partial \theta} \end{aligned}$$

and,

$$\begin{aligned} \bar{\partial} &= \frac{\partial}{\partial \bar{z}} = \frac{\partial r}{\partial \bar{z}} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \bar{z}} \frac{\partial}{\partial \theta} \\ &= \frac{z}{2\sqrt{z\bar{z}}} \frac{\partial}{\partial r} + \frac{1}{2i} \frac{\bar{z}}{z} \frac{\partial(z/\bar{z})}{\partial \bar{z}} \frac{\partial}{\partial \theta} \\ &= \frac{z}{2\sqrt{z\bar{z}}} \frac{\partial}{\partial r} + \frac{1}{2i} \frac{\bar{z}}{z} \left(-\frac{z}{\bar{z}^2}\right) \frac{\partial}{\partial \theta} \\ &= \frac{z}{2r} \frac{\partial}{\partial r} - \frac{1}{2i\bar{z}} \frac{\partial}{\partial \theta} \\ &= \frac{e^{i\theta}}{2} \frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir} \frac{\partial}{\partial \theta} \end{aligned}$$

we can now recast or generators in polar coordinates as following:

$$\begin{aligned} l_0 &= -z\partial = -re^{i\theta} \left[\frac{e^{-i\theta}}{2} \frac{\partial}{\partial r} + \frac{e^{-i\theta}}{2ir} \frac{\partial}{\partial \theta} \right] \\ &= -\frac{r}{2} \left[\partial_r - \frac{i}{r} \partial_\theta \right] \\ \bar{l}_0 &= -\bar{z}\bar{\partial} \\ &= -re^{-i\theta} \left[\frac{e^{i\theta}}{2} \frac{\partial}{\partial r} - \frac{e^{i\theta}}{2ir} \frac{\partial}{\partial \theta} \right] \\ &= -\frac{r}{2} \left[\partial_r + \frac{i}{r} \partial_\theta \right] \end{aligned}$$

It is now clear that the following combination

$$l_0 + \bar{l}_0 = -r\partial_r$$

is the generator of dilatation and

$$i(l_0 - \bar{l}_0) = \partial_\theta$$

is the generator of rotation. For $n = -1$, the generators associated with the infinitesimal transformation $z \rightarrow z + \epsilon_1 z^2$ looks like:

$$l_1 = -z^2 \partial$$

These are nothing but Mobius transformation.

$$z' = \frac{az + b}{cz + d}$$

For translation we have $a = 1, c = 0$ and $d = 1$ and we have $ad - bc = 1$. For dilatation and rotation, we have

$$\begin{aligned} z' &= z - \epsilon_0 z \\ &= (1 - \epsilon_0) z \\ &= \frac{1 - \epsilon_0}{\sqrt{1 - \epsilon_0}} z + 0 = \frac{\sqrt{1 + \epsilon_0} z + 0}{0z + \frac{1}{\sqrt{1 + \epsilon_0}}} \end{aligned}$$

with $a = \sqrt{1 + \epsilon_0}, b = 0, c = 0$ and $d = \frac{1}{\sqrt{1 + \epsilon_0}}$ so that $ad - bc = 1$. For SCT, we have

$$\begin{aligned} z' &= z + \epsilon_1 z^2 \\ &= z(1 - \epsilon_1 z)^{-1} \\ &= \frac{z}{1 - \epsilon_1 z} = \frac{z + 0}{-\epsilon_1 z + 1} \end{aligned}$$

Since ϵ_n is in general a complex number. These are the set of 2×2 complex matrices with unit determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with } ad - bc = 1$$

These matrices form the group $SL(2, \mathbb{C})$. But under the mapping $a \rightarrow -a, b \rightarrow -b, c \rightarrow -c$ and $d \rightarrow -d$, they generate the same transformation. Therefore, the conformal transformation in 2d forms the group $SL(2, \mathbb{C})/\mathbb{Z}_2$. In the Lorentzian case the group is replaced by $SL(2, R) \times SL(2, R) = SO(2, 2)$ where one factor of $SL(2, R)$ pertains to left-movers and the other to right-movers.

The **global conformal group** in dimensions greater than two is comprised of translations, rotations, scaling, and special conformal transformations, as well as dimensions equal to two, as long as the space is compactified. If singularities are included, functions with poles are allowed, the symmetry gets larger. In the last lecture we introduced global conformal transformations/symmetries of some manifold M that form a (symmetry) group G which can be promoted to a symmetry group of some quantum system, where the kinematics of the system are described by a Hilbert space \mathcal{H} . The quantum system is said to be globally conformally invariant if there is some unitary representation, operators U that act on the Hilbert space,

$$U : G \rightarrow U(\mathcal{H}). \quad (4.1)$$

just a reminder that in the generalized Minkowski space $\mathbb{R}^{p,q}$, the structure of the group of global conformal transformations G consists of compositions of translations, dilations, rotations(boosts), and special conformal transformations (SCTs).

Here we now study the case where $d = 2$, which will expand our notion of what a symmetry is and will allow us to define local, infinitesimal conformal transformations.

For example, a global conformal transformation, a $1 - 1$ differentiable map from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, may consist of a dilation and a rotation and look like

For contrast, consider an infinitesimal conformal transformation $\text{id} + \epsilon X$, where $X = X(x)$ is a vector field, the derivative of a diffeomorphism, that acts on the two-dimensional Minkowski space as

This transformation preserves all of the right angles in the untransformed Minkowski space,

and the action is close to the identity, such that $|x - x'| \sim \mathcal{O}(\epsilon)$. The transformation $\text{id} + \epsilon X$ is conformal to first order in ϵ , as is required by the definition of infinitesimal.

Although the vector field $X(x)$ is a generator of the infinitesimal conformal transformation, it does not necessarily define a global transformation via exponentiation, as it just may not be well defined globally.

To begin to make sense of this, consider in quantum mechanics, where we talk about quantum systems symmetric under a group G with Hilbert space

$$(\mathcal{H}, U : G \rightarrow U(\mathcal{H})). \quad (4.2)$$

So, in quantum mechanics, we are reduced to finding these unitary representations of G . If G is finite, it does not make sense to speak infinitesimally (e.g., one-hundredth of a reflection).

We assume G is a manifold, and then we may as well go as far to assume that G is a Lie group with an associated Lie algebra \mathfrak{g} , which consists of vector fields that exponentiate to the Lie group. Then the quantum system is symmetric under the Lie algebra if you get a representation

$$(\mathcal{H}, \pi : \mathfrak{g} \rightarrow L(\mathcal{H})) \quad (4.3)$$

Where $L(\mathcal{H})$ is the set of (bounded and unbounded) linear operators, and π generates a unitary operator on the Hilbert space, such that $\pi(X) = e^{isX}$, $s \in \mathbb{R}$.

Note that for an infinite-dimensional group, (1) the operator e^{isX} may not be continuous, and (2) the Lie algebra may not exponentiate to a Lie group, which we will encounter in conformal field theory. In other words, in contrast to when we used infinitesimal quantities to build global representations, we find that infinitesimal conformal transformations don't necessarily exponentiate to a group.

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc = 1. \quad (4.4)$$

The set of global conformal transformations allowed in this basis are for $s \in \mathbb{R}$,

$$\text{Translation:} \quad e^{s\ell_{-1}} \equiv \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \quad \equiv z \rightarrow z - s \quad (4.5)$$

$$\text{Dilation:} \quad e^{s(\ell_0 + \bar{\ell}_0)} \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \equiv z \rightarrow e^{-s} z \quad (4.6)$$

$$\text{Rotation:} \quad e^{is(\bar{\ell}_0 - \ell_0)} \equiv \begin{pmatrix} \exp(i\frac{\theta}{2}) & 0 \\ 0 & \exp(-i\frac{\theta}{2}) \end{pmatrix} \quad \equiv z \rightarrow e^{is} z \quad (4.7)$$

$$\text{Special Conformal:} \quad e^{s\ell_1} \equiv \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \equiv z \rightarrow \frac{z}{1 + cz}. \quad (4.8)$$

Note that for $\mathbb{R}^{d,0}$, $d > 2$, the local transformations are also global! Also note that in one-dimensional spacetime, $(1,0)$ or $(0,1)$, conformal transformations are all monotonic increasing functions $\mathbb{R} \rightarrow \mathbb{R}$.

4.3 Virasoro Algebra

In some cases, the regularization scheme used to quantize a classical theory does not preserve the original symmetry. The classical generators, which we obtain by solving the Killing equations, satisfy a certain algebra—for example, the Witt algebra. However, in the quantum system, the requirement of normal ordering to render observables finite modifies the algebraic structure. In the next chapter, we will learn how to construct the corresponding quantum operators systematically using the operator formalism. As a result, these quantum operators no longer satisfy the Witt algebra but instead form the Virasoro algebra. This is unavoidable; it's a reflection of the fact that the classical symmetry is realized projectively in the quantum theory.

The Virasoro algebra is the central extension of Witt algebra. To centrally extend the algebra, a new generator—called the center—is introduced, which commutes with all other generators. To distinguish from the generators of Witt algebra, we use the notation L_n to denote the generators of Virasoro algebra. The algebra is then given as:

$$[L_n, L_m] = (n - m)L_{n+m} + \underbrace{c\rho(n, m)}_{\text{complex number}}$$

This algebra contains a finite-dimensional subalgebra generated by $L_{0,\pm 1}, \bar{L}_{0,\pm 1}$. These are the generators that are well defined all over the complex plane and form the global conformal group in two dimensions just as we saw in Witt algebra. The rest of the generators are local. The object $\rho(n, m)$ which we have added to centrally extend the algebra has certain properties: it is a number which depends on n and m , it commutes with all L_n , it satisfies recursion relation and it vanishes for certain n and m . To see this, let us first note that

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + c\rho(n, m) \\ [L_m, L_n] &= -(n - m)L_{n+m} + c\rho(m, n) \end{aligned}$$

adding above expressions

$$\begin{aligned} [L_n, L_m] + [L_m, L_n] &= (n - m)L_{n+m} + c\rho(n, m) - (n - m)L_{n+m} + c\rho(m, n) \\ \rho(n, m) + \rho(m, n) &= 0 \end{aligned}$$

We have shown that $\rho(n, m)$ is anti-symmetric under the exchange of $n \leftrightarrow m$. Now let us set $m = 0$,

$$\begin{aligned} [L_n^{\text{old}}, L_0] &= nL_n^{\text{old}} + c\rho(n, 0) \\ &= n \left[L_n^{\text{old}} + \frac{c}{n}\rho(n, 0) \right] \end{aligned}$$

Redefining the generators for $n \neq 0$ as:

$$L_n^{\text{new}} = L_n^{\text{old}} + \frac{c}{n}\rho(n, 0)$$

and

$$L_0^{\text{new}} = L_0^{\text{old}}$$

The algebra of the translated generators should not changed:

$$\begin{aligned} [L_n^{\text{old}}, L_0] &= \left[L_n^{\text{new}} - \frac{c}{n}\rho(n, 0), L_0 \right] \\ [L_n^{\text{new}}, L_0] - \left[\frac{c}{n}\rho(n, 0), L_0 \right] &= nL_n^{\text{old}} + c\rho(n, 0) \\ [L_n^{\text{new}}, L_0] &= nL_n^{\text{new}} \end{aligned}$$

This proves that $\rho(n, 0) = 0$ for $n \neq 0$. For $n = 1, m = -1$ we have:

$$\begin{aligned} [L_1^{\text{old}}, L_{-1}^{\text{old}}] &= 2L_0^{\text{old}} + c\rho(1, -1) \\ &= 2 \left[L_0^{\text{old}} + \frac{c}{2}\rho(1, -1) \right] \\ &= 2L_0^{\text{new}} \end{aligned}$$

This, proves that $\rho(1, -1) = 0$, to find other n and m values for which $\rho(n, m)$ vanishes, we will utilize the Jacobi identity.

$$[L_n, [L_m, L_r]] + [L_r, [L_n, L_m]] + [L_m, [L_r, L_n]] = 0$$

For $r = 0$

$$\begin{aligned} [L_n, mL_m] + [L_0, (n - m)L_{n+m}] + [L_m, -nL_n] &= 0 \\ m[(n - m)L_{n+m} + c\rho(n, m)] + (n - m)[-(n + m)L_{n+m}] - n[(m - n)L_{m+n} + c\rho(m, n)] &= 0 \\ \underbrace{[m(n - m) - (n - m)(n + m) - n(m - n)]}_{=0} L_{m+n} - c(m + n)\rho(m, n) &= 0 \end{aligned}$$

Hence,

$$(m + n)\rho(m, n) = 0$$

It means, $\rho(m, n)$ can only be non zero. if $m + n = 0$. So, $\rho(m, n) = \rho(n, -n)\delta_{m+n,0}$. Next to find the recursion relation, we utilize,

$$\begin{aligned} [L_n, [L_1, L_{-1-n}]] + [L_{-1-n}, [L_n, L_1]] + [L_1, [L_{-1-n}, L_n]] &= 0 \\ (n+2)[L_n, L_{-n}] + (n-1)[L_{-1-n}, L_{n+1}] - (2n+1)[L_1, L_{-1}] &= 0 \\ (n+2)[2nL_0 + c\rho(n, -n)] + (n-1)[-2(n+1)L_0 + c\rho(-1-n, 1+n)] - (2n+1)2L_0 &= 0 \\ \underbrace{[2n(n+2) - 2(n-1)(n+1) - 2(2n+1)]L_0 + c[(n+2)\rho(n, -n) - (n-1)\rho(n+1, -n-1)]}_{=0} &= 0 \end{aligned}$$

Hence the recursion relation,

$$\rho(n+1, -n-1) = \frac{n+2}{n-1}\rho(n, -n)$$

under $n \rightarrow n+1$,

$$\rho(n, -n) = \frac{n+1}{n-2}\rho(n-1, -n+1)$$

We can see that for $n=1$

$$\rho(1, -1) = \frac{2}{-1}\rho(0, 0) = 0$$

For $n=2$

$$\rho(2, -2) = \frac{3}{0}\rho(1, -1) = \text{indeterminate}$$

So, we only need to fix $\rho(2, -2)$ and all the other $\rho(n, m)$ gets fixed from that via this recursion relation.

$$\begin{aligned} \rho(n, -n) &= \frac{n+1}{n-2}\rho(n-1, -n+1) \\ &= \frac{n+1}{n-2} \cdots \frac{4}{1}\rho(2, -2) \\ &= \frac{1}{3!} \frac{(n+1)!}{(n-2)!}\rho(2, -2) = {}^nC_3 \rho(2, -2) \end{aligned}$$

For free boson we normally like to set $c=1$ which forces the normalize of $\rho(2, -2) = 1/2$ since $c \times \rho(n, m) = \xi c \times \frac{\rho(n, m)}{\xi}$ needs to stay constant.

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + {}^nC_3\rho(2, -2) \\ &= (n-m)L_{n+m} + c \frac{(n+1)!}{6(n-2)!} \frac{1}{2} \\ &= (n-m)L_{n+m} + c \frac{(n+1)!}{12(n-2)!} \end{aligned}$$

A short remark about the central charge that sometimes c and \bar{c} aren't complex conjugate of each other which allows us to interpret z and \bar{z} as being independent variables even though they are related to each other via complex conjugation operation.

4.4 How Fields transform under $SL(2, \mathbb{C})$

There are four types of fields we will be concerned about in 2d CFT. They are chiral fields that only depend on $z(\bar{h}=0)$ and anti-chiral fields which only depend on $\bar{z}(h=0)$. It is also common to use the terminology holomorphic and anti-holomorphic in order to distinguish between chiral and anti-chiral quantities. There is a special kind of field which has the same transformation law for both Global conformal transformation as well as local conformal transformation, we call them Primary fields. Under conformal transformation $z \rightarrow f(z)$ and $\bar{z} \rightarrow \bar{f}(\bar{z})$, these Primary fields transform as (passive):

$$\phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z} \right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}} \right)^{\bar{h}} \phi(z, \bar{z}) \Big|_{z=f(z), \bar{z}=\bar{f}(\bar{z})} = \left(\frac{\partial f}{\partial z} \right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}} \right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z}))$$

where (h, \bar{h}) are conformal dimensions/weight of the field $\phi(z, \bar{z})$. Sometimes this transformation law (active) is also expressed as:

$$\phi'(z, \bar{z}) = \left(\frac{d}{dz} f^{-1}(z) \right)^h \left(\frac{d}{d\bar{z}} \bar{f}^{-1}(\bar{z}) \right)^{\bar{h}} \phi\left(f^{-1}(z), \bar{f}^{-1}(\bar{z})\right)$$

Define $\zeta = f^{-1}(z)$, $\bar{\zeta} = \bar{f}^{-1}(\bar{z})$.

$$\phi'(z, \bar{z}) = \left(\frac{d\zeta}{dz} \right)^h \left(\frac{d\bar{\zeta}}{d\bar{z}} \right)^{\bar{h}} \phi(\zeta, \bar{\zeta}) \quad \text{with } \zeta = f^{-1}(z), \bar{\zeta} = \bar{f}^{-1}(\bar{z}).$$

Now write the derivative of the inverse in terms of the derivative of f :

$$\frac{d\zeta}{dz} \Big|_{\zeta=f^{-1}(z)} = \frac{1}{\frac{d}{d\zeta} f(\zeta) \Big|_{\zeta=f^{-1}(z)}}, \quad \frac{d\bar{\zeta}}{d\bar{z}} \Big|_{\bar{\zeta}=\bar{f}^{-1}(\bar{z})} = \frac{1}{\frac{d}{d\bar{\zeta}} \bar{f}(\bar{\zeta}) \Big|_{\bar{\zeta}=\bar{f}^{-1}(\bar{z})}}$$

$$\therefore \phi'(z, \bar{z}) = \left(\frac{1}{\frac{d}{d\zeta} f(\zeta) \Big|_{\zeta=f^{-1}(z)}} \right)^h \left(\frac{1}{\frac{d}{d\bar{\zeta}} \bar{f}(\bar{\zeta}) \Big|_{\bar{\zeta}=\bar{f}^{-1}(\bar{z})}} \right)^{\bar{h}} \phi\left(f^{-1}(z), \bar{f}^{-1}(\bar{z})\right).$$

Now replace the left-hand argument $f^{-1}(z)$ by z (relabel $\zeta \mapsto z, \bar{\zeta} \mapsto \bar{z}$):

$$\phi'(z, \bar{z}) = \left(\frac{d}{dz} f(z) \Big|_{z=f^{-1}(z)} \right)^{-h} \left(\frac{d}{d\bar{z}} \bar{f}(\bar{z}) \Big|_{\bar{z}=\bar{f}^{-1}(\bar{z})} \right)^{-\bar{h}} \phi\left(f^{-1}(z), \bar{f}^{-1}(\bar{z})\right).$$

These are active transformation because the RHS has the form $\pi_{ab} \phi_b(\Lambda^{-1}x)$. To see how the transformation works, we consider the following example, $z' = \lambda z$

$$\begin{aligned} \phi'(z, \bar{z}) &= \left(\frac{\partial \lambda z}{\partial z} \right)^h \left(\frac{\partial \bar{\lambda} \bar{z}}{\partial \bar{z}} \right)^{\bar{h}} \phi(\lambda z, \lambda \bar{z}) \\ &= \lambda^h \lambda^{\bar{h}} \phi(\lambda z, \lambda \bar{z}) \end{aligned}$$

$$\phi(z', \bar{z}') = \lambda^{-h} \lambda^{-\bar{h}} \phi'(z, \bar{z})$$

under infinitesimal conformal transformation $z' = f(z) = z + \epsilon$,

$$\begin{aligned} \phi'(z, \bar{z}) &= [1 + \partial\epsilon]^h [1 + \bar{\partial}\bar{\epsilon}]^{\bar{h}} \phi(z + \epsilon, \bar{z} + \bar{\epsilon}) \\ &= [1 + \partial\epsilon]^h [1 + \bar{\partial}\bar{\epsilon}]^{\bar{h}} [\phi(z, \bar{z}) + \epsilon\partial\phi(z, \bar{z}) + \bar{\epsilon}\bar{\partial}\phi(z, \bar{z}) + \mathcal{O}(\epsilon^2)] \\ &= \phi(z, \bar{z}) + \epsilon\partial\phi(z, \bar{z}) + \bar{\epsilon}\bar{\partial}\phi(z, \bar{z}) + h\phi(z, \bar{z})\partial\epsilon + \bar{h}\phi(z, \bar{z})\bar{\partial}\bar{\epsilon} + \mathcal{O}(\epsilon^2) \\ &= [1 + (\epsilon\partial + h\partial\epsilon) + (\bar{\epsilon}\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon})] \phi(z, \bar{z}) \end{aligned}$$

So, the variation $\delta\phi(z, \bar{z})$ can be given as:

$$\sum_{n=-\infty}^{\infty} \epsilon_n [L_n, \phi] + \bar{\epsilon}_n [\bar{L}_n, \phi] = \delta\phi(z, \bar{z}) = [(\epsilon\partial + h\partial\epsilon) + (\bar{\epsilon}\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon})] \phi(z, \bar{z})$$

In the active point of view,

$$\begin{aligned} \phi'(z, \bar{z}) &= [1 + \partial\epsilon]^{-h} [1 + \bar{\partial}\bar{\epsilon}]^{-\bar{h}} \phi(z - \epsilon, \bar{z} - \bar{\epsilon}) \\ &= [1 + \partial\epsilon]^{-h} [1 + \bar{\partial}\bar{\epsilon}]^{-\bar{h}} [\phi(z, \bar{z}) - \epsilon\partial\phi(z, \bar{z}) - \bar{\epsilon}\bar{\partial}\phi(z, \bar{z}) + \mathcal{O}(\epsilon^2)] \\ &= \phi(z, \bar{z}) - \epsilon\partial\phi(z, \bar{z}) - \bar{\epsilon}\bar{\partial}\phi(z, \bar{z}) - h\phi(z, \bar{z})\partial\epsilon - \bar{h}\phi(z, \bar{z})\bar{\partial}\bar{\epsilon} + \mathcal{O}(\epsilon^2) \\ &= [1 - (\epsilon\partial + h\partial\epsilon) - (\bar{\epsilon}\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon})] \phi(z, \bar{z}) \end{aligned}$$

The other kind is quasi-primary field, they only transform as primary field under Global conformal transformation.

