Ramanujan transformations and Robin defects

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