4.5 Energy Momentum Tensor

Usually, a Field Theory is defined in terms of a Lagrangian action from which one can derive various objects and properties of the theory. In particular, the energy-momentum tensor can be deduced from the variation of the action with respect to the metric and so it encodes the behaviour of the theory under infinitesimal transformations $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$.

$$S = \int d^D x \mathcal{L}$$

$$\delta S = \int d^D x \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \int d^D x T_{\mu\nu} \delta g^{\mu\nu}$$

Under $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$, $\delta g^{\mu\nu} = \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu$.

$$\begin{aligned} \delta S &= \int d^D x T_{\mu\nu} (\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) \\ &= \int d^D x T_{\mu\nu} \frac{2}{D} (\partial^\rho \epsilon_\rho) g^{\mu\nu} = \frac{2}{D} \int d^D x T_\mu^\mu (\partial^\rho \epsilon_\rho) = 0 \implies T_\mu^\mu = 0 \end{aligned}$$

This tell us that for the conformal invariance to hold, the stress energy tensor has to be traceless.

$$\begin{aligned} \delta S &= \int d^D x T_{\mu\nu} (\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) \\ &= 2 \int d^D x T_{\mu\nu} \partial^\mu \epsilon^\nu \\ &= 2 \int d^D x \partial^\mu (T_{\mu\nu} \epsilon^\nu) - 2 \int d^D x (\partial^\mu T_{\mu\nu}) \epsilon^\nu = 0 \implies \partial^\mu T_{\mu\nu} = 0 \end{aligned}$$

Hence, we conclude that stress energy tensor is canonically conserved as well as traceless in the presence of conformal symmetry. Next we see how this condition gets translated in complex coordinates. The goal is to find $T_{zz}, T_{z\bar{z}}, T_{\bar{z}z}$ and $T_{\bar{z}\bar{z}}$.

$$\begin{aligned} T_{zz} &= \frac{\partial x^0}{\partial z} \frac{\partial x^0}{\partial z} T_{00} + 2 \frac{\partial x^0}{\partial z} \frac{\partial x^1}{\partial z} T_{01} + \frac{\partial x^1}{\partial z} \frac{\partial x^1}{\partial z} T_{11} \\ &= \frac{1}{4} T_{00} + 2 \frac{1}{2} \frac{1}{2i} T_{01} - \frac{1}{4} T_{11} \\ &= \frac{1}{4} [T_{00} - 2i T_{01} - T_{11}] \\ T_{z\bar{z}} &= \frac{\partial x^0}{\partial z} \frac{\partial x^0}{\partial \bar{z}} T_{00} + \frac{\partial x^0}{\partial z} \frac{\partial x^1}{\partial \bar{z}} T_{01} + \frac{\partial x^1}{\partial z} \frac{\partial x^0}{\partial \bar{z}} T_{10} + \frac{\partial x^1}{\partial z} \frac{\partial x^1}{\partial \bar{z}} T_{11} \\ &= \frac{1}{2} \frac{1}{2} T_{00} + \frac{1}{2} \frac{-1}{2i} T_{01} + \frac{1}{2i} \frac{1}{2} T_{10} + \frac{1}{2i} \frac{-1}{2i} T_{11} \\ &= \frac{1}{4} [T_{00} + T_{11}] = \frac{1}{4} \delta^{\mu\nu} T_{\mu\nu} = \frac{1}{4} T_\mu^\mu = 0 \\ T_{\bar{z}\bar{z}} &= \frac{\partial x^0}{\partial \bar{z}} \frac{\partial x^0}{\partial \bar{z}} T_{00} + 2 \frac{\partial x^0}{\partial \bar{z}} \frac{\partial x^1}{\partial \bar{z}} T_{01} + \frac{\partial x^1}{\partial \bar{z}} \frac{\partial x^1}{\partial \bar{z}} T_{11} \\ &= \frac{1}{4} T_{00} + 2 \frac{1}{2} \frac{-1}{2i} T_{01} - \frac{1}{4} T_{11} \\ &= \frac{1}{4} [T_{00} + 2i T_{01} - T_{11}] \end{aligned}$$

Next we investigate the form of $\partial^\mu T_{\mu\nu}$ in complex coordinates.

$$\begin{aligned}\partial_0 T_{00} + \partial_1 T_{10} &= 0 \\ \partial_0 T_{01} + \partial_1 T_{11} &= 0\end{aligned}\tag{3.5}$$

We can now calculate $\partial_{\bar{z}} T_{zz}$:

$$\bar{\partial} T_{zz} = \frac{1}{4} (\bar{\partial} T_{00} - 2i \bar{\partial} T_{01} - \bar{\partial} T_{11})$$

using (3.1) and $T_{\mu}^{\mu} = T_{00} + T_{11} = 0$

$$\begin{aligned}&= \frac{1}{8} [(\partial_0 + i\partial_1) T_{00} - 2i(\partial_0 + i\partial_1) T_{01} - (\partial_0 + i\partial_1) T_{11}] \\ &= \frac{1}{8} [(\partial_0 T_{00} + 2\partial_1 T_{01} - \partial_0 T_{11}) + i(\partial_1 T_{00} - 2\partial_0 T_{01} - \partial_1 T_{11})] \\ &= \frac{1}{8} [2(\partial_0 T_{00} + \partial_1 T_{01}) - 2i(\partial_1 T_{11} + \partial_0 T_{01})] = 0\end{aligned}$$

In the last step we used (3.5). Similarly one can show

$$\partial T_{\bar{z}\bar{z}} = 0$$

Thus, T_{zz} is the chiral field we discussed earlier and $T_{\bar{z}\bar{z}}$ is the anti-chiral field. So in complex coordinates, the stress energy tensor looks like:

$$T_{\mu\nu}(z, \bar{z}) = \begin{bmatrix} 0 & T(z) \\ \bar{T}(\bar{z}) & 0 \end{bmatrix}$$

4.6 Ward Identities

The consequence of a symmetry of the action and measure on correlation functions may also be expressed via the so-called Ward identities. An infinitesimal transformation may be written in terms of the generator as:

$$\phi'(x) = \phi(x) - i\omega_a G_a \phi(x)$$

where ω_a is a collection of infinitesimal, constant parameters. We will consider the above variation of fields in the correlation function. The action is not invariant under such local transformation and its variation is given by:

$$\delta S = \int d^D x (\partial_{\mu} T^{\mu\nu}) \epsilon_{\nu}$$

where j_a^{μ} is the conserved current. The correlation function can be given as:

$$\langle \phi(x_1) \phi(x_2) \dots \phi(x_n) \rangle = \frac{1}{Z} \int D\phi \phi(x_1) \phi(x_2) \dots \phi(x_n) e^{-S[\phi]}$$

We use the invariance of correlators under the transformation to argue,

$$\begin{aligned}\langle \phi(x_1) \phi(x_2) \dots \phi(x_n) \rangle &= \langle \phi'(x_1) \phi'(x_2) \dots \phi'(x_n) \rangle \\ &= \frac{1}{Z} \int D\phi' \phi'(x_1) \phi'(x_2) \dots \phi'(x_n) e^{-S[\phi']} = \frac{1}{Z} \int D\phi' \phi'(x_1) \phi'(x_2) \dots \phi'(x_n) e^{-S[\phi]} \\ &= \frac{1}{Z} \int D\phi' [\phi(x_1) \phi(x_2) \dots \phi(x_n) + \delta(\phi(x_1) \phi(x_2) \dots \phi(x_n))] e^{-S[\phi] - \int d^D x \partial_{\mu} T^{\mu\nu} \epsilon_{\nu}(x)} \\ &= \frac{1}{Z} \int D\phi [\phi(x_1) \phi(x_2) \dots \phi(x_n) + \delta(\phi(x_1) \phi(x_2) \dots \phi(x_n))] e^{-S[\phi]} \left(1 - \int d^D x \partial_{\mu} T^{\mu\nu} \epsilon_{\nu}(x) + \dots \right)\end{aligned}$$

When expanded to first order in $\omega_a(x)$, the above yields

$$\begin{aligned} \langle \delta(\phi(x_1)\phi(x_2)\dots\phi(x_n)) \rangle &= \int dx \langle \partial_\mu T^{\mu\nu} \phi(x_1)\phi(x_2)\dots\phi(x_n) \rangle \epsilon_\nu(x) \\ -i \sum_{i=1}^N [\phi(x_1)\dots G_i^\mu \phi(x_i)\dots\phi(x_n)] \epsilon_\mu(x_i) &= \int dx \langle \partial_\mu T^{\mu\nu} \phi(x_1)\phi(x_2)\dots\phi(x_n) \rangle \epsilon_\nu(x) \\ -i \int d^D x \epsilon_\mu(x) \delta^D(x-x_i) \sum_{i=1}^N [\phi(x_1)\dots G_i^\mu \phi(x_i)\dots\phi(x_n)] &= \int dx \langle \partial_\mu T^{\mu\nu} \phi(x_1)\phi(x_2)\dots\phi(x_n) \rangle \epsilon_\nu(x) \end{aligned}$$

Hence,

$$\frac{\partial}{\partial x^\mu} \langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\dots\phi(x_n) \rangle = -i \sum_{i=1}^N \delta(x-x_i) \langle \phi(x_1)\dots G_i^\mu \phi(x_i)\dots\phi(x_n) \rangle$$

4.7 Translation

The generator of Translation $P^\mu = -i\partial^\mu$ is given as:

$$\frac{\partial}{\partial x^\mu} \langle T^{\mu\nu}(x)\phi(x_1)\phi(x_2)\dots\phi(x_n) \rangle = - \sum_{i=1}^N \delta(x-x_i) \langle \phi(x_1)\dots \partial_i^\mu \phi(x_i)\dots\phi(x_n) \rangle$$

4.8 Rotation

The generator of Rotation is $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) + S^{\mu\nu}$ and the associated conserved current is $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$.

$$\frac{\partial}{\partial x^\mu} \langle j^{\mu\nu\rho}(x)\phi(x_1)\dots\phi(x_n) \rangle = -i \sum_{i=1}^N \delta(x-x_i) \langle \phi(x_1)\dots J_i^{\nu\rho} \phi(x_i)\dots\phi(x_n) \rangle$$

$$\frac{\partial}{\partial x^\mu} \langle (T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\phi(x_1)\dots\phi(x_n) \rangle = -i \sum_{i=1}^N \delta(x-x_i) \langle \phi(x_1)\dots [i(x^\nu\partial^\rho - x^\rho\partial^\nu) + S^{\nu\rho}]_i \phi(x_i)\dots\phi(x_n) \rangle$$

$$x^\rho \frac{\partial}{\partial x^\mu} \langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n) \rangle + \delta_\mu^\rho \langle T^{\mu\nu}\phi(x_1)\dots\phi(x_n) \rangle - x^\nu \frac{\partial}{\partial x^\mu} \langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n) \rangle - \delta_\mu^\nu \langle T^{\mu\rho}\phi(x_1)\dots\phi(x_n) \rangle$$

$$= -i \sum_{i=1}^N \delta(x-x_i) \langle \phi(x_1)\dots [i(x^\nu\partial^\rho - x^\rho\partial^\nu) + S^{\nu\rho}]_i \phi(x_i)\dots\phi(x_n) \rangle$$

$$\langle (T^{\rho\nu} - T^{\nu\rho})\phi(x_1)\dots\phi(x_n) \rangle = -i \sum_{i=1}^N \delta(x-x_i) \langle \phi(x_1)\dots S_i^{\nu\rho} \phi(x_i)\dots\phi(x_n) \rangle$$

4.9 Dilatation

The generator for Dilatation is $D = x \cdot \partial$ and the associated conserved current is $j^\mu = T^\mu_\nu x^\nu$.

$$S[\phi] = \int d^2x \mathcal{L}(\phi^A, \partial_\mu \phi^A)$$

$$x^\mu = (1 + \varepsilon)x^\mu, \quad \delta x^\mu = \varepsilon x^\mu, \quad \bar{\delta} \phi^A = -\varepsilon \Delta_A \phi^A$$

$$\delta_{\text{tot}} \phi^A = \bar{\delta} \phi^A + \delta x^\nu \partial_\nu \phi^A$$

From Noether Theorem

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \delta \phi^A - T^\mu{}_\nu \delta x^\nu$$

For dilatation, (ε factor suppressed)

$$j_D^\mu = - \sum_A \Delta_A \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \phi^A - x_\nu T^{\mu\nu}$$

Let's define $V^\mu \equiv - \sum_A \Delta_A \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \phi^A$, then

$$\begin{aligned} j_D^\mu &= V^\mu - x_\nu T^{\mu\nu} \\ \partial_\mu j_D^\mu &= \partial_\mu V^\mu - \partial_\mu (x_\nu T^{\mu\nu}) \\ &= \partial_\mu V^\mu - T^\mu{}_\mu - x_\nu \partial_\mu T^{\mu\nu} \\ &= \partial_\mu V^\mu + T^\mu{}_\mu \quad (\partial_\mu T^{\mu\nu} = 0) \\ \partial_\mu j_D^\mu &= 0 \iff T^\mu{}_\mu + \partial_\mu V^\mu = 0 \end{aligned}$$

The related Ward identity becomes:

$$\partial_\mu \langle T^\mu{}_\nu x^\nu \phi(x_1) \dots \phi(x_n) \rangle = - \sum_i \delta(x - x_i) \left[x_i^\nu \frac{\partial}{\partial x_i^\nu} \langle \phi(x_1) \dots \phi(x_n) \rangle + \Delta_i \langle \phi(x_1) \dots \phi(x_n) \rangle \right]$$

which becomes

$$\langle T^\mu{}_\mu \phi(x_1) \dots \phi(x_n) \rangle = - \sum_i \delta(x - x_i) \Delta_i \langle \phi(x_1) \dots \phi(x_n) \rangle$$

4.10 In complex coordinates

We wish to rewrite these identities in terms of complex coordinates and complex components. But, we will first derive the relevant identities for doing so.

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

The metric in Euclidean space could be given as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In complex coordinates, it becomes:

$$\begin{aligned} g^{\alpha\beta} &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \end{aligned}$$

The anti-symmetric levi civita tensor in cartesian coordinates has the following form:

$$\epsilon_{\text{cartesian}}^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

using the tensorial transformation law for the above coordinate transformation,

$$\begin{aligned} \epsilon^{\alpha\beta} &= J_\mu^\alpha \epsilon_{\text{cartesian}}^{\mu\nu} J_\nu^\beta \\ &= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \end{aligned}$$

Lowering the indices:

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{-i}{2} & 0 \end{pmatrix}$$

For the delta functions, we remind ourselves that it is not defined as:

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$

But by the expression

$$\int_M d^2x \delta(x) f(z) = f(0)$$

Any function which satisfies this property is a valid representation of delta function. The ∞ of $\delta(x)$ at $x = 0$ in the above mentioned representation is defined by the integral $\int_{-\infty}^{\infty} \delta(x) dx = 1$. For more detail refer to section 7.2 of the textbook Green's function by G.F. Roach. It just so happens to be, that

$$\delta(x) = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}} = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z}$$

satisfies the required property. This identity is justified as follows. We consider a vector F^μ whose divergence is integrated within a region M of the complex plane bounded by the contour ∂M . Gauss's theorem may be applied:

$$\int_M d^2x \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu F^\mu \quad (3.6)$$

where $d\xi_\mu$ is an outward-directed differential of circumference, orthogonal to the boundary ∂M of the domain of integration. It is more convenient to use a counterclockwise differential ds^ρ , parallel to the contour ∂M : $d\xi_\mu = \epsilon_{\mu\rho} ds^\rho$. In terms of complex coordinates, the above surface integral is nothing but a contour integral, where the (anti)holomorphic component of ds^ρ is $dz(d\bar{z})$:

$$\int_M d^2x \partial_\mu F^\mu = \int_{\partial M} ds^\sigma \epsilon_{\sigma\mu} F^\mu \quad (3.7)$$

$$\begin{aligned} &= \int_{\partial M} (dz \epsilon_{z\bar{z}} F^{\bar{z}} + d\bar{z} \epsilon_{\bar{z}z} F^z) \\ &= \frac{i}{2} \int_{\partial M} (-dz F^{\bar{z}} + d\bar{z} F^z) \end{aligned} \quad (3.8)$$

Here the contour ∂M circles counterclockwise. If $F^{\bar{z}}$ (F^z) is holomorphic (antiholomorphic), then Cauchy's theorem may be applied; otherwise the contour ∂M must stay fixed. We consider then a holomorphic function $f(z)$ and check the correctness of the first representation in Eq. (5.33) by integrating it against $f(z)$ within a neighborhood M of the origin:

$$\begin{aligned} \int_M d^2x \delta(x) f(z) &= \frac{1}{\pi} \int_M d^2x f(z) \partial_{\bar{z}} \frac{1}{z} \\ &= \frac{1}{\pi} \int_M d^2x \partial_{\bar{z}} \left(\frac{f(z)}{z} \right) \\ &= \frac{1}{2\pi i} \int_{\partial M} dz \frac{f(z)}{z} \\ &= f(0) \end{aligned} \quad (3.9)$$

In the second equation we have used the assumption that $f(z)$ is analytic within M , and in the third equation we have used the form (3.7) of Gauss's theorem with $F^{\bar{z}} = f(z)/\pi z$ and $F^z = 0$, and in the last equation we used Cauchy's residue theorem. Since the original ward identity was covariant, we now only need the following object to write the Ward identities in complex coordinates:

$$\begin{aligned}
\partial_\mu \langle T_z^\mu \phi(z_1) \dots \phi(z_n) \rangle &= g^{\alpha\beta} \partial_\alpha \langle T_{\beta z} \phi(z_1) \dots \phi(z_n) \rangle = 2\partial_z \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2\partial_{\bar{z}} \langle T_{zz} \phi(z_1) \dots \phi(z_n) \rangle \\
\partial_\mu \langle T_{\bar{z}}^\mu \phi(z_1) \dots \phi(z_n) \rangle &= g^{\alpha\beta} \partial_\alpha \langle T_{\beta\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle = 2\partial_z \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2\partial_{\bar{z}} \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle \\
\langle T_\mu^\mu \phi(z_1) \dots \phi(z_n) \rangle &= g^{\alpha\beta} \langle T_{\alpha\beta} \phi(z_1) \dots \phi(z_n) \rangle = 2 \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2 \langle T_{\bar{z}z} \phi(z_1) \dots \phi(z_n) \rangle \\
\epsilon_{\mu\nu} \langle T^{\mu\nu} \phi(z_1) \dots \phi(z_n) \rangle &= \epsilon^{\alpha\beta} \langle T_{\alpha\beta} \phi(z_1) \dots \phi(z_n) \rangle = -2 \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2 \langle T_{\bar{z}z} \phi(z_1) \dots \phi(z_n) \rangle
\end{aligned}$$

The Ward identities are then explicitly written as:

$$\begin{aligned}
2\pi\partial_z \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2\pi\partial_{\bar{z}} \langle T_{zz} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_i \partial_z \frac{1}{z-w_i} \partial_{w_i} \langle \phi(z_1) \dots \phi(w_i) \dots \phi(z_n) \rangle \\
2\pi\partial_z \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2\pi\partial_{\bar{z}} \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_i \partial_z \frac{1}{\bar{z}-\bar{z}_i} \partial_{\bar{z}_i} \langle \phi(z_1) \dots \phi(w_i) \dots \phi(z_n) \rangle \\
2 \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2 \langle T_{\bar{z}z} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_{i=1}^N \delta(z-w_i) \Delta_i \langle \phi(z_1) \dots \phi(z_n) \rangle \\
-2 \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle + 2 \langle T_{\bar{z}z} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_{i=1}^N \delta(z-w_i) s_i \langle \phi(z_1) \dots \phi(z_n) \rangle
\end{aligned}$$

Adding and subtracting the last two expressions,

$$\begin{aligned}
2 \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_{i=1}^N \delta(z-w_i) \frac{\Delta_i + s_i}{2} \langle \phi(z_1) \dots \phi(z_n) \rangle \\
2\pi \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_i \partial_z \frac{1}{z-w_i} h_i \langle \phi(z_1) \dots \phi(z_n) \rangle \tag{3.10}
\end{aligned}$$

$$\begin{aligned}
2 \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_{i=1}^N \delta(z-w_i) \frac{\Delta_i - s_i}{2} \langle \phi(z_1) \dots \phi(z_n) \rangle \\
2\pi \langle T_{z\bar{z}} \phi(z_1) \dots \phi(z_n) \rangle &= - \sum_i \partial_z \frac{1}{\bar{z}-\bar{w}_i} \bar{h}_i \langle \phi(z_1) \dots \phi(z_n) \rangle \tag{3.11}
\end{aligned}$$

Inserting these relations in the first two Ward identities:

$$\partial_{\bar{z}} \left\{ \langle T(z, \bar{z}) \phi(z_1) \dots \phi(z_n) \rangle - \sum_{i=1}^n \left[\frac{1}{z-w_i} \partial_{w_i} \langle \phi(z_1) \dots \phi(z_n) \rangle + \frac{h_i}{(z-w_i)^2} \langle \phi(z_1) \dots \phi(z_n) \rangle \right] \right\} = 0 \tag{3.12}$$

$$\partial_z \left\{ \langle \bar{T}(z, \bar{z}) \phi(z_1) \dots \phi(z_n) \rangle - \sum_{i=1}^n \left[\frac{1}{\bar{z}-\bar{w}_i} \partial_{\bar{w}_i} \langle \phi(z_1) \dots \phi(z_n) \rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2} \langle \phi(z_1) \dots \phi(z_n) \rangle \right] \right\} = 0 \tag{3.13}$$

where we have introduced a renormalized energy-momentum tensor

$$T = -2\pi T_{zz}, \quad \bar{T} = -2\pi T_{\bar{z}\bar{z}} \tag{3.14}$$

Thus the expressions between braces in (3.12) and ((3.13)) are respectively holomorphic and antiholomorphic; we may write

$$\langle T(z) \phi(z_1) \dots \phi(z_n) \rangle = \sum_{i=1}^n \left\{ \frac{1}{z-w_i} \partial_{w_i} \langle \phi(z_1) \dots \phi(z_n) \rangle + \frac{h_i}{(z-w_i)^2} \langle \phi(z_1) \dots \phi(z_n) \rangle \right\} + \text{reg.} \tag{3.15}$$

where "reg." stands for a holomorphic function of z , regular at $z = w_i$. There is a similar expression for the antiholomorphic counterpart. The Ward identity shows that the correlator

of the field $T(z)$ with primary fields $\phi(w_i, \bar{w}_i)$ becomes singular as z approaches the points w_i . The OPE of the energy-momentum tensor with primary fields is written simply by removing the brackets $\langle \dots \rangle$, it being understood that OPE is meaningful only within correlation functions.

4.11 Dilation Operator in CFT viewed as 'hamiltonian' ?

From the commutation relations for the conformal Lie algebra, we may infer that the dilation operator plays the same role as the Hamiltonian in CFTs. The appropriate commutation relations are

$$[D, P_\mu] = iP_\mu \quad \text{and} \quad [D, K_\mu] = -iK_\mu,$$

so that P_μ and K_μ are raising and lowering operators, respectively, for the operator D .

This is analogous to the operators \hat{a} and \hat{a}^\dagger being creation and annihilation operators for \hat{H} when discussing the energy spectra of the n -dimensional harmonic oscillator.

My question is, while \hat{a} and \hat{a}^\dagger raise and lower the energy by one unit ($\pm\hbar\omega$) for each application of the operator onto eigenstates of \hat{H} , what is being raised and lowered when we apply P_μ and K_μ onto the eigenvectors of D ?

Secondly, what exactly do we mean by the eigenvectors of D ? Are they fields in space-time?

Using the notation of Di Francesco in his book 'Conformal Field Theory', the fields transform under a dilation like $F(\Phi(x)) = \lambda^{-\Delta}\Phi(x)$, where λ is the scale of the coordinates and Δ is the scaling dimension of the fields.

Can I write $F(\Phi(x)) = D\Phi(x) = \lambda^{-\Delta}\Phi(x)$ to make the eigenvalue equation manifest? The commutation relations

$$[D, P_\mu] = +iP_\mu, \quad [D, K_\mu] = -iK_\mu$$

show that P_μ and K_μ raise and lower the ****conformal dimension**** of a state. In other words, if you have a state $|\phi\rangle$ of dimensions Δ , so that $D|\phi\rangle = i\Delta|\phi\rangle$, then

$$\begin{aligned} DP_\mu|\phi\rangle &= [D, P_\mu]|\phi\rangle + P_\mu D|\phi\rangle \\ &= iP_\mu|\phi\rangle + P_\mu(i\Delta|\phi\rangle) \\ &= i(\Delta + 1)P_\mu|\phi\rangle. \end{aligned}$$

While the generators of the conformal group act on the fields (after all they generate a symmetry), I find it easier to think about the action on a state, like the $|\phi\rangle$ above. According to the state-operator correspondence, such states can be obtained by acting by local operators on the vacuum. P_μ is the generator of translations, and hence acts on a local operator $\phi(x)$ as a derivative $[P_\mu, \phi(x)] = i\partial_\mu\phi(x)$.

The equation above then tells you that the derivative carries conformal dimension 1.

The notation you propose at the end of your question seems a bit dangerous. D is frequently used for denoting the **infinitesimal generator** of dilatations, but the function F gives the action of the corresponding **group element**. That being said, you are of course free to introduce whatever notation you find convenient, as long as you make it clear – both to yourself and to others – what you mean.

4.12 Free Fields and Operator Product Expansion

The operator product expansion, or OPE, is the representation of a product of operators (at positions z and w , respectively) by a sum of terms, each being a single operator, well-defined as $z \rightarrow w$, multiplied by a c-number function of $z - w$, possibly diverging as $z \rightarrow w$, and which embodies the infinite fluctuations as the two positions tend toward each other. For a single primary field ϕ of conformal dimension h and \bar{h} , we have the OPE from Ward identity as:

$$T(z)\phi(w, \bar{w}) \sim \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w\phi(w, \bar{w})$$

$$T(\bar{z})\phi(w, \bar{w}) \sim \frac{\bar{h}}{(\bar{z} - \bar{w})^2} \phi(w, \bar{w}) + \frac{1}{\bar{z} - \bar{w}} \partial_{\bar{w}} \phi(w, \bar{w}) \quad (3.16)$$

whenever appearing in OPEs, the symbol \sim will mean equality modulo expressions regular as $z \rightarrow w$. Of course, the OPE contains also an infinite number of regular terms which, for the Energy Momentum tensor, can not be obtained from the conformal ward identity. In general, we would write the OPE of two fields $A(z)$ and $B(w)$ as

$$A(z)B(w) = \sum_{n=-\infty}^{\infty} \frac{\{AB\}_n(w)}{(z-w)^n}$$

where the composite $\{AB\}_n(w)$ are non singular at $z = w$. For instance, $\{T\phi\}_1 = \partial_w \phi(w)$. We stress that so far, quantities appearing in (3.16) are not operators but simply fields occurring within correlation functions. We will now proceed with specific examples, in order to familiarize ourselves with basic techniques and with simple but important systems.

4.13 The Free Boson

From the point of view of the canonical or path integral formalism, the simplest conformal field theory is that of a free massless boson ϕ , with the following action:

$$S = \frac{1}{2}g \int d^2x \partial_\mu \phi \partial^\mu \phi$$

where g is some normalization parameter that we leave unspecified at the moment. The two-point function, or propagator can be found by comparing with

$$S = \frac{1}{2} \int d^d x d^d y \phi(x) A(x, y) \phi(y)$$

we have

$$A(x, y) = -g \delta^{(2)}(x - y) \square \quad (3.17)$$

We can calculate the two-point function $K(x, y) \equiv \langle \varphi(x_1) \varphi(x_2) \rangle = A^{-1}$ by solving the following equation:

$$\begin{aligned} \int d^2u A(x, u) K(u, y) &= \delta^{(2)}(x - y) \\ - \int d^2u g \delta^{(2)}(x - u) \square K(u, y) &= \delta^{(2)}(x - y) \\ -g \square K(x, y) &= \delta^{(2)}(x - y) \end{aligned}$$

Because of rotation and translation invariance, the propagator $K(x, y)$ should depend only on the distance separating the two points. Thus, we can write $K(x, y) \equiv K(\rho)$ with $\rho = |x - y|$, and integrate over x within a disk of radius ρ around y . We find

$$\begin{aligned} 1 &= 2\pi g \int_0^\rho d\rho \rho \left(-\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho K'(\rho)) \right) \\ &= 2\pi g (-\rho K'(\rho)) \end{aligned}$$

The solution of the two-point function for massless free boson can be obtained up to an additive constant,

$$\langle \phi(x) \phi(y) \rangle = -\frac{1}{4\pi g} \ln(x - y)^2 + \text{const} \quad (3.18)$$

In terms of complex coordinates, this is

$$\langle \phi(z, \bar{z}) \phi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} \{ \ln(z - w) + \ln(\bar{z} - \bar{w}) \} + \text{const}$$

The holomorphic and anti-holomorphic components can be separated by taking the derivatives

$\partial\phi$ and $\bar{\partial}\phi$:

$$\begin{aligned}\langle\partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w})\rangle &= -\frac{1}{4\pi g}\partial_w\partial_z\{\ln(z-w)\} \\ &= -\frac{1}{4\pi g}\partial_w\frac{1}{z-w} = -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \\ \langle\partial_{\bar{z}}\phi(z,\bar{z})\partial_{\bar{w}}\phi(w,\bar{w})\rangle &= -\frac{1}{4\pi g}\partial_{\bar{w}}\partial_{\bar{z}}\{\ln(\bar{z}-\bar{w})\} \\ &= -\frac{1}{4\pi g}\partial_{\bar{w}}\frac{1}{\bar{z}-\bar{w}} = -\frac{1}{4\pi g}\frac{1}{(\bar{z}-\bar{w})^2}\end{aligned}$$

In the following, we will focus on holomorphic field $\partial\phi \equiv \partial_z\phi$.

$$\begin{aligned}T_{\mu\nu} &= g\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right] \\ &= g\partial_\mu\phi\partial_\nu\phi - \frac{g}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\ &= g\partial_\mu\phi\partial_\nu\phi - \frac{g}{2}\eta_{\mu\nu}\times 2\eta^{z\bar{z}}\partial_z\phi\underbrace{\partial_{\bar{z}}\phi}_{=0} \\ &= g\partial_\mu\phi\partial_\nu\phi\end{aligned}$$

Then,

$$T(z) = -2\pi T_{zz} = -2\pi g : \partial\phi\partial\phi : \quad \text{and} \quad \bar{T}(\bar{z}) = 0$$

Like all composite field, the energy momentum tensor has to be normal ordered, in order to ensure the vanishing of its vacuum expectation value. More explicitly, the exact meaning of above expression is

$$T(z) = -2\pi g \lim_{w \rightarrow z} [\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle]$$

The OPE of $T(z)$ with $\partial\phi$ may be calculated from Wick's theorem: ¹

$$\begin{aligned}T(z)\partial\phi(w) &= -2\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \\ &\sim -2\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) - 2\pi g : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \\ &\sim \frac{\partial\phi(z)}{(z-w)^2}\end{aligned}$$

By expanding $\phi(z)$ around w , we arrive at the OPE

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)}$$

This shows that $\partial\phi$ is a primary field with conformal dimension $h = 1$. Wick's theorem also allows us to calculate the OPE of energy-momentum tensor with itself:

$$\begin{aligned}T(z)T(w) &= 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\ &\sim 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : + 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\ &\quad + 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : + 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\ &\quad + 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : + 4\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\ &\sim 8\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : + 16\pi^2 g^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\ &\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial\phi(z)\partial\phi(w) :}{(z-w)^2}\end{aligned}$$

Expanding $\phi(z)$ around w , we arrive at the OPE

$$\begin{aligned} &\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial\phi(w)\partial\phi(w) :}{(z-w)^2} - \frac{4\pi g : \partial^2\phi(w)\partial\phi(w) :}{(z-w)} \\ &\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \end{aligned}$$

We immediately see that the energy momentum tensor is not strictly a primary field, because of the anomalous term $1/2(z-w)^4$ which does not appear in (3.16).

4.14 Free Fermion

Now we consider another simple model: free fermion. In two dimensions, the action of a free Majorana fermion is

$$S = \frac{1}{2}g \int d^2x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \quad (3.19)$$

where the gamma matrices γ^μ satisfy the so-called Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (3.20)$$

and we impose the Majorana condition ($\Psi^* = \Psi$) to the fermionic field to remove a half of the degrees of freedom. In the Euclidean space $\eta^{\mu\nu} = \text{diag}(1, 1)$, we take a basis of Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.21)$$

and therefore,

$$\begin{aligned} \gamma^0 \gamma^\mu \partial_\mu &= \gamma^0 (\gamma^0 \partial_0 + \gamma^1 \partial_1) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial + i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{\partial} \right] \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \partial_0 - i\partial_1 \\ \partial_0 + i\partial_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2\partial_z \\ 2\partial_{\bar{z}} & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \end{aligned}$$

Using this basis, we can express the action as

$$S = g \int d^2x (\bar{\psi} \partial \bar{\psi} + \psi \bar{\partial} \psi) \quad (3.22)$$

where we write the two-component spinor Ψ as $(\psi, \bar{\psi})$. Since the equations of motion are $\partial \bar{\psi} = 0$ and $\bar{\partial} \psi = 0$, $\psi(z)$ and $\bar{\psi}(\bar{z})$ holomorphic and antiholomorphic field, respectively.

Now let us calculate the two-point function as in the free fermion

$$K_{ij} = \langle \Psi_i(x) \Psi_j(y) \rangle \quad (i, j = 1, 2) \quad (3.23)$$

The action can be expressed by

$$S = \frac{1}{2} \int d^2x d^2y \Psi_i(x) A_{ij}(x, y) \Psi_j(y) \quad (3.24)$$

where the kernel is

$$A_{ij}(x, y) = g \delta(x - y) (\gamma^0 \gamma^\mu)_{ij} \partial_\mu \quad (3.25)$$

Recalling that propagator K_{ij} is the inverse of A_{ij} . Therefore, we can write the equation for K as

⁰¹ Since we are Wick contracting the normal ordered product of operators, the only term that it will give rise to are crosscontractions, since the wick contraction of operators which are already normal ordered vanishes.

$$\int d^2u A(x, u) K(u, y) = \delta^{(2)}(x - y)$$

$$g \int d^2u \delta^{(2)}(x - u) (\gamma^0 \gamma^\mu)_{ij} \frac{\partial}{\partial x^\mu} K(u, y) = \delta^{(2)}(x - y) \delta_{ij}$$

$$g (\gamma^0 \gamma^\mu)_{ik} \frac{\partial}{\partial x^\mu} K_{kj}(x, y) = \delta(x - y) \delta_{ij}$$

In terms of complex coordinates, this becomes

$$2g \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} \langle \psi(z, \bar{z}) \psi(w, \bar{w}) \rangle & \langle \psi(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle \\ \langle \bar{\psi}(z, \bar{z}) \psi(w, \bar{w}) \rangle & \langle \bar{\psi}(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle \end{pmatrix} = \frac{1}{\pi} \begin{pmatrix} \bar{\partial} \frac{1}{z-w} & 0 \\ 0 & \partial_{\bar{z}-\bar{w}} \end{pmatrix} \quad (3.26)$$

where we have used the complex form of the δ -function:

$$\delta(x) = \frac{1}{\pi} \bar{\partial} \frac{1}{z} = \frac{1}{\pi} \partial \frac{1}{\bar{z}}$$

Therefore, we obtain the two-point functions for the fermionic fields

$$\langle \psi(z, \bar{z}) \psi(w, \bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{z - w}$$

$$\langle \bar{\psi}(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{\bar{z} - \bar{w}}$$

$$\langle \psi(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle = 0$$

These, after differentiation, imply

$$\langle \partial_z \psi(z, \bar{z}) \psi(w, \bar{w}) \rangle = -\frac{1}{2\pi g} \frac{1}{(z - w)^2}$$

$$\langle \partial_z \psi(z, \bar{z}) \partial_w \psi(w, \bar{w}) \rangle = -\frac{1}{\pi g} \frac{1}{(z - w)^3}$$

Thus the OPE of two holomorphic fields can be written as:

$$\psi(z) \psi(w) \sim \frac{1}{2\pi g} \frac{1}{z - w} \quad (3.27)$$

In order to see whether the fermion field is a primary field or not, we can calculate its OPE with the energymomentum tensor. By using (??) in complex-coordinate form, we can calculate all the components of the energy-momentum tensor.

$$T^{\bar{z}\bar{z}} = \frac{\partial \mathcal{L}}{\partial (\partial_{\bar{z}} \Phi)} \partial^{\bar{z}} \Phi = g^{\bar{z}\bar{z}} \frac{\partial \mathcal{L}}{\partial (\partial \bar{\Phi})} \partial \bar{\Phi} = 2 \frac{\partial \mathcal{L}}{\partial (\partial \bar{\Phi})} \partial \bar{\Phi} = 2g \psi \partial \psi$$

$$T^{zz} = \frac{\partial \mathcal{L}}{\partial (\partial_z \Phi)} \partial^z \Phi = g^{z\bar{z}} \frac{\partial \mathcal{L}}{\partial (\partial \Phi)} \bar{\partial} \Phi = 2 \frac{\partial \mathcal{L}}{\partial (\partial \Phi)} \bar{\partial} \Phi = 2g \bar{\psi} \partial \bar{\psi}$$

$$T^{z\bar{z}} = \frac{\partial \mathcal{L}}{\partial (\partial_z \Phi)} \partial^{\bar{z}} \Phi - g^{z\bar{z}} \mathcal{L} = 2 \frac{\partial \mathcal{L}}{\partial (\partial \Phi)} \partial \bar{\Phi} - 2\mathcal{L} = -2g \psi \bar{\partial} \psi$$

The traceless condition $T^{z\bar{z}}$ is preserved when taking into account the equation of motion, as we have discussed. The holomorphic part is defined as:

$$T(z) = -\pi g : \psi(z) \partial \psi(z) : \quad (3.28)$$

The normal-ordering product can be written in an equivalent way for the free field as follow:

$$: \psi \partial \psi : (z) = \lim_{w \rightarrow z} (\psi(z) \partial \psi(w) - \langle \psi(z) \partial \psi(w) \rangle) \quad (3.29)$$

which is the same expression as in bosonic field theory. Then we can calculate the OPE between the fermion field and energy-momentum tensor directly.

$$\begin{aligned}
T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) \\
&= -\pi g : \psi(z)\partial\psi(z) : \psi(w) - \pi g : \psi(z)\partial\psi(z) : \psi(w) \\
&\sim \frac{1}{2} \frac{\psi(z)}{(z-w)^2} + \frac{1}{2} \frac{\partial\psi(z)}{(z-w)} \\
&\sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}
\end{aligned}$$

In contracting $\psi(z)$ with $\psi(w)$ we have carried $\psi(w)$ over $\partial\psi(z)$, thus introducing a $(-)$ sign by Pauli's principle. The OPE immediately tells us that in the free fermion model, ψ is a primary field with conformal dimension $1/2$. TT OPE in this model can also be obtained directly by calculation.

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \quad (3.30)$$

4.15 Boundary Conformal Field Theory: bc CFT

The action of the bc CFT is as follows;

$$S = \frac{1}{2\pi} \int d^2z b\bar{\partial}c \quad (2.5.1)$$

where b and c are anti-commuting primary fields and for the action to be conformally invariant, we need the conformal weights of b and c to be as follows;

$$h_b = \lambda, \quad h_c = 1 - \lambda$$

The equations of motion of these fields are as follows;

$$\bar{\partial}b = 0 \Rightarrow b = b(z), \quad \partial\bar{c} = 0 \Rightarrow c = c(z)$$

and hence, both of the fields are holomorphic. We can now work out the propagators of the theory that have to satisfy the following equation;

$$\bar{\partial}\langle b(z)c(0) \rangle = 2\pi\delta^{(2)}(z) \Rightarrow \bar{\partial}\langle b(z)c(0) \rangle = \bar{\partial}\frac{1}{z} \Rightarrow \langle b(z)c(0) \rangle = \frac{1}{z} \quad (2.5.2)$$

where we used the identity;

$$\bar{\partial}\frac{1}{z} = 2\pi\delta^{(2)}(z)$$

Using (2.5.2), we get the bc OPE;

$$\begin{aligned}
b(z)c(0) \sim \frac{1}{z} &\Rightarrow b(z_1)c(z_2) \sim \frac{1}{z_1 - z_2}, \quad c(z_1)b(z_2) \sim \frac{1}{z_1 - z_2} \\
&\Rightarrow \partial b(z_1)c(z_2) \sim -\frac{1}{(z_1 - z_2)^2}, \quad \partial c(z_1)b(z_2) \sim -\frac{1}{(z_1 - z_2)^2}
\end{aligned} \quad (2.5.3)$$

We now derive the stress tensor of the theory. It can be done by using Noether's theorem (do this approach) but it can be done by another method. Since $T(z)$ should have a conformal weight of 2, there are two possible terms that can appear in $T(z)$ and they are as follows;

$$T(z) = \alpha : b\partial c : (z) + \beta : \partial b c : (z) \quad (2.5.4)$$

where α and β are undetermined coefficients. We can fix α and β by requiring that $b(z)$ and $c(z)$ have the correct weights.

$$\begin{aligned}
T(z)b(0) &\sim \alpha : b\partial c : (z)b(0) + \beta : \partial b c : (z)b(0) = -\frac{\alpha b(z)}{z^2} + \frac{\beta \partial b(z)}{z} \sim -\frac{\alpha b(0)}{z^2} + \frac{(\beta - \alpha)\partial b(0)}{z} \\
T(z)c(0) &\sim -\alpha : \partial c b : (z)c(0) - \beta : c\partial b : (z)c(0) = \frac{\beta c(z)}{z^2} - \frac{\alpha \partial c(z)}{z} \sim \frac{\beta c(0)}{z^2} + \frac{(\beta - \alpha)\partial c(0)}{z}
\end{aligned}$$

and thus, we see that $\alpha = -\lambda, \beta = 1 - \lambda$ and it gives the correct $\beta - \alpha$ value in the residues. Using these values in 2.5.4, we get;

$$T(z) = -\lambda : b\partial c : (z) + (1 - \lambda) : \partial bc : (z) = : \partial bc : (z) - \lambda \partial (: bc : (z)) \quad (2.5.5)$$

Similarly, the TT OPE can be calculated but we will calculate the $1/z^4$ term only to work out the central charge.

$$\begin{aligned} \frac{1}{z^4} \text{ term} &= \lambda^2 : b\partial c : (z) : b\partial c : (0) - \lambda(1 - \lambda) : b\partial c : (z) : \partial bc : (0) \\ &\quad - \lambda(1 - \lambda) : \partial bc : (z) : b\partial c : (0) + (1 - \lambda)^2 : \partial bc : (z) : \partial bc : (0) \\ &= -\frac{\lambda^2}{z^4} + \frac{2\lambda(1 - \lambda)}{z^4} + \frac{2\lambda(1 - \lambda)}{z^4} - \frac{(1 - \lambda)^2}{z^4} = \frac{1 - 3(2\lambda - 1)^2}{2z^4} \Rightarrow c = 1 - 3(2\lambda - 1)^2 \quad (2.5.6) \end{aligned}$$

Talk about ghost number symmetry

4.16 The Ghost System

In string theory applications, there appears another simple system, with the following action:

$$S = \frac{1}{2}g \int d^2x b_{\mu\nu} \partial^\mu c^\nu$$

where $b_{\mu\nu}$ is a traceless symmetric tensor, and where both c^μ and $b_{\mu\nu}$ are fermions (anti-commuting fields). These fields are called ghosts because they are not fundamental dynamical fields, but rather represent a jacobian arising from a change of variables in some functional integrals. More precisely, they are known as reparametrization ghosts. The role of these ghost fields is to cancel the unphysical gauge degrees of freedom.

The equation of motion are

$$\partial^\alpha b_{\alpha\mu} = 0 \quad \text{and} \quad \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0$$

In holomorphic form we write $c = c^z$ and $\bar{c} = c^{\bar{z}}$. The only nonzero components of the traceless symmetric tensor $b_{\mu\nu}$ are $b = b_{zz}$ and $\bar{b} = b_{\bar{z}\bar{z}}$. The equations of motion are then

$$\begin{aligned} \bar{\partial} b &= 0 & \partial \bar{b} &= 0 \\ \bar{\partial} c &= 0 & \partial \bar{c} &= 0 & \partial c &= -\bar{\partial} \bar{c} \end{aligned}$$

The propagator is calculated in the usual way, by writing the action as:

$$\begin{aligned} S &= \frac{1}{2} \int d^2x d^2y b_{\mu\nu}(x) A_\alpha^{\mu\nu} c^\alpha(y) \\ A_\alpha^{\mu\nu} &= \frac{1}{2} g \delta_\alpha^\nu \delta(x - y) \partial^\mu \end{aligned}$$

where we must consider (μ, ν) as a single composite index, symmetric under the exchange of μ and ν . The factor of $\frac{1}{2}$ in front of $A_\alpha^{\mu\nu}$ compensates the double counting of each pair (μ, ν) in the sum, which should be avoided since $b^{\mu\nu}$ is the same degree of freedom as $b^{\nu\mu}$. Again the propagator is $K = A^{-1}$, satisfying

$$\frac{1}{2} g \delta_\alpha^\mu \partial^\nu K_{\mu\nu}^\beta(x, y) = \delta(x - y) \delta_{\alpha\beta}$$

or, in complex representation

$$g \partial_{\bar{z}} K_{zz}^\beta = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z - w} \delta_{\beta z}$$

which implies

$$\langle b(z)c(w) \rangle = K_{zz}^z(z, w) = \frac{1}{\pi g} \frac{1}{z - w}$$

In OPE form, this is

$$b(z)c(w) \sim \frac{1}{\pi g} \frac{1}{z-w}$$

from which we immediately derive

$$\begin{aligned} \langle c(z)b(w) \rangle &= -\langle b(w)c(z) \rangle = \frac{1}{\pi g} \frac{1}{z-w} \\ \langle b(z)\partial_w c(w) \rangle &= -\langle \partial_z c(z)b(w) \rangle = \partial_w \left(\frac{1}{\pi g} \frac{1}{z-w} \right) = \frac{1}{\pi g} \frac{1}{(z-w)^2} \\ \langle \partial_z b(z)c(w) \rangle &= -\frac{1}{\pi g} \frac{1}{(z-w)^2} \\ \langle \partial_z b(z)\partial_w c(w) \rangle &= -\frac{2}{\pi g} \frac{1}{(z-w)^3} \end{aligned}$$

The canonical energy-momentum tensor for this system is

$$T_B^{\mu\nu} = \frac{1}{2}g [b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta]$$

The Belinfante tensor is

$$T_B^{\mu\nu} = \frac{1}{2}g [b^{\mu\alpha}\partial^\nu c_\alpha + b^{\nu\alpha}\partial^\mu c_\alpha + \partial_\alpha b^{\mu\nu}c^\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta]$$

The normal ordered holomorphic component is obtained from the above by setting $\mu = \nu = 1$, that is, by considering $T^{\bar{z}\bar{z}} = 4T_{zz}$:

$$T(z) = \pi g : (2\partial c b + c\partial b) :$$

The OPE for this stress energy tensor with c is again calculated using Wick's theorem:

$$\begin{aligned} T(z)c(w) &= \pi g : (2\partial c b + c\partial b) : c(w) \\ &= \pi g : 2\partial c b : c(w) + \pi g : c\partial b : c(w) \\ &\sim \frac{2\partial_z c(z)}{z-w} - \frac{c(z)}{(z-w)^2} \\ &\sim \frac{2\partial_w c(w)}{z-w} - \frac{c(w) + (z-w)\partial_w c(w)}{(z-w)^2} \\ &\sim \frac{\partial_w c(w)}{z-w} - \frac{c(w)}{(z-w)^2} \\ T(z)b(w) &= \pi g : (2\partial c b + c\partial b) : b(w) \\ &= -\pi g : 2b\partial c : b(w) - \pi g : \partial b c : b(w) \\ &\sim \frac{2b(z)}{(z-w)^2} - \frac{\partial_z b(z)}{(z-w)} \\ &\sim \frac{2b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)} \end{aligned}$$

One must be careful because b and c are anticommuting so that interchange of fields flips the sign: one should anticommutate the fields being paired until they are next to each other before doing the Wick contraction. The OPE of T with itself, may contain more terms, which add up to following:

$$\begin{aligned} T(z)T(w) &= \pi^2 g^2 : (2\partial c(z)b(z) + c(z)\partial b(z)) :: (2\partial c(w)b(w) + c(w)\partial b(w)) : \\ &= \pi^2 g^2 : 2\partial c(z)b(z) :: (2\partial c(w)b(w) + c(w)\partial b(w)) : + \pi^2 g^2 : c(z)\partial b(z) :: (2\partial c(w)b(w) + c(w)\partial b(w)) : \\ &= 4\pi^2 g^2 : \partial c(z)b(z) :: \partial c(w)b(w) : + 2\pi^2 g^2 : \partial c(z)b(z) :: c(w)\partial b(w) : + 2\pi^2 g^2 : c(z)\partial b(z) :: \partial c(w)b(w) : \\ &\quad + \pi^2 g^2 : c(z)\partial b(z) :: c(w)\partial b(w) : \end{aligned}$$

$$\begin{aligned}
&= 4\pi^2 g^2 : \overbrace{\partial c(z)b(z)} :: \overbrace{\partial c(w)b(w)} : + 4\pi^2 g^2 : \overbrace{\partial c(z)b(z)} :: \overbrace{\partial c(w)b(w)} : + 4\pi^2 g^2 : \overbrace{\partial c(z)b(z)} :: \overbrace{\partial c(w)b(w)} : \\
&+ 2\pi^2 g^2 : \overbrace{\partial c(z)b(z)} :: \overbrace{c(w)\partial b(w)} : + 2\pi^2 g^2 : \overbrace{\partial c(z)b(z)} :: \overbrace{c(w)\partial b(w)} : + 2\pi^2 g^2 : \overbrace{\partial c(z)b(z)} :: \overbrace{c(w)\partial b(w)} : \\
&+ 2\pi^2 g^2 : \overbrace{c(z)\partial b(z)} :: \overbrace{\partial c(w)b(w)} : + 2\pi^2 g^2 : \overbrace{c(z)\partial b(z)} :: \overbrace{\partial c(w)b(w)} : + 2\pi^2 g^2 : \overbrace{c(z)\partial b(z)} :: \overbrace{\partial c(w)b(w)} : \\
&+ \pi^2 g^2 : \overbrace{c(z)\partial b(z)} :: \overbrace{c(w)\partial b(w)} : + \pi^2 g^2 : \overbrace{c(z)\partial b(z)} :: \overbrace{c(w)\partial b(w)} : + \pi^2 g^2 : \overbrace{c(z)\partial b(z)} :: \overbrace{c(w)\partial b(w)} :
\end{aligned}$$

The end result is

$$\begin{aligned}
T(z)T(w) &= \frac{-4}{(z-w)^4} + \frac{4\pi g : \partial c(z)b(w) :}{(z-w)^2} - \frac{4\pi g : b(z)\partial c(w) :}{(z-w)^2} - \frac{4}{(z-w)^4} + \frac{2\pi g : \partial c(z)\partial b(w) :}{z-w} - \frac{4\pi g : b(z)c(w) :}{(z-w)^3} \\
&\quad - \frac{4}{(z-w)^4} - \frac{4\pi g : c(z)b(w) :}{(z-w)^3} + \frac{2\pi g : \partial b(z)\partial c(w) :}{z-w} - \frac{1}{(z-w)^4} - \frac{\pi g : c(z)\partial b(w) :}{(z-w)^2} + \frac{\pi g : \partial b(z)c(w) :}{(z-w)^2} + \dots
\end{aligned}$$

After some Taylor expansions to turn $f(z)$ functions into $f(w)$ functions, together with a little collecting of terms, this can be written as,

$$T(z)T(w) = \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

4.17 Central Charge

The specific models treated in the last section lead us naturally to the following general OPE of the energymomentum tensor.

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

where the constant c , not to be confused with ghost field c_μ , depends on the specific model under study: it is equal to 1 for the free boson, 1/2 for the free fermion, -26 for the reparametrization ghosts, and -2 for the simple ghost system. This model dependent constant term is called the central charge. Except for this anomalous term, the OPE simply means that T is a quasi-primary field with conformal dimension $h = 2$.

The central charge may not be determined from symmetry considerations: its value is determined by the short-distance behavior of the theory. For free fields, as seen in the previous section, it is determined by applying Wick's theorem on the normal-ordered energy-momentum tensor. When two decoupled systems (e.g., two free fields) are put together, the energy-momentum tensor of the total system is simply the sum of the energy-momentum tensors associated with each part, and the associated central charge is simply the sum of the central charges of the parts. Thus, the central charge is somehow an extensive measure of the number of degrees of freedom of the system.

4.18 Transformation of the Energy-Momentum Tensor

The departure of OPE from the general form (3.16) means that the energy-momentum tensor does not exactly transform like a primary field of dimension 2, contrary to what we expect classically. This happens because the normal ordering is not invariant under conformal transformation. According to conformal ward identity

$$\begin{aligned}
\delta_\epsilon T(w) &= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) T(z) T(w) \\
&= -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.} \right] \\
&= -\frac{c/2}{3!} \partial_w^3 \epsilon(w) - 2T(w) \partial_w \epsilon(w) - \epsilon(w) \partial_w T(w)
\end{aligned}$$

In the third equation, we used Cauchy's integral formula for derivatives. The "exponentiation" of this infinitesimal variation to a finite transformation $z \rightarrow w(z)$ is:

$$T'(w) = \left(\frac{dw}{dz} \right)^{-2} \left[T(z) - \frac{c}{12} \{w; z\} \right] \quad (3.31)$$

where we have introduced the Schwarzian derivative:

$$\{w; z\} = \frac{\frac{d^3w}{dz^3}}{\frac{dw}{dz}} - \frac{3}{2} \left(\frac{\frac{d^2w}{dz^2}}{\frac{dw}{dz}} \right)^2$$

Instead of giving the long and technical proof of the last statement, we shall derive the above for free boson system. We write the free boson energy-momentum tensor as

$$T(z) = -2\pi g \lim_{\delta \rightarrow 0} : \partial\phi\partial\phi : = -2\pi g \lim_{\delta \rightarrow 0} \left[\partial\phi \left(z + \frac{1}{2}\delta \right) \partial\phi \left(z - \frac{1}{2}\delta \right) + \frac{1}{4\pi\delta^2} \right]$$

Consider the transformation $z \rightarrow w(z)$. Since ϕ has conformal dimension zero $\partial\phi$ transforms as

$$\partial_z\phi = \frac{\partial w}{\partial z} \frac{\partial\phi'(w)}{\partial w} = w^{(1)}\partial_w\phi'(w)$$

(here we denote the n-th derivative of w by $w^{(n)}$ in order to lighten the notation). Hence $T(z)$ transforms as:

$$T(z) = -2\pi g \lim_{\delta \rightarrow 0} \left[w^{(1)} \left(z + \frac{1}{2}\delta \right) w^{(1)} \left(z - \frac{1}{2}\delta \right) \partial_w\phi' \left(w \left(z - \frac{1}{2}\delta \right) \right) \partial_w\phi' \left(w \left(z + \frac{1}{2}\delta \right) \right) + \frac{1}{4\pi\delta^2} \right] \quad (3.32)$$

we will use the following to simplify the above:

$$w(z + \delta/2) \simeq w(z) + \frac{\delta}{2}\partial_z w(z) + \frac{1}{2!} \left(\frac{\delta}{2} \right)^2 \partial_z^2 w(z) + \frac{1}{3!} \left(\frac{\delta}{2} \right)^3 \partial_z^3 w(z) + \dots$$

$$\partial_z w(z + \delta/2) \simeq \partial_z w(z) + \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!} \left(\frac{\delta}{2} \right)^2 \partial_z^3 w(z) + \dots$$

$$\begin{aligned} [w(z + \delta/2) - w(z - \delta/2)]^2 &= (\partial_z w(z))^2 \delta^2 + \frac{1}{12} (\partial_z^3 w \partial_z w(z)) \delta^4 + \mathcal{O}(\delta^6) \\ &= (w^{(1)}\delta)^2 \left[1 + \frac{1}{12} \frac{w^{(3)}}{w^{(1)}} \delta^2 + \dots \right] \end{aligned}$$

so the inverse is then,

$$\frac{1}{[w(z + \delta/2) - w(z - \delta/2)]^2} = \frac{1}{\delta^2} \frac{1}{(\partial_z w(z))^2} - \frac{1}{12} \frac{\partial_z^3 w(z)}{(\partial_z w(z))^3} + \mathcal{O}(\delta^2) \quad (3.33)$$

$$\begin{aligned} \partial_z w(z + \delta/2) \partial_z w(z - \delta/2) &= \left[\partial_z w(z) + \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!} \left(\frac{\delta}{2} \right)^2 \partial_z^3 w(z) + \dots \right] \\ &\quad \left[\partial_z w(z) - \frac{\delta}{2}\partial_z^2 w(z) + \frac{1}{2!} \left(\frac{\delta}{2} \right)^2 \partial_z^3 w(z) + \dots \right] \\ &= (w^{(1)})^2 - \left(\frac{\delta}{2} \right)^2 (w^{(2)})^2 + \left(\frac{\delta}{2} \right)^2 w^{(1)} w^{(3)} + \dots \quad (3.34) \end{aligned}$$

Then,

$$\begin{aligned} T(z) &= \lim_{\delta \rightarrow 0} \left[w^{(1)} \left(z + \frac{\delta}{2} \right) w^{(1)} \left(z - \frac{\delta}{2} \right) \right. \\ &\quad \left. \times \left\{ -2\pi g : \partial_w\phi'(w) \partial_w\phi'(w) : + \frac{1}{2 [w(z + \frac{\delta}{2}) - w(z - \frac{\delta}{2})]^2} \right\} - \frac{1}{2\delta^2} \right] \end{aligned}$$

$$\begin{aligned}
&= (w^{(1)}(z))^2 T'(w) \\
&\quad + \lim_{\delta \rightarrow 0} \left[\frac{w^{(1)}(z + \frac{\delta}{2}) w^{(1)}(z - \frac{\delta}{2})}{2 [w(z + \frac{\delta}{2}) - w(z - \frac{\delta}{2})]^2} - \frac{1}{2\delta^2} \right] \\
&= (w^{(1)}(z))^2 T'(w) \\
&\quad + \lim_{\delta \rightarrow 0} \left[\left\{ (w^{(1)})^2 - \frac{\delta^2}{4} (w^{(2)})^2 + \frac{\delta^2}{4} w^{(1)} w^{(3)} + \dots \right\} \right. \\
&\quad \quad \times \left. \left\{ \frac{1}{2\delta^2 (w^{(1)})^2} - \frac{1}{24} \frac{w^{(3)}}{(w^{(1)})^3} + \dots \right\} - \frac{1}{2\delta^2} \right] \\
&= (w^{(1)}(z))^2 T'(w) \\
&\quad + \lim_{\delta \rightarrow 0} \left[\frac{1}{2\delta^2} - \frac{1}{8} \left(\frac{w^{(2)}}{w^{(1)}} \right)^2 - \frac{1}{24} \frac{w^{(3)}}{w^{(1)}} + \frac{1}{8} \frac{w^{(3)}}{w^{(1)}} + \dots - \frac{1}{2\delta^2} \right] \\
&= (w^{(1)}(z))^2 T'(w) + \frac{1}{12} \left[\frac{w^{(3)}}{w^{(1)}} - \frac{3}{2} \left(\frac{w^{(2)}}{w^{(1)}} \right)^2 \right] \tag{4.9}
\end{aligned}$$

Here we use (3.33) and (3.34) in the second step for simplification. There is another way to look at this derivation which sheds light on the origin of schwarzian derivative term. Let us focus on how the propagator transforms under conformal transformation:

$$\begin{aligned}
\langle \phi(w(z_1)) \phi(w(z_2)) \rangle &= -\ln |w(z_1) - w(z_2)|^2 \\
&= -\ln \left| z_{12} \frac{w(z_1) - w(z_2)}{z_1 - z_2} \right|^2 \\
&= -\ln |z_{12}|^2 - \ln \left| \frac{w(z_1) - w(z_2)}{z_{12}} \right|^2 \\
&= \langle \phi(z_1) \phi(z_2) \rangle - \ln \left| \frac{w(z_1) - w(z_2)}{z_{12}} \right|^2
\end{aligned}$$

Let us write $w_1 \equiv w(z_1)$ and $w_2 \equiv w(z_2)$. One can show that for $|z_{12}| = |z_1 - z_2|$ small:

$$\begin{aligned}
\partial_{z_1} \partial_{z_2} \ln \left| \frac{w(z_1) - w(z_2)}{z_{12}} \right| &= \partial_{z_1} \partial_{z_2} \ln |w(z_1) - w(z_2)|^2 - \partial_{z_1} \partial_{z_2} \ln |z_{12}|^2 \\
&= \partial_{z_1} \left[\frac{-w^{(1)}(z_2)}{w(z_1) - w(z_2)} \right] - \frac{1}{z_{12}^2} \\
&= \frac{w^{(1)}(z_2) w^{(1)}(z_1)}{(w(z_1) - w(z_2))^2} - \frac{1}{z_{12}^2} \\
&= \frac{2}{12} \left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2} w_2} - \frac{3}{2} \left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2} w_2} \right)^2 \right] + \mathcal{O}(z_{12})
\end{aligned}$$

Since only the $z_{12} \rightarrow 0$ limit is of interest we can drop all terms on the right-hand side that vanish in this limit. Substituting the result of this into the above we learn that:

$$\lim_{z_1 \rightarrow z_2} \partial_{z_1} \partial_{z_2} \langle \phi(w(z_1)) \phi(w(z_2)) \rangle = \lim_{z_1 \rightarrow z_2} \partial_{z_1} \partial_{z_2} \langle \phi(z_1) \phi(z_2) \rangle - \frac{2}{12} \left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2} w_2} - \frac{3}{2} \left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2} w_2} \right)^2 \right]$$

The non-invariance of the vacuum and thus correlator is really what opens up the possibility of a trace anomaly $\langle U^{-1} T(w) U \rangle \equiv \langle T'(w) \rangle \neq 0$ and it is not at all coincidental. The conformal

transformations which do change the vacuum are those that have non-vanishing Schwarzian derivative, and thus an extra inhomogeneous central charge term. Using this we can derive the transformation law for energy-momentum tensor:

$$\begin{aligned}
T^{(w)}(z_2) &\equiv: \lim_{z_1 \rightarrow z_2} \left[-\frac{1}{2} \partial_{z_1} \phi(z_1) \partial_{z_2} \phi(z_2) \right] : w \\
&= \lim_{z_1 \rightarrow z_2} \left[-\frac{1}{2} (\partial_{z_1} \phi(z_1) \partial_{z_2} \phi(z_2) - \partial_{z_1} \partial_{z_2} \langle \phi(w(z_1)) \phi(w(z_2)) \rangle) \right] \\
&= \lim_{z_1 \rightarrow z_2} \left\{ -\frac{1}{2} \left(\partial_{z_1} \phi(z_1) \partial_{z_2} \phi(z_2) - \partial_{z_1} \partial_{z_2} \langle \phi(z_1) \phi(z_2) \rangle + \frac{2}{12} \left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2} w_2} - \frac{3}{2} \left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2} w_2} \right)^2 \right] \right) \right\} \\
&= \lim_{z_1 \rightarrow z_2} \left\{ -\frac{1}{2} (\partial_{z_1} \phi(z_1) \partial_{z_2} \phi(z_2) - \partial_{z_1} \partial_{z_2} \langle \phi(z_1) \phi(z_2) \rangle) \right\} - \frac{1}{12} \left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2} w_2} - \frac{3}{2} \left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2} w_2} \right)^2 \right] \\
&=: \lim_{z_1 \rightarrow z_2} \left[-\frac{1}{2} \partial_{z_1} \phi(z_1) \partial_{z_2} \phi(z_2) \right] : z - \frac{1}{12} \left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2} w_2} - \frac{3}{2} \left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2} w_2} \right)^2 \right] \\
&= T^{(z)}(z_2) - \frac{1}{12} \left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2} w_2} - \frac{3}{2} \left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2} w_2} \right)^2 \right]
\end{aligned} \tag{3.35}$$

where we noted in the last two lines that:

$$\begin{aligned}
T^{(z)}(z_2) &\equiv: \lim_{z_1 \rightarrow z_2} \left[-\frac{1}{2} \partial_{z_1} \phi(z_1) \partial_{z_2} \phi(z_2) \right] : z \\
&= \lim_{z_1 \rightarrow z_2} \left\{ -\frac{1}{2} (\partial_{z_1} \phi(z_1) \partial_{z_2} \phi(z_2) - \partial_{z_1} \partial_{z_2} \langle \phi(z_1) \phi(z_2) \rangle) \right\}
\end{aligned} \tag{3.36}$$

as shown above. So we learn that a finite holomorphic change in normal ordering, $z \rightarrow w(z)$, with fixed coordinates, z_2 , of the energy-momentum tensor is given by:

$$T^{(w)}(z_2) = T^{(z)}(z_2) - \frac{1}{12} \left[\frac{\partial_{z_2}^3 w_2}{\partial_{z_2} w_2} - \frac{3}{2} \left(\frac{\partial_{z_2}^2 w_2}{\partial_{z_2} w_2} \right)^2 \right] \tag{3.37}$$

Chapter 5

Operator Formalism

In the previous chapter, conformal symmetry was seen to impose constraints on correlation functions in the form of Ward identities. These identities were conveniently expressed using operator product expansions between the energy-momentum tensor and local fields, but the OPEs were understood only as a shorthand for singularities inside correlators. Nothing required a Hilbert space or an operator formalism: in principle, everything could have been computed directly in the path integral by evaluating Green's functions and extracting their shortdistance behavior. Up to this point, all we really needed was a way to compute the two-point correlator, whether by solving the Schwinger-Dyson equations or by brute-force path integration. The OPEs then followed from the explicit Green's function.

From here onward the viewpoint changes. Instead of relying on an explicit propagator, we will systematically construct an operator formalism in which OPEs can be obtained directly from symmetry and representation theory. This is a qualitatively new approach: even when the Green's function is not accessible by solving the equations of motion or performing the path integral, the operator method still allows us to determine the structure of OPEs and, in turn, recover the correlators themselves.

The Hilbert space is what makes this possible. Once a Hilbert space is defined, local fields are no longer only insertions in correlation functions - they correspond to states, and operators act as maps on this space. This representation turns the OPE into an actual operator identity, rather than a mnemonic for propagator singularities. The OPE then expresses how the action of one operator near another decomposes into the basis of states, giving us algebraic control that was absent in the purely path-integral description.

To set this up, we must specify a notion of "time," since operators evolve with respect to it. In Euclidean space, the natural choice is to take the radial coordinate as time, leading to radial quantization. States are then defined on concentric circles, time evolution is dilation, and contour integrals of the stress tensor implement the Virasoro algebra. Within this framework, commutators appear as contour manipulations, and the OPE becomes a universal computational tool, independent of explicit Green's functions.

5.0.1 Radial quantization

In the operator formalism one must first distinguish a time direction from a space direction. In Minkowski spacetime this choice is canonical, but in Euclidean space it is arbitrary. A Hilbert space by itself is just a complete inner product space; what makes it physically meaningful is the specification of a time direction, which singles out a Hamiltonian as the generator of time translations. The Hamiltonian defines a vacuum state (its lowest-energy eigenstate) and organizes the remaining states as excitations built on top of it.

This structure underlies any quantum theory: begin with a vacuum, generate excitations, and classify states by the Hamiltonian spectrum. In two-dimensional conformal field theory we can choose a different notion of "time": the radial direction from the origin. To motivate this further, consider \mathbb{R}^d in spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_{d-1} = r^2 \left[\frac{dr^2}{r^2} + d\Omega_{d-1} \right]$$

Now let $t = \log(r)$ so that

$$ds^2 = e^{2t} [dt^2 + d\Omega_{d-1}]$$

which is conformally related to the metric on $\mathbb{R} \times S^{d-1}$. Now if we consider a CFT on \mathbb{R}^d , the theory should be invariant under rescaling of metric, so studying that theory on \mathbb{R}^d should be equivalent to studying the theory on $\mathbb{R} \times S^{d-1}$.

From a Minkowski space point of view (in particular in the context of string theory), we will initially define our theory on an infinite space-time cylinder, with time t going from $-\infty$ to $+\infty$ along the "flat" direction of the cylinder, and space being compactified with x going from 0 to L , and the point $(0, t)$ and (L, t) being identified. If we continue to Euclidian space, the cylinder is described by a single complex coordinate

$$w = t + ix, \quad x \sim x + L,$$

with Hamiltonian $H = -i\partial_t$. We can then consider the conformal map of this cylinder to the plane via

$$z = e^{\frac{2\pi}{L}w}$$

Here $t \rightarrow -\infty$ maps to $z = 0$, $t \rightarrow +\infty$ to $z = \infty$, and the compact x direction gives angular periodicity. A very interesting feature of this map is that it takes circles of constant radius in \mathbb{R}^2 to constant t slices on $\mathbb{R} \times S$. As a consequence, the dilatation operator on \mathbb{R}^2 which maps circles onto circles with different radius, corresponds to time translation on $\mathbb{R} \times S$, so it behave as Hamiltonian.

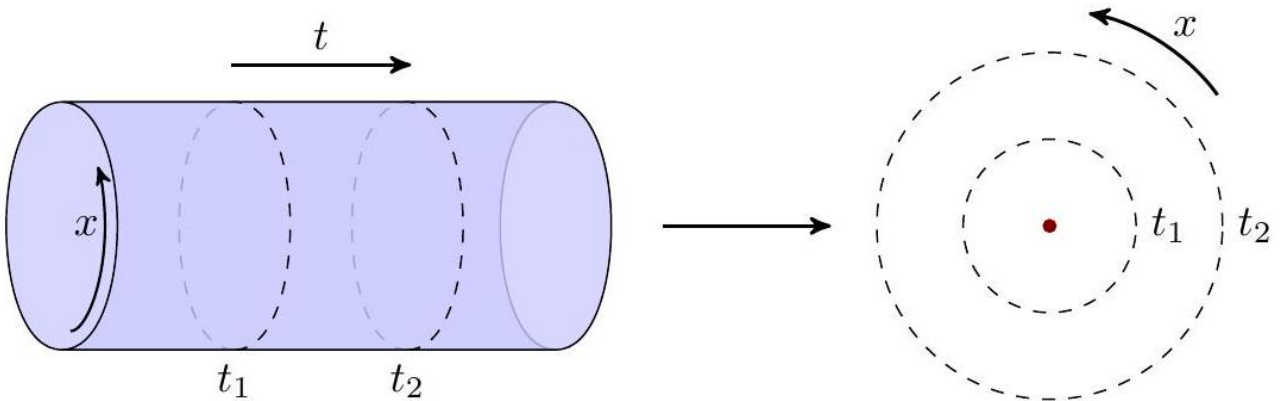


Figure 5.1: Conformal map from the cylinder to the complex plane.

Since periodicity in x is reflected in the periodicity of w and \bar{w} . All operators on cylinder admit the following expansion:

$$\phi(w, \bar{w}) = \sum_{m,n \in \mathbb{Z}} \phi_{n,m} e^{-nw} e^{-m\bar{w}} \quad (4.1)$$

where $\phi_{n,m}$ do not have any (w, \bar{w}) dependence. Consider the transformation of primary field living on cylinder to z -plane under the conformal mapping $z = e^w$:

$$\begin{aligned} \Phi(z, \bar{z}) &= (\partial_z w)^h (\partial_{\bar{z}} \bar{w})^{\bar{h}} \Phi(w, \bar{w}) \\ &= z^{-h} \bar{z}^{-\bar{h}} \sum_{m,n \in \mathbb{Z}} \phi_{n,m} e^{-nw} e^{-m\bar{w}} \\ &= z^{-h} \bar{z}^{-\bar{h}} \sum_{m,n \in \mathbb{Z}} \phi_{m,m} z^{-n} \bar{z}^{-m} \\ &= \sum_{m,n} \phi_{n,m} z^{-n-h} \bar{z}^{-m-\bar{h}} \end{aligned}$$

In free-field theories the vacuum is annihilated by positive-frequency modes; in interacting theories we assume asymptotic fields behave freely, e.g.

$$\phi_{\text{in}} \propto \lim_{t \rightarrow -\infty} \phi(x, t).$$

In radial quantization, this translates into

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \rightarrow 0} \phi(z, \bar{z})|0\rangle.$$

Expanding the operator in modes, we have

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \rightarrow 0} \sum_{m, n} \phi_{n, m} z^{-n-h} \bar{z}^{-m-\bar{h}} |0\rangle.$$

For this expression to be non-singular as $z \rightarrow 0$, the modes with negative powers of z and \bar{z} must annihilate the vacuum. This requires

$$n + h > 0 \Rightarrow n > -h, \quad m + \bar{h} > 0 \Rightarrow m > -\bar{h},$$

so that

$$\phi_{n, m} |0\rangle = 0 \quad \text{for } n > -h \text{ and } m > -\bar{h}$$

Thus, only the non-negative powers contribute. For $n + h < 0$ and $m + \bar{h} < 0$, we have

$$\lim_{z, \bar{z} \rightarrow 0} z^{-n-h} \bar{z}^{-m-\bar{h}} |0\rangle = 0$$

The only non-trivial contribution comes from the modes with

$$n + h = 0, \quad m + \bar{h} = 0$$

yielding

$$|\phi_{\text{in}}\rangle = \phi_{-h, -\bar{h}} |0\rangle.$$

Once the Hilbert space is tied to a vacuum and a Hamiltonian (here, the dilatation operator), operators can be classified by scaling dimensions, excitations arranged systematically, and operator product expansions formulated as exact operator identities—rather than being inferred indirectly from correlation functions.

5.1 The Hermitian product

On this Hilbert space we must also define a bilinear product, which we do indirectly by defining an asymptotic "out" state, together with the action of Hermitian conjugation on conformal fields.

$$\langle \phi_{\text{out}} | = |\phi_{\text{in}}\rangle^\dagger = \left(\lim_{z, \bar{z} \rightarrow 0} \phi(z, \bar{z}) |0\rangle \right)^\dagger$$

In Minkowski space, Hermitian conjugation does not affect the space-time coordinates. Things are different in Euclidian space, since the Euclidian time $\tau = it$ must be reversed ($\tau \rightarrow -\tau$) upon Hermitian conjugation if t is to be left unchanged. In radial quantization this corresponds to the mapping:

$$\begin{aligned} w \equiv e^{\tau+ix} &\rightarrow e^{-\tau+ix} \\ &= e^{-(\tau-ix)} = \frac{1}{e^{\tau+ix}} = \frac{1}{\bar{z}} \end{aligned}$$

Since ϕ is a primary field,

$$\begin{aligned}
\phi(z, \bar{z}) &= \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \phi(w, \bar{w}) \\
&= \left(-\frac{1}{z^2}\right)^h \left(-\frac{1}{\bar{z}^2}\right)^{\bar{h}} \phi(w, \bar{w}) \\
&= (-1)^h (w^{2h}) (-1)^{-\bar{h}} (\bar{w}^{2\bar{h}}) \phi(w, \bar{w}) \\
&= (-1)^{h+\bar{h}} w^{2h} \bar{w}^{2\bar{h}} \phi(w, \bar{w})
\end{aligned}$$

$$\phi(w, \bar{w}) = (-1)^{h+\bar{h}} w^{-2h} \bar{w}^{-2\bar{h}} \phi(z, \bar{z})$$

Since $|\phi_{\text{in}}\rangle$ were defined using $\lim_{z \rightarrow 0}$, so $\lim_{w \rightarrow 0}$ could be used to define $|\phi_{\text{out}}\rangle$ state:

$$\langle \phi_{\text{out}} | = \lim_{z, \bar{z} \rightarrow \infty} \langle 0 | \phi\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) = \lim_{w, \bar{w} \rightarrow 0} (-1)^{h+\bar{h}} w^{-2h} \bar{w}^{-2\bar{h}} \langle 0 | \phi\left(\frac{1}{w}, \frac{1}{\bar{w}}\right)$$

relabelling $w = z$,

$$\langle \phi_{\text{out}} | = \lim_{z, \bar{z} \rightarrow 0} (-1)^{h+\bar{h}} z^{-2h} \bar{z}^{-2\bar{h}} \langle 0 | \phi\left(\frac{1}{z}, \frac{1}{\bar{z}}\right)$$

The factor $(-1)^{h+\bar{h}}$ is coming due to spin, if we ignore that:

$$\langle \phi_{\text{out}} | = \lim_{z, \bar{z} \rightarrow 0} z^{-2h} \bar{z}^{-2\bar{h}} \langle 0 | \phi\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) = \lim_{z, \bar{z} \rightarrow 0} \langle 0 | \phi(z, \bar{z})^\dagger$$

Hence, we conclude

$$\phi(z, \bar{z})^\dagger = z^{-2h} \bar{z}^{-2\bar{h}} \phi\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) \quad (4.2)$$

Or we could alternatively do Fourier expansion on the radial plane, the adjoint property then reads

$$\begin{aligned}
\Phi(z, \bar{z})^\dagger &= \bar{z}^{-2h} z^{-2\bar{h}} \Phi(1/\bar{z}, 1/z) \\
&= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} \left(\frac{1}{\bar{z}}\right)^{-n-h} \left(\frac{1}{z}\right)^{-m-\bar{h}} \\
&= \bar{z}^{-2h} z^{-2\bar{h}} \sum_{m,n} \phi_{n,m} (\bar{z})^{n+h} (z)^{m+\bar{h}} \\
&= \sum_{m,n} \phi_{n,m} z^{m-\bar{h}} \bar{z}^{n-h}
\end{aligned}$$

$$\sum_{m,n} \phi_{n,m}^\dagger \overline{(z^{-n-h} \bar{z}^{-m-\bar{h}})} = \sum_{m,n} \phi_{-n,-m} z^{-m-\bar{h}} \bar{z}^{-n-h} \implies \phi_{n,m}^\dagger = \phi_{-n,-m}$$

The inner product is well defined as well:

$$\begin{aligned}
\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle &= \lim_{z, \bar{z}, w, \bar{w} \rightarrow 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0 | \phi(1/\bar{z}, 1/z) \phi(w, \bar{w}) | 0 \rangle \\
&= \lim_{\xi, \bar{\xi} \rightarrow \infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0 | \phi(\bar{\xi}, \xi) \phi(0, 0) | 0 \rangle \\
&= \lim_{\xi, \bar{\xi} \rightarrow \infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \frac{C_{12}}{\bar{\xi}^{2h} \xi^{2\bar{h}}} = C_{12}
\end{aligned}$$

Unless the prefactors in (4.2) were missing the limit would have been ill defined.

5.1.1 Radial Ordering

Within radial quantization, the time ordering that appears in the definition of correlation functions becomes a radial ordering,

$$\mathcal{R}(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w| \\ \phi_2(w)\phi_1(z) & |z| < |w| \end{cases} \quad (4.3)$$

As usual, we will always omit the radial-ordered operator in the correlation function as well as in the OPE expansion. One consequence after specifying time direction is that we can relate OPE to commutation relations. For this, let us consider the contour integral around w for two holomorphic fields $a(z)$ and $b(w)$. If the contour is not radially ordered over the whole path, we can decompose it into contributions that are radially ordered:

$$\oint_w dza(z)b(w) = \underbrace{\oint_{|z|>|w|} dza(z)b(w)}_{\text{radial ordering: } z \text{ outside } w} - \underbrace{\oint_{|z|<|w|} dzb(w)a(z)}_{\text{radial ordering: } z \text{ inside } w} = [A, b(w)] \quad (4.4)$$

where the operator A is the contour integral of $a(z)$ at a fixed time

$$A = \oint dza(z) \quad (4.5)$$

Here we take the contours C_1 and C_2 at fixed-radius $|w| + \epsilon$ and $|w| - \epsilon$ with a small positive number ϵ as illustrated in Figure 4.2. Then, with $B = \oint dzb(z)$, we can generalize the relation (??) to

$$[A, B] = \oint_0 dw[A, b(w)] = \oint_0 dw \oint_w dza(z)b(w) \quad (4.6)$$

5.2 Virasoro Algebra

5.2.1 Conformal Generators

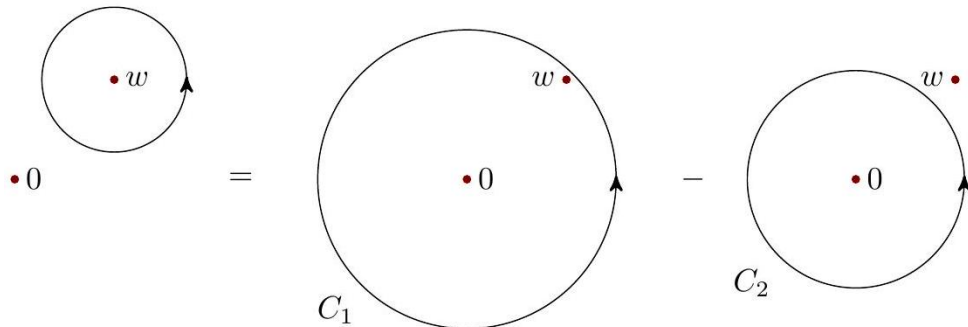
The ward identity

$$\delta_\epsilon \langle \phi_1 \dots \phi_n \rangle = \int_M d^2x \partial_\mu \langle T^{\mu\nu} \epsilon_\nu \phi_1 \dots \phi_n \rangle$$

could be expressed in complex coordinates using (3.7):

$$\delta_{\epsilon, \bar{\epsilon}} \langle \phi_1 \dots \phi_n \rangle = \frac{i}{2} \int_C [-dz \langle T^{\bar{z}z} \epsilon_{\bar{z}} \phi_1 \dots \phi_n \rangle + d\bar{z} \langle T^{zz} \epsilon_z \phi_1 \dots \phi_n \rangle]$$

Figure 5.2: Subtraction of contours



$$= -\frac{1}{2\pi i} \int_C dz \langle T(z) \epsilon \phi_1 \dots \phi_n \rangle + \frac{1}{2\pi i} \int_C d\bar{z} \langle \bar{T}(\bar{z}) \bar{\epsilon} \phi_1 \dots \phi_n \rangle$$

where we used,

$$T = -2\pi T_{zz} = -2\pi g_{z\mu} g_{z\nu} T^{\mu\nu} = -2\pi \frac{1}{4} T^{\bar{z}\bar{z}} = -\frac{\pi}{2} T^{\bar{z}\bar{z}}$$

$$\bar{T} = -2\pi T_{\bar{z}\bar{z}} = -2\pi g_{\bar{z}\mu} g_{\bar{z}\nu} T^{\mu\nu} = -2\pi \frac{1}{4} T^{zz} = -\frac{\pi}{2} T^{zz}$$

and

$$\epsilon = \epsilon^z = g^{z\mu} \epsilon_\mu = 2\epsilon_{\bar{z}}$$

$$\bar{\epsilon} = \epsilon^{\bar{z}} = g^{\bar{z}\mu} \epsilon_\mu = 2\epsilon_z$$

If we apply, (4.4) and (4.6) to the conformal ward identity. Let $\epsilon(z)$ be holomorphic component of an infinitesimal conformal change of coordinates. We then define the conformal charge

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) \quad (4.7)$$

with the help of (4.4), the conformal ward identity translates into

$$\delta_\epsilon \phi = -[Q_\epsilon, \phi]$$

which means that the operator Q_ϵ is the generator of conformal transformation. We may expand energy-momentum tensor according to (4.1):

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \quad L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z)$$

$$\bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n \quad \bar{L}_n = \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z})$$

we may also expand the infinitesimal conformal change $\epsilon(z)$ as:

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} z^{n+1} \epsilon_n$$

Then the expression, (4.7) becomes:

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz \left(\sum_{n \in \mathbb{Z}} z^{n+1} \epsilon_n \right) T(z)$$

$$= \sum_{n \in \mathbb{Z}} \epsilon_n \frac{1}{2\pi i} \oint dz z^{n+1} T(z)$$

$$= \sum_{n \in \mathbb{Z}} \epsilon_n L_n$$

The mode operators L_n and L_m of the energy-momentum tensor are the generators of the local conformal transformations on the Hilbert space, exactly like l_n and l_m of Witt algebra. The next part is to find the algebra obeyed by L_n and L_m which we described in previous chapter as the central extension of Witt algebra. We will now prove that here:

$$\begin{aligned}
[L_m, L_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)] \\
&= \oint \frac{dw}{2\pi i} w^{n+1} \oint_{C(w)} \frac{dz}{2\pi i} z^{m+1} \mathcal{R}(T(z)T(w)) \\
&= \oint \frac{dw}{2\pi i} w^{n+1} \oint_{C(w)} \frac{dz}{2\pi i} z^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \right] \\
&= \oint \frac{dw}{2\pi i} w^{n+1} \left[(m+1)m(m-1)w^{m-2} \frac{c}{2 \cdot 3!} + 2(m+1)w^m T(w) + w^{m+1} \partial_w T(w) \right] \\
&= \oint \frac{dw}{2\pi i} \left[\frac{c}{12} (m^3 - m) w^{m+n-1} + 2(m+1)w^{m+n+1} T(w) + w^{m+n+2} \partial_w T(w) \right] \\
&= \oint \frac{dw}{2\pi i} \left[\frac{c}{12} (m^3 - m) w^{m+n-1} + 2(m+1)w^{m+n+1} T(w) + \partial_w \{w^{m+n+2} T(w)\} - \partial_w w^{m+n+2} T(w) \right] \\
&= \frac{c}{12} (m^3 - m) \delta_{m,-n} + 2(m+1)L_{m+n} + 0 - \oint \frac{dw}{2\pi i} (m+n+2)T(w)w^{m+n+1} \\
&= \frac{c}{12} (m^3 - m) \delta_{m,-n} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n} \\
&= (m-n)L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m,-n}
\end{aligned}$$

Collectively,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m,-n} \\
[L_m, \bar{L}_n] &= 0 \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m,-n}
\end{aligned}$$

5.3 The Hilbert Space

The vacuum state $|0\rangle$ in CFT must be invariant under global conformal transformation. This means the generator of transformation

$$\frac{az+b}{cz+d}$$

which are $L_{\pm 1}$ and L_0 must annihilate the vacuum. This, in turn is recovered from the condition that $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ are well defined as $z, \bar{z} \rightarrow 0$,

$$\begin{aligned}
\lim_{z \rightarrow 0} T(z)|0\rangle &= \lim_{z \rightarrow 0} \sum_{n \in \mathbb{Z}} z^{-n-2} L_n |0\rangle \\
\lim_{\bar{z} \rightarrow 0} \bar{T}(\bar{z})|0\rangle &= \lim_{\bar{z} \rightarrow 0} \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n |0\rangle
\end{aligned}$$

Here, $z^{-(n+2)}$ diverges at $z = 0$ if $n+2 > 0$, i.e. if $n > -2$, or equivalently $n \geq -1$. Which implies, we need

$$\begin{aligned}
L_n |0\rangle &= 0 \\
\bar{L}_n |0\rangle &= 0 \quad (\text{for } n \geq -1)
\end{aligned}$$

This includes a subcondition regarding invariance of $|0\rangle$ under global conformal group. It also implies the vanishing of the vacuum expectation value of the energy-momentum tensor: ¹

$$\langle 0|T(z)|0\rangle = \left(\sum_{n=-\infty}^{-1} z^{-n-2} \langle 0|L_n \rangle \right) |0\rangle + \sum_{n=0}^{\infty} z^{-n-2} \langle 0|(L_n|0\rangle) = 0$$

$$\langle 0|\bar{T}(\bar{z})|0\rangle = \left(\sum_{n=-\infty}^{-1} z^{-n-2} \langle 0|\bar{L}_n \rangle \right) |0\rangle + \sum_{n=0}^{\infty} z^{-n-2} \langle 0|(\bar{L}_n|0\rangle) = 0$$

Primary fields, when acting on vacuum, create asymptotic states, eigenstates of the Hamiltonian. A simple demonstration follows from the OPE (3.16) between $T(z)$ and a primary field $\phi(z, \bar{z})$ of dimension (h, \bar{h}) , translated into operator language:

$$\begin{aligned} [L_n, \phi(w, \bar{w})] &= \frac{1}{2\pi i} \oint_w dz z^{n+1} T(z) \phi(w, \bar{w}) \\ &= \frac{1}{2\pi i} \oint_w dz z^{n+1} \left[\frac{h\phi(w, \bar{w})}{(z-w)^2} + \frac{\partial\phi(w, \bar{w})}{z-w} + \text{reg.} \right] \\ &= h(n+1)w^n \phi(w, \bar{w}) + w^{n+1} \partial\phi(w, \bar{w}) \quad (n \geq -1) \end{aligned}$$

The antiholomorphic counterpart of this relation is

$$[\bar{L}_n, \phi(w, \bar{w})] = \bar{h}(n+1)\bar{w}^n \phi(w, \bar{w}) + \bar{w}^{n+1} \bar{\partial}\phi(w, \bar{w}) \quad (n \geq -1)$$

After applying these relations to the asymptotic state

$$|h, \bar{h}\rangle \equiv \phi(0, 0)|0\rangle$$

we conclude that

$$\begin{aligned} L_0|h, \bar{h}\rangle &= L_0\phi(0, 0)|0\rangle \\ &= \phi(0, 0)L_0|0\rangle + [L_0, \phi(0, 0)]|0\rangle \\ &= 0 + h\phi(0, 0)|0\rangle + 0 = h|h, \bar{h}\rangle \end{aligned}$$

compactly,

$$L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle, \quad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle.$$

Thus $|h, \bar{h}\rangle$ is an eigenstate of the Hamiltonian. Likewise, we have

$$L_n|h, \bar{h}\rangle = 0, \quad \bar{L}_n|h, \bar{h}\rangle = 0, \quad \text{if } n > 0$$

Excited states above the asymptotic state $|h, \bar{h}\rangle$ may be obtained by applying ladder operators. The generators L_{-m} ($m > 0$) increases the eigenstate of L_0 a.k.a conformal weight by virtue of virasoro algebra

$$[L_0, L_{-m}] = (0+m)L_{-m} = mL_{-m}$$

This means the excited states may be obtained by successive applications of these operators on the asymptotic state $|h\rangle$:

$$L_{-k_1} L_{-k_2} \dots L_{-k_n} |0\rangle \quad (1 \geq k_1 \geq \dots \geq k_n)$$

This state is the eigenstate of L_0 with eigenvalue

$$h' = h + k_1 + k_2 + \dots + k_n = h + N$$

and these are called descendants of the asymptotic state $|h\rangle$ and the integer N is called the level of the descendant.

5.3.1 Free Boson

This section gives a detailed account of the canonical quantization of the free boson on the cylinder. The mode expansions are obtained, after imposing the appropriate boundary conditions. The mapping from the cylinder to the complex plane is used to define the conformal generators and, in particular, the vacuum energies. Freefield theories are of special importance not only because they can be solved explicitly, but also because they are the building blocks of more complicated models, or can be shown to be equivalent to interesting statistical models.

⁰¹ If the vacuum is not invariant under local conformal transformation i.e. if vacuum is not annihilated by L_n for $n > 1$, then we can not ensure the vanishing of vev of energy-momentum tensor.

5.4 Mode Expansion

We let $\varphi(x, t)$ be a free Bose field defined on a cylinder of circumference L :

$$\varphi(x + L, t) \equiv \varphi(x, t)$$

This field may be Fourier expanded as follows:

$$\varphi(x, t) = \sum_n e^{2\pi i n x / L} \varphi_n(t)$$

with Fourier coefficients

$$\varphi_n(t) = \frac{1}{L} \int_0^L dx e^{-2\pi i n x / L} \varphi(x, t)$$

Next, we try to find the Fourier coefficients of kinetic part.

$$\begin{aligned} \partial_t \varphi(x, t) &= \sum_n e^{2\pi i n x / L} \dot{\varphi}_n(t) \\ (\partial_t \varphi)^2 &= \sum_{n, m} e^{2\pi i (n+m)x / L} \dot{\varphi}_n \dot{\varphi}_m \end{aligned}$$

Only the $m = -n$ term survives the integral over x , so

$$\int_0^L dx (\partial_t \varphi)^2 = L \sum_n \dot{\varphi}_n \dot{\varphi}_{-n} \quad (4.8)$$

Similarly,

$$\begin{aligned} \partial_x \varphi(x, t) &= \sum_n \frac{2\pi i n}{L} e^{2\pi i n x / L} \varphi_n(t) \\ (\partial_x \varphi)^2 &= \sum_{n, m} \frac{2\pi i n}{L} \frac{2\pi i m}{L} e^{2\pi i (n+m)x / L} \varphi_n \varphi_m \\ \int_0^L dx (\partial_x \varphi)^2 &= L \sum_n \left(\frac{2\pi n}{L} \right)^2 \varphi_n \varphi_{-n} \end{aligned} \quad (4.9)$$

Putting them in the free field Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} g \int_0^L dx [(\partial_t \varphi)^2 - (\partial_x \varphi)^2] \\ &= \frac{gL}{2} \sum_n \left(\dot{\varphi}_n \dot{\varphi}_{-n} - \left(\frac{2\pi n}{L} \right)^2 \varphi_n \varphi_{-n} \right) \end{aligned}$$

The momentum conjugate to φ_n is

$$\pi_n = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_n} = gL \dot{\varphi}_{-n} \quad [\varphi_n, \pi_m] = i\delta_{nm}$$

The Hamiltonian is

$$\begin{aligned} H &= \sum_n \pi_n \dot{\varphi}_n - \mathcal{L} \\ &= \frac{1}{2gL} \sum_n (\pi_n \pi_{-n} + (2\pi n g)^2 \varphi_n \varphi_{-n}) \end{aligned}$$

Thus the system reduces to a sum of decoupled harmonic oscillators with frequencies $\omega_n = \frac{2\pi|n|}{L}$. The usual procedure is to define creation and annihilation operators \tilde{a}_n and \tilde{a}_n^\dagger :

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi g|n|}} (2\pi g|n| \phi_n + i\pi_{-n}) \quad (4.10)$$

such that $[\tilde{a}_n, \tilde{a}_m^\dagger] = \delta_{n,m}$ and $[\tilde{a}_n, \tilde{a}_m] = 0$. This ofcourse does not work for zero mode ϕ_0 . Instead we shall use the following operators:

$$a_n = \begin{cases} -i\sqrt{n}\tilde{a}_n & (n > 0) \\ i\sqrt{-n}\tilde{a}_{-n}^\dagger & (n < 0) \end{cases}, \quad \bar{a}_n = \begin{cases} -i\sqrt{n}\tilde{a}_{-n} & (n > 0) \\ i\sqrt{-n}\tilde{a}_n^\dagger & (n < 0) \end{cases}$$

and we treat the zero mode ϕ_0 separately. The associated commutation relations are

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_n, \tilde{a}_m] = 0, \quad [\tilde{a}_n, \tilde{a}_m] = n\delta_{n+m,0} \quad (4.11)$$

The Hamiltonian is then expressible as

$$H = \frac{1}{2gL} \sum_n (\pi_n \pi_{-n} + (2\pi gn)^2 \varphi_n \varphi_{-n}) = \frac{\pi_0^2}{2gL} + \frac{(2\pi g0)^2}{2gL} \phi_0^2 + \frac{1}{2gL} \sum_{n \neq 0} (\pi_n \pi_{-n} + (2\pi gn)^2 \varphi_n \varphi_{-n}) \quad (4.12)$$

We start with the oscillator operators:

$$\begin{aligned} \tilde{a}_n &= \frac{1}{\sqrt{4\pi gn}} (2\pi gn \varphi_n + i\pi_{-n}), & \tilde{a}_n^\dagger &= \frac{1}{\sqrt{4\pi gn}} (2\pi gn \varphi_{-n} - i\pi_n) \\ \tilde{a}_{-n} &= \frac{1}{\sqrt{4\pi gn}} (2\pi gn \varphi_{-n} + i\pi_n), & \tilde{a}_{-n}^\dagger &= \frac{1}{\sqrt{4\pi gn}} (2\pi gn \varphi_n - i\pi_{-n}) \\ 4\pi gn \tilde{a}_n^\dagger \tilde{a}_n &= (2\pi gn)^2 \varphi_{-n} \varphi_n + \pi_n \pi_{-n} + i[\text{cross terms}] \\ 4\pi gn \tilde{a}_{-n}^\dagger \tilde{a}_{-n} &= (2\pi gn)^2 \varphi_n \varphi_{-n} + \pi_{-n} \pi_n - i[\text{same cross terms}] \end{aligned}$$

adding above equations,

$$2\pi gn (\tilde{a}_n^\dagger \tilde{a}_n + \tilde{a}_{-n}^\dagger \tilde{a}_{-n}) = 2 [(2\pi gn)^2 \varphi_n \varphi_{-n} + \pi_n \pi_{-n}] \implies 2\pi gn (\tilde{a}_n^\dagger \tilde{a}_n + \tilde{a}_{-n}^\dagger \tilde{a}_{-n}) = \pi_n \pi_{-n} + (2\pi gn)^2 \varphi_n \varphi_{-n}$$

Substituting it in (4.12),

$$H = \frac{1}{2gL} \pi_0^2 + \frac{\pi}{L} \sum_{n \neq 0} n (\tilde{a}_n^\dagger \tilde{a}_n + \tilde{a}_{-n}^\dagger \tilde{a}_{-n}).$$

$$a_n = -i\sqrt{n}\tilde{a}_n, \quad a_{-n} = a_n^\dagger = i\sqrt{n}\tilde{a}_n^\dagger, \quad \bar{a}_n = -i\sqrt{n}\tilde{a}_{-n}, \quad \bar{a}_{-n} = \bar{a}_n^\dagger = i\sqrt{n}\tilde{a}_{-n}^\dagger \quad (n > 0).$$

So

$$n\tilde{a}_n^\dagger \tilde{a}_n = (i\sqrt{n}\tilde{a}_n^\dagger) (-i\sqrt{n}\tilde{a}_n) = a_{-n} a_n, \quad n\tilde{a}_{-n}^\dagger \tilde{a}_{-n} = (i\sqrt{n}\tilde{a}_{-n}^\dagger) (-i\sqrt{n}\tilde{a}_{-n}) = \tilde{a}_{-n} \tilde{a}_n$$

$$H = \frac{1}{2gL} \pi_0^2 + \frac{\pi}{L} \sum_{n \neq 0} (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n) \quad (4.13)$$

The commutation relation (4.11), leads to

$$\begin{aligned} [H, a_{-m}] &= \frac{\pi}{L} \sum_{n \neq 0} [a_{-n} a_n, a_{-m}] \\ &= \frac{\pi}{L} \sum_{n \neq 0} (a_{-n} [a_n, a_{-m}] + [a_{-n}, a_{-m}] a_n) \\ &= \frac{\pi}{L} \sum_{n \neq 0} (a_{-n} n \delta_{n-m,0} + (-n) \delta_{-n-m,0} a_n) \\ &= \frac{2\pi}{L} m a_{-m} = -i\dot{a} \end{aligned}$$

which means for a_{-m} ($m > 0$), when applied to an eigenstate of H of energy E , produces another eigenstate with energy $E + 2m\pi/L$. Since the fourier modes are

$$\varphi_n = \frac{i}{n\sqrt{4\pi g}} (a_n - \bar{a}_{-n})$$

The mode expansion at $t = 0$ may be written as:

$$\varphi(x) = \varphi_0 + \sum_{n \neq 0} \varphi_n e^{\frac{2\pi i n x}{L}} = \varphi_0 + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n - \bar{a}_{-n}) e^{2\pi i n x/L}$$

The time evolution of the operators ϕ_0, a_n and \bar{a}_n in the Heisenberg picture follow from the above Hamiltonian:

$$[H, \phi_0] = \frac{1}{2gL} [\pi_0^2, \phi_0] = -\frac{\pi_0 [\phi_0, \pi_0] + [\phi_0, \pi_0] \pi_0}{2gL} = -\frac{i}{gL} \pi_0 \quad (\text{using } [\varphi_0, \pi_0] = i)$$

Hence,

$$\varphi_0(t) = \varphi_0(0) + \frac{1}{gL} \pi_0 t$$

$$a_n(t) = a_n(0) e^{-2\pi i n t/L}$$

$$\bar{a}_n(t) = \bar{a}_n(0) e^{-2\pi i n t/L}$$

In terms of constant operators, the mode expansion of the field at arbitrary time is then,

$$\varphi_{\text{cyl}} = \varphi_0 + \frac{1}{gL} \pi_0 t + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n e^{-2\pi i n (t-x)/L} + \bar{a}_n e^{-2\pi i n (t+x)/L}) \quad (4.14)$$

where 'cyl' means the field defined on the cylinder.

5.5 Relation between Cylinder and Plane

Now, we move to Euclidean space-time ($t = -i\tau$) by taking $w = \tau - ix$ and $\bar{w} = \tau + ix$. The mode expansion of boson field on the cylinder now is then

$$\varphi_{\text{cyl}}(w, \bar{w}) = \varphi_0 - i \frac{1}{2gL} \pi_0 (w + \bar{w}) + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n e^{-2\pi n w/L} + \bar{a}_n e^{-2\pi n \bar{w}/L}) \quad (4.15)$$

with now $w \sim w + iL$. Using a conformal transformation as in Figure 4.1, we map all the operators from the cylinder to the complex plane:

$$z = e^{2\pi w/L}, \quad \bar{z} = e^{2\pi \bar{w}/L}$$

we finally obtain the expansion by simply replacing w by z .

$$\varphi_{\text{pl}}(z, \bar{z}) = \varphi_0 - \frac{i}{4\pi g} \pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \quad (4.16)$$

The propagator could be calculated by assuming $|z| > |z'|$ and considering the product

$$\begin{aligned} \varphi_{\text{pl}}(w, \bar{w}) \varphi_{\text{pl}}(w', \bar{w}') &= \left[\varphi_0 - i \frac{1}{4\pi g} \pi_0 \ln |z|^2 + i \frac{1}{\sqrt{4\pi g}} \sum_{m \neq 0} \frac{1}{m} (a_m z^{-m} + \bar{a}_m \bar{z}^{-m}) \right] \\ &\quad \times \left[\varphi_0 - i \frac{1}{4\pi g} \pi_0 \ln |z'|^2 + i \frac{1}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} (a_n z'^{-n} + \bar{a}_n \bar{z}'^{-n}) \right]. \end{aligned}$$

We will move annihilation operators (a_k, \bar{a}_k with $k > 0$) and π_0 to the right to reach normal ordering; nonvanishing c-number contributions come from $[\varphi_0, \pi_0] = i$ and $[a_m, a_n] = m\delta_{m+n,0}, [\bar{a}_m, \bar{a}_n] = m\delta_{m+n,0}$. Let's first focus on these terms

$$\begin{aligned}
& \left(\varphi_0 - i \frac{1}{4\pi g} \pi_0 \ln |z|^2 \right) \left(\varphi_0 - i \frac{1}{4\pi g} \pi_0 \ln |z'|^2 \right) \\
&= \varphi_0^2 - i \frac{1}{4\pi g} \left(\varphi_0 \pi_0 \ln |z'|^2 + \pi_0 \varphi_0 \ln |z|^2 \right) - \left(\frac{1}{4\pi g} \right)^2 \pi_0^2 \ln |z|^2 \ln |z'|^2 \\
&= \varphi_0^2 - i \frac{1}{4\pi g} \left(\varphi_0 \pi_0 \ln |z'|^2 + (\varphi_0 \pi_0 - i) \ln |z|^2 \right) - \left(\frac{1}{4\pi g} \right)^2 \pi_0^2 \ln |z|^2 \ln |z'|^2 \\
&= \varphi_0^2 - i \frac{1}{4\pi g} \varphi_0 \pi_0 \left(\ln |z'|^2 + \ln |z|^2 \right) - \frac{1}{4\pi g} \ln |z|^2 - \left(\frac{1}{4\pi g} \right)^2 \pi_0^2 \ln |z|^2 \ln |z'|^2 \\
&= \varphi_0^2 - i \frac{1}{4\pi g} \varphi_0 \pi_0 \left(\ln |z'|^2 + \ln |z|^2 \right) - \underbrace{\left(\frac{1}{4\pi g} \right)^2 \pi_0^2 \ln |z|^2 \ln |z'|^2 - \frac{1}{4\pi g} \ln |z|^2}_{\text{normal ordered}} \\
&=: \left(\varphi_0 - i \frac{1}{4\pi g} \pi_0 \ln |z|^2 \right) \left(\varphi_0 - i \frac{1}{4\pi g} \pi_0 \ln |z'|^2 \right) : - \frac{1}{4\pi g} \ln |z|^2,
\end{aligned}$$

The zero-mode reordering produces the c-number $-\frac{1}{4\pi g} \ln |z|^2$. Next, we'll consider holomorphic oscillator pieces

$$i \frac{1}{\sqrt{4\pi g}} \sum_{m \neq 0} \frac{1}{m} a_m z^{-m} \quad \text{and} \quad i \frac{1}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} a_n z'^{-n}.$$

We can split the sum in two parts,

$$\sum_{m \neq 0} \frac{1}{m} a_m z^{-m} = \sum_{m=-\infty}^{-1} \frac{1}{m} a_m z^{-m} + \sum_{m=1}^{\infty} \frac{1}{m} a_m z^{-m}$$

The only non-normal-ordered combination producing a c-number is the product of the $m > 0$ part from the first bracket with the $n < 0$ part from the second bracket. That term (including prefactors) is

$$\left(i \frac{1}{\sqrt{4\pi g}} \right)^2 \left(\sum_{m=1}^{\infty} \frac{1}{m} a_m z^{-m} \right) \left(\sum_{n=-\infty}^{-1} \frac{1}{n} a_n z'^{-n} \right) = -\frac{1}{4\pi g} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{-1} \frac{1}{mn} a_m a_n z^{-m} z'^{-n}.$$

Using $a_m a_n =: a_n a_m : + [a_m, a_n] =: a_n a_m : + m \delta_{m+n,0}$. The only contributing commutator in the sum over n occurs when $n = -m$. Extracting that:

$$-\frac{1}{4\pi g} \sum_{m=1}^{\infty} \frac{1}{m(-m)} m z^{-m} z'^m = -\frac{1}{4\pi g} \sum_{m=1}^{\infty} \frac{-1}{m} \left(\frac{z'}{z} \right)^m = +\frac{1}{4\pi g} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{z'}{z} \right)^m.$$

Thus the holomorphic oscillators give the c-number

$$\frac{1}{4\pi g} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{z'}{z} \right)^m$$

By identical steps for \bar{a}_m modes we obtain

$$\frac{1}{4\pi g} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\bar{z}'}{\bar{z}} \right)^m$$

Collecting the zero-mode, holomorphic and anti-holomorphic c-number contributions:

$$-\frac{1}{4\pi g} \ln |z|^2 + \frac{1}{4\pi g} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{z'}{z} \right)^m + \frac{1}{4\pi g} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\bar{z}'}{\bar{z}} \right)^m.$$

We can use $-\ln(1-x) = \sum_{m=1}^{\infty} x^m/m$ to simplify the series expansion (valid for $|z'| < |z|$):

$$\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{z'}{z} \right)^m = -\ln \left(1 - \frac{z'}{z} \right), \quad \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\bar{z}'}{\bar{z}} \right)^m = -\ln \left(1 - \frac{\bar{z}'}{\bar{z}} \right).$$

So the total c-number becomes

$$-\frac{1}{4\pi g} \ln |z|^2 - \frac{1}{4\pi g} \ln \left| 1 - \frac{z'}{z} \right|^2 = -\frac{1}{4\pi g} \ln \left| z \left(1 - \frac{z'}{z} \right) \right|^2 = -\frac{1}{4\pi g} \ln |z - z'|^2$$

Therefore

$$\varphi_{\text{pl}}(w, \bar{w}) \varphi_{\text{pl}}(w', \bar{w}') =: \varphi_{\text{pl}}(w, \bar{w}) \varphi_{\text{pl}}(w', \bar{w}') : -\frac{1}{4\pi g} \ln |z - z'|^2$$

Since we know that the normal ordered product vanishes inside correlator, the propagator could be simply read off as:

$$\langle \varphi_{\text{pl}}(w, \bar{w}) \varphi_{\text{pl}}(w', \bar{w}') \rangle = -\frac{1}{4\pi g} \ln |z - z'|^2$$

This is the same as (3.18). From now on, we will drop the subscript "pl." We know that φ is not itself a primary field, ² but its derivatives $\partial\varphi$ and $\bar{\partial}\bar{\varphi}$ are. Now we concentrate on the holomorphic field $\partial\varphi$

$$i\partial\varphi(z) = \frac{1}{4\pi g} \frac{\pi_0}{z} + \frac{1}{4\pi g} \sum_{n \neq 0} a_n z^{-n-1} = \frac{1}{4\pi g} a_0 z^{0-1} + \frac{1}{4\pi g} \sum_{n \neq 0} a_n z^{-n-1}$$

We may write the zero modes by a_0 and \bar{a}_0 :

$$a_0 \equiv \bar{a}_0 \equiv \pi_0$$

Now the commutation relations of (4.11) can be extended to include the zero mode operator without changing the form of the algebra. The mode expansion of $\partial\varphi$ is consistent with the expansion of a primary field with $h = 1$ then becomes:

$$i\partial\varphi(z) = \sum_n a_n z^{-n-1} \quad (4.17)$$

Often the normalization of $g = \frac{1}{4\pi}$ is used in this kind of work. Using this mode expansion, we can also compute the two-point function of $\partial\varphi$. For ($|z| > |w|$) :

$$\langle \varphi(z) \partial\varphi(w) \rangle = \sum_{m, n \neq 0} \frac{1}{n} \langle a_n a_m \rangle z^{-n} w^{-m-1}$$

According to the commutation relation (4.11) and the fact that a_m and a_{-m} are annihilation and creation operators respectively, it follows that ³

$$\langle \varphi(z) \partial\varphi(w) \rangle = \sum_{n>0} \sum_{m \neq 0} \frac{n \delta_{n,-m} z^{-n} w^{-m-1}}{n} = \frac{1}{w} \sum_{n>0} (w/z)^n = \frac{1}{w} \frac{w/z}{1 - w/z} = \frac{1}{z - w} \quad (4.18)$$

Its differentiation with respect to z provides the two-point function.

$$\langle \partial\varphi(z) \partial\varphi(w) \rangle = -\frac{1}{(z - w)^2} \quad (4.19)$$

The holomorphic energy-momentum tensor is given by

$$\begin{aligned}
T(z) &= -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) : \\
&= \frac{1}{2} \sum_{n,m \in \mathbb{Z}} z^{-n-m-2} : a_n a_m : = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} : a_{k-m} a_m : \right) z^{-k-2} \\
&= \left(\frac{1}{2} a_0^2 + \sum_{m>0} : a_{-m} a_m : \right) z^{-2} + \sum_{k \neq 0} z^{-k-2} \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_{k-m} a_m : \\
&= \sum_{k \in \mathbb{Z}} z^{-k-2} L_k
\end{aligned}$$

which implies

$$\begin{aligned}
L_k &= \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_{k-m} a_m : \quad (k \neq 0) \\
L_0 &= \sum_{m>0} a_{-m} a_m + \frac{1}{2} a_0^2
\end{aligned}$$

The Hamiltonian (4.13) now can be written as,

$$H = \frac{2\pi}{L} (L_0 + \bar{L}_0) \quad (4.20)$$

This confirms the role of L_0 and \bar{L}_0 as a Hamiltonian. Since we place the free boson on a cylinder of size L , energy is proportional to $\frac{2\pi}{L}$, which is called the finite-size scaling. The mode operator a_m plays a similar role to L_m with respect to L_0 , because of the commutation relation $[L_0, a_{-m}] = m a_{-m}$. Therefore its effect on the conformal dimension is the same as that of L_m .

By definition, the normal ordering prescription implies that $\langle T(z) \rangle = 0$.⁴ We can always define the normal ordering product by subtracting all the singular terms from the OPE. In fact the general expectation value for energy-momentum tensor (3.32) can be written as

$$\langle T(z) \rangle = -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\langle \partial\varphi(z+\epsilon)\partial\varphi(z) \rangle + \frac{1}{\epsilon^2} \right). \quad (\text{with } g = \frac{1}{4\pi})$$

We will see that this relation is useful when we consider the anti-periodic boundary condition later. By plugging in the two-point function from (4.26), we see that $\langle T(z) \rangle = 0$, which implies that $(L_0)_{\text{pl}}$ vanishes on the vacuum. We now map the theory back to a cylinder $z \rightarrow w = \frac{L}{2\pi} \ln z$, by using (3.37):

$$T_{\text{cyl}}(w) = \left(\frac{dw}{dz} \right)^{-2} \left[T_{\text{pl}}(z) - \frac{1}{12} \{w, z\} \right] = \left(\frac{L}{2\pi z} \right)^{-2} \left[T_{\text{pl}}(z) - \frac{1}{12} \frac{1}{2z^2} \right] = \left(\frac{2\pi}{L} \right)^2 \left[T_{\text{pl}}(z) z^2 - \frac{1}{24} \right] \quad (4.21)$$

where we use the central charge $c = 1$ for the free boson. Taking the expectation value on both sides, we have

$$\langle T_{\text{cyl}} \rangle = -\frac{1}{24} \left(\frac{2\pi}{L} \right)^2 \quad (4.22)$$

On cylinder, we do Fourier expansion according to (4.1)

$$T(w) = \sum_{n \in \mathbb{Z}} e^{nw} L_n$$

The mode expansion of the energy-momentum tensor tell us that this nonzero expectation value implies a constant term in the expression for L_0 in terms of modes.

⁰² Refer to equation (5.25) in Sénéchal's CFT with $h = 0$ as shown in Polchinski's String Theory, Vol. 1 (eq. 2.4.17), the corresponding two-point function should be constant rather than logarithmic.

³ $\langle a_n a_m \rangle = \langle 0 | a_m a_n | 0 \rangle + n \delta_{n, -m} \langle 0 | 0 \rangle = n \delta_{n, -m}$ for $n > 0$

$$L_{0,\text{cyl}} = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n - \frac{1}{24} \quad (4.23)$$

The Hamiltonian is now written as

$$H = \frac{2\pi}{L} (L_{0,\text{cyl}} + \bar{L}_{0,\text{cyl}}) \quad (4.24)$$

Actually, in the general case, Hamiltonian of a theory defined on a cylinder with central charge c can be written as

$$H = \frac{2\pi}{L} \left(L_{0,\text{pl}} + \bar{L}_{0,\text{pl}} - \frac{c}{12} \right) \quad (4.25)$$

Thus, we can infer that the central charge c shows up as the vacuum energy of a theory on a cylinder, and this is one instance of finite-size effects.

In the antiperiodic case, $\varphi(x+L, t) = -\varphi(x, t)$, this boundary condition implies that the field φ is doublevalued on the cylinder. Once the cylinder is mapped onto the plane, this amounts to defining the theory on a pair of Riemann sheets. The summation index starts at $n = \frac{1}{2}$ and takes half-integer values thereafter. The vacuum expectation values is taken in one of the two ground states and

$$\begin{aligned} \langle \varphi(z) \partial \varphi(w) \rangle &= \sum_{m, n \neq 0} \frac{1}{n} \langle a_n a_m \rangle z^{-\frac{2n+1}{2}} w^{-\frac{2m+1}{2}-1} = \sum_{n>0} \sum_{m \neq 0} \frac{n \delta_{n, -m} z^{-\frac{2n+1}{2}} w^{-\frac{2m+1}{2}-1}}{n} \\ &= \frac{1}{w} \sum_{n \geq 0} (w/z)^{\frac{2n+1}{2}} = \frac{1}{w} \frac{\sqrt{w/z}}{1 - w/z} = \sqrt{\frac{z}{w}} \frac{1}{z - w} \end{aligned}$$

Its differentiation with respect to z provides the two-point function.

$$\langle \partial \varphi(z) \partial \varphi(w) \rangle = -\frac{1}{2} \frac{\sqrt{z/w} + \sqrt{w/z}}{(z - w)^2} \quad (4.26)$$

This expression has branch cut at $z = 0, \infty$ and $w = 0, \infty$; the antiperiodic boundary condition on ϕ as z circles around the origin is incorporated in the square roots. The periodic and antiperiodic two point function coincide in the limit $z \rightarrow w$, meaning the short distance physics is independent of the choice of boundary conditions. The vacuum energy density obtained from the normal ordering prescription is then,

$$\langle T(z)_{\text{pl}} \rangle = \frac{1}{16z^2}$$

Since L_0 is the coefficient of $1/z^2$ in the mode expansion of the energy-momentum tensor, this nonzero expectation value implies a constant term in the expression for L_0 .

$$L_{0,\text{pl}} = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n + \frac{1}{16}$$

5.5.1 Vertex Operators and discussions on path integral formalism of CFT :

Since canonical scaling dimension (h, \bar{h}) of the bosonic field φ vanishes, it is possible to construct an infinite variety of local fields related to φ without introducing a scale, namely the so called vertex operators. These operators become relevant in string theory because their action on absolute vacuum creates the conformal vacuum state of the theory. They are indeed primary fields and defined as:

$$\mathcal{V}_\alpha =: e^{i\alpha\varphi(z, \bar{z})} :$$

This document synthesizes a complete discussion regarding the derivation and subtleties of the

⁰⁴ However, as we will see in the next section, this is not the case for the fermionic theory with anti-periodic boundary condition.

state-operator correspondence in Conformal Field Theory (CFT) and String Theory. It addresses two primary areas of inquiry: the general path integral derivation of the correspondence (based on David Tong's notes), and the specific mathematical puzzle regarding the mapping of the unit operator to the vacuum state in Polchinski's *String Theory*.

The discussion begins with a paraphrase of the path integral derivation found in Chapter 4 of David Tong's string theory notes. We start with the standard formula for the time-evolution of a wave-function in the path integral formalism:

$$\psi[\phi_f(x), t_f] = \int_{\phi(x, t_i)=\phi_i(x)} [d\phi_i(x)]_{\phi(x, t_f)=\phi_f(x)} \int_{\phi(x, t_i)=\phi_i(x)} [d\phi(x, t)] \exp \left[i\hbar \int_{t_i}^{t_f} dt' L \right] \psi[\phi_i(x), t_i]. \quad (5.1)$$

Consider a radially quantized CFT where the time direction is radial. We take $t_i = 0$. Since this corresponds to the origin of the radial plane, the *a priori* function $\phi_i(x)$ reduces to a number ϕ_i . The path integral reduces to:

$$\psi[\phi_f(x), t_f] = \int d\phi_i \int_{\phi(x, t_f)=\phi_f(x)}^{\phi(0)=\phi_i} [d\phi(x, t)] \exp \left[i\hbar \int_0^{t_f} dt' L \right] \psi(\phi_i, 0). \quad (5.2)$$

Tong states that the only effect of the initial state is to change the weighting of the path integral at the point $z = 0$, which is exactly the definition of a local operator inserted at that point. The question arises: precisely why does this weighting constitute a local operator, and how can we explicitly construct the operator for a given state?

To resolve this, we must establish a dictionary that translates the path integral formalism to the operator formalism. Matrix elements of time-ordered local operator strings translate to path integrals with field insertions. For example:

$$\langle \psi_2 | T O_1[\phi(x_1), x_1] O_2[\phi(x_2), x_2] | \psi_1 \rangle \rightarrow \int_{BC} \mathcal{D}\phi \dots O_1[\phi(x_1), x_1] O_2[\phi(x_2), x_2] e^{iS}. \quad (5.3)$$

Here, BC stands for suitable boundary conditions which are related to the states $|\psi_i\rangle$. The functions O_i must depend explicitly on the fields.

In the specific case of the wave-function evolution mentioned above, the integral over $d\phi_i$ can be absorbed into the measure. This implies that there is no boundary condition at $x = 0$ anymore because we are summing over all possible values there. The weight $\psi(\phi_i, 0)$ can then be viewed as a function of the field evaluated at $x = 0$, which is precisely a local operator evaluated at $x = 0$.

We can define the vacuum state $|0\rangle$ by setting the functional $\Psi[\phi(0), 0] = 1$. The wave-functional becomes:

$$|0\rangle \leftrightarrow \psi_0[\phi(x), t] = \int^{\phi(x, t)=\phi(x)} \mathcal{D}\phi e^S \times 1. \quad (5.4)$$

For an arbitrary state $|\psi\rangle$, we have:

$$|\psi\rangle \leftrightarrow \psi[\phi(x), t] = \int^{\phi(x, t)=\phi(x)} \mathcal{D}\phi e^S \Psi[\phi(0), 0] \leftrightarrow \Psi[\phi(0), 0] |0\rangle. \quad (5.5)$$

Thus, every state $|\psi\rangle$ can be created out of a local operator $\Psi[\phi(0), 0]$ acting on the $|0\rangle$ state.

Inserting a local operator effectively means multiplying the integrand of the path integral by an operator at a fixed position. In radial quantization, inserting the operator at $z = 0$ (the initial point in time) makes it play the role of a weight factor. In the general path integral formula where t_i is arbitrary, the operator is delocalized over the boundary condition. By restricting $t_i = 0$, we localize the dependence to a single point.

In field theory, states are functions on the space of boundary conditions on a spatial slice. For a theory on the complex plane with radial time, spatial slices are circles. Thus, quantum states are functions on the space of boundary conditions on a circle of fixed radius.

Let \mathcal{H} be the space of states. A path integral on an annulus A defines a map $T_A : \mathcal{H} \rightarrow \mathcal{H}$. This corresponds to evolving a state from the inner boundary to the outer boundary. However, if we consider a disc D instead of an annulus, we only have one boundary. Inserting a local functional $O(\phi(0), \partial_z O(0))$ at the origin and performing the path integral over the disc yields a quantum state in \mathcal{H} (since fixing the boundary condition on the rim and integrating gives a number). Thus, the path integral on the disc with a local functional at the origin defines a map:

$$T_D : \{\text{space of local functionals at the origin}\} \rightarrow \mathcal{H}. \quad (5.6)$$

This map is the essence of the state-operator correspondence.

We can be even more explicit. Consider the vacuum state functional $\Psi_0[\phi_f]$. If we act on this with a local operator $O(0)$ (a local function of the fields $O[\phi(0)]$), the resulting state is:

$$\Psi_O[\phi_f] = \int d\phi_i \int_{\phi(x,t_f)=\phi_f}^{\phi(0)=\phi_i} [d\phi(x,t)] \exp \left[i\hbar \int_0^{t_f} dt' L \right] O[\phi_i]. \quad (5.7)$$

Here we attach the operator O to the wavefunction via the identification $O[\phi_i] = \psi(\phi_i, 0)$.

In Section 2.8 of Polchinski's *String Theory*, a specific issue arises regarding the unit operator $\mathbb{1}$. The author argues that $\mathbb{1}$ corresponds to the vacuum state $|0\rangle$. He further claims that for this operator, ∂X^μ is holomorphic inside the contour, which implies that operators α_m^μ with $m \geq 0$ must vanish.

This causes confusion because the standard mode expansion is:

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \alpha_m^\mu z^{-m-1}. \quad (5.8)$$

Usually, this expansion contains singular parts ($1/z^m$) at the origin. The question is: what mathematical condition requires ∂X^μ to have no poles in the specific case of the identity operator?

The resolution lies in the topology of the worldsheet. The field ∂X^μ is defined on a given worldsheet surface. Eq (2.1.12) in Polchinski states that ∂X^μ is holomorphic on this surface.

The expansion of a holomorphic function depends on the underlying topology. If the surface is the Riemann sphere, the only holomorphic functions are constants. If the surface is a cylinder (as in free string propagation), we relax the condition at the origin, allowing the expansion $\sum_{m \in \mathbb{Z}} a_m z^m$. Even though this has "poles" at $z = 0$, it is holomorphic on the cylinder (which is the plane without the origin).

The state-operator correspondence maps the cylinder to the complex plane. Inserting the identity operator $\mathbb{1}$ at the origin is equivalent to "filling in the hole" of the cylinder, effectively converting the topology from a cylinder to a full plane. On the full complex plane (including the origin), a holomorphic function cannot have poles at $z = 0$. Therefore, the coefficients associated with the singularity (negative m in the expansion $\sum a_m z^m$, or $m \geq 0$ in the α mode indexing) must vanish. This identifies the state as the ground state, annihilated by α_m^μ for $m \geq 0$.

Before the state-operator correspondence is applied, the map from the cylinder to the plane ($z = e^w$) introduces a coordinate singularity at the origin. Local operators are singular at the origin precisely because of this map. The correspondence observes that if we substitute the singularity at the origin with a local operator, we map states from $t = -\infty$ on the cylinder to the origin of the plane. The "singular behavior" is determined by the operator we insert. The identity operator $\mathbb{1}$ is unique because it is characterized by having *no singularity at all*. Other operators, such as $:\partial X^\mu \bar{\partial} X^\nu e^{ik \cdot X}:$, reproduces specific singularities.

Another perspective relies on operator equations. We know that $\bar{\partial} \partial X^\mu =$ contact terms. Contact terms arise only if there are operators inside the contour. If the operator inside the contour is the unit operator $\mathbb{1}$, there are no non-trivial operators to generate contact terms. Thus, $\bar{\partial} \partial X^\mu = 0$ everywhere inside the contour, implying ∂X^μ is strictly holomorphic at the origin.

A rigorous proof that the singular modes vanish for the identity operator can be constructed using the path integral. We wish to show that the expectation value of the contour integral of the singular modes is zero. Let Ω be the object that must vanish for the identity to be the vacuum:

$$\Omega = \int \mathcal{D}X e^{-S} \oint \frac{dz}{2\pi i} z^m \partial X^\mu(z, \bar{z}) \quad \text{for } m \geq 0. \quad (5.9)$$

We note that inside the path integral, we integrate over all functions $X^\mu(z, \bar{z})$, not just holomorphic solutions to the equations of motion.

Using the divergence theorem in the complex plane, we rewrite the contour integral over a circle ∂C of radius R :

$$\oint_{\partial C} \frac{dz}{2\pi i} z^m \partial X = \frac{1}{2\pi} \int_C d^2 z z^m \partial \bar{\partial} X = \frac{1}{2\pi} \int d^2 z z^m \partial \bar{\partial} X \Theta(R^2 - |z|^2). \quad (5.10)$$

Substituting this into Ω , we can express the insertion as a derivative with respect to a source

parameter a :

$$\Omega = \left. \frac{d}{da} \right|_{a=0} \int \mathcal{D}X \exp \left\{ \frac{1}{2\pi} \int d^2z \left(\frac{1}{\alpha'} X \partial \bar{\partial} X + a z^m \partial \bar{\partial} X \Theta(R^2 - |z|^2) \right) \right\}. \quad (5.11)$$

We perform a change of variables in the path integral:

$$X \rightarrow X' = X + \frac{a\alpha'}{2} z^m \Theta(R^2 - |z|^2). \quad (5.12)$$

The path integral measure $\mathcal{D}X$ is invariant under this shift ($\mathcal{D}X = \mathcal{D}X'$). Substituting X' into the action and using integration by parts, the linear terms in a cancel. The integral becomes:

$$\int d^2z z \frac{1}{\alpha'} X' \partial \bar{\partial} X' + O(a^2). \quad (5.13)$$

Therefore, Ω becomes:

$$\Omega = \left. \frac{d}{da} \right|_{a=0} \left[\int \mathcal{D}X' \exp \left(\frac{1}{2\pi\alpha'} \int d^2z X' \partial \bar{\partial} X' \right) + O(a^2) \right] = 0. \quad (5.14)$$

This proves that the path integral insertion of $\oint z^m \partial X$ vanishes for $m \geq 0$ when inserted on the vacuum.

5.6 Algebraic Argument

Finally, we can view this through the commutator definition provided in Polchinski Eq (2.6.15). For an operator A at the origin and charge Q :

$$|A\rangle \cong [Q, A]. \quad (5.15)$$

If we take $Q = \alpha_m^\mu$ (for $m \geq 0$) and $A = \mathbb{1}$, we have:

$$0 = [\alpha_m^\mu, 1] \propto \text{Res}_{z \rightarrow 0} z^m \partial X^\mu 1. \quad (5.16)$$

Since the commutator of any operator with the identity is zero, the residue must vanish. This implies:

$$\alpha_m^\mu 1 = 0 \quad \text{for } m \geq 0. \quad (5.17)$$

This confirms that the state corresponding to the identity operator is annihilated by the positive modes, which is the definition of the ground state $|0\rangle$. Conversely, higher states are defined by residues that do not vanish; for example, $\partial^m X^\mu(0)$ is defined via $\alpha_{-m}^\mu |1\rangle$.

Unitary CFTs

In the study of Conformal Field Theories, we typically require unitarity, which implies the existence of a positive definite inner product. However, a common confusion arises from the conformal algebra itself, specifically regarding the Hermiticity of the dilatation operator D .

Consider the commutator between the dilatation operator D and the momentum generator P_μ :

$$[D, P_\mu] = iP_\mu \quad (5.18)$$

This commutation relation suggests that P_μ acts as a ‘‘ladder operator’’ for D , shifting its eigenvalue by $+i$. If we assume the theory is unitary, D must be a Hermitian operator ($D^\dagger = D$), and thus its eigenvalues must be real.

However, if we have an eigenstate $|\Delta\rangle$ with a real eigenvalue Δ , the logic follows:

$$D(P_\mu |\Delta\rangle) = P_\mu D|\Delta\rangle + [D, P_\mu]|\Delta\rangle = P_\mu(\Delta|\Delta\rangle) + iP_\mu|\Delta\rangle = (\Delta + i)(P_\mu|\Delta\rangle) \quad (5.19)$$

This implies that $P_\mu|\Delta\rangle$ is an eigenstate with eigenvalue $\Delta + i$. Since Δ is real, $\Delta + i$ is complex. This seems to contradict the requirement that Hermitian operators must have real eigenvalues. Does this mean unitary representations of the conformal algebra cannot exist?

The paradox is resolved by looking closer at the properties of the Hilbert space and the definition of the operators.

1. Algebraic Consistency Check

First, we can check if the algebra itself forbids D from being Hermitian. Let us assume $D^\dagger = D$ and $P_\mu^\dagger = P_\mu$ (standard Hermitian momentum). Taking the Hermitian conjugate of the

commutator:

$$\begin{aligned} [D, P_\mu]^\dagger &= (iP_\mu)^\dagger \\ [P_\mu^\dagger, D^\dagger] &= -iP_\mu^\dagger \\ [P_\mu, D] &= -iP_\mu \end{aligned} \tag{5.20}$$

This result is perfectly consistent with the original relation $[D, P_\mu] = iP_\mu$. Thus, there is no algebraic contradiction in assuming D is Hermitian.

2. Non-Normalizable States

The argument that P_μ creates states with complex eigenvalues relies on the assumption that there exist *normalizable* eigenstates of D that are also in the domain of P_μ .

It turns out that **no such normalizable eigenstates exist**. For instance, one might try to construct an eigenstate using the operator-state correspondence:

$$|\Psi\rangle = \mathcal{O}(0)|0\rangle \tag{5.21}$$

While this is an eigenstate of D , it is strictly speaking not a normalizable state in the Hilbert space (its norm $\langle 0|\mathcal{O}(0)\mathcal{O}(0)|0\rangle$ diverges). Because these states do not live in the valid Hilbert space, the logic that leads to imaginary eigenvalues breaks down.

3. Euclidean vs. Lorentzian Signature

A source of confusion often stems from moving between signatures.

- In **Euclidean** radial quantization, we often define conjugation such that $P_\mu^\dagger = K_\mu$. In this specific formalism, D is treated as anti-Hermitian (eigenvalues are pure imaginary).
- In **Lorentzian** signature (the physical spacetime), the conformal group is $SO(2, d)$. Here, unitary representations are well-defined and classified, and D can indeed be treated as Hermitian without contradiction.

5.7 Intersection of Critical Lines with Different Central Charges

This note summarizes the investigation into the multicritical point where two different Conformal Field Theories (CFTs) intersect. Specifically, we look at the XY spin chain in a transverse field.

The Hamiltonian of the system is given by the anisotropic XY model:

$$H = - \sum_i \left(\frac{1+\gamma}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-\gamma}{2} \sigma_i^y \sigma_{i+1}^y + h \sigma_i^z \right) \tag{5.22}$$

where $\gamma, h \geq 0$. This model exhibits two distinct critical lines corresponding to different universality classes:

- **Ising Line** ($h = 1, \gamma > 0$): Central charge $c = 1/2$.
- **XX Line** ($\gamma = 0, |h| < 1$): Central charge $c = 1$. This is the free boson or ‘‘Luttinger liquid’’ phase.

The intersection occurs at $\gamma = 0$ and $h = 1$. The question arises: does the central charge jump discontinuously, or is there a flow to a new fixed point?

For a standard CFT, the entanglement entropy $S(l)$ for a subsystem of length l in a system of size L scales as:

$$S(l) = \frac{c}{3} \ln \left(\frac{L}{\pi} \sin \frac{\pi l}{L} \right) + C \tag{5.23}$$

Initially, numerical fitting suggested $c \approx 1$ at the multicritical point. However, this was due to a coding error (missing a factor of $1/2$ in the Hamiltonian). The corrected calculation shows that the entanglement entropy vanishes:

$$S(l) \approx 0 \tag{5.24}$$

This suggests the ground state is a simple product state, which we derive below.

To understand why $S = 0$, we rewrite the Hamiltonian at the multicritical point using Pauli ladder operators.

We use the standard relations for raising and lowering operators $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$:

$$\frac{1}{2}(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) = \sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ \quad (5.25)$$

$$\sigma_n^z = \sigma_n^+ \sigma_n^- - \sigma_n^- \sigma_n^+ = 1 - 2\sigma_n^- \sigma_n^+ \quad (5.26)$$

Substituting these into the Hamiltonian at $\gamma = 0, h = 1$ (and ignoring irrelevant constants):

$$\begin{aligned} H &= - \sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \sigma_n^z) \\ &= - \sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + (1 - 2\sigma_n^- \sigma_n^+)) \end{aligned} \quad (5.27)$$

Rearranging the terms to complete the square, we obtain a ‘‘frustration-free’’ form:

$$H = \sum_n (\sigma_n^- - \sigma_{n+1}^-)(\sigma_n^+ - \sigma_{n+1}^+) \quad (5.28)$$

We define a new bond operator Γ_n :

$$\Gamma_n := \sigma_n^+ - \sigma_{n+1}^+ \quad (5.29)$$

The Hamiltonian then takes the positive-definite form:

$$H = \sum_n \Gamma_n^\dagger \Gamma_n \quad (5.30)$$

Since H is a sum of positive terms ($A^\dagger A \geq 0$), a state with zero energy must be annihilated by every Γ_n .

The condition $\Gamma_n |\psi\rangle = 0$ implies $(\sigma_n^+ - \sigma_{n+1}^+) |\psi\rangle = 0$. The fully polarized ferromagnetic state satisfies this:

$$|\psi\rangle = |\uparrow\uparrow \dots \uparrow\rangle = \bigotimes_n |\uparrow\rangle_n \quad (5.31)$$

Since $\sigma^+ |\uparrow\rangle = 0$, this state is annihilated by all terms. Being a product state, its entanglement entropy is exactly zero.

Why does the central charge formula fail? The central charge is well-defined only for relativistic theories with linear dispersion $E \sim k$. At this multicritical point, the dispersion becomes quadratic:

$$E(k) \sim k^2 \quad (5.32)$$

This implies a dynamical critical exponent $z = 2$.

- **Lorentz Invariance Broken:** $z \neq 1$ breaks Lorentz symmetry.
- **No Conformal Invariance:** Scale invariance exists, but without Lorentz invariance, it does not imply conformal invariance in the standard sense.
- **Universality Class:** This point belongs to the **U(1) Dilute Bose Gas** universality class, distinct from the $c = 1$ and $c = 1/2$ lines.

Acknowledgements

I am grateful to Prof. Subhabrata Majumdar (TIFR, Mumbai) Prof. Shiraz Minwalla(TIFR, Mumbai), Dr. Abhijit Gadde(TIFR, Mumbai), for insightful discussions and guidance and support. I also thank beautiful notes of Hassaan Saleem(SUNY,Albany) and Chandra Prakash(IIT,Guahati) for fruitful discussions, particularly on technical derivations.

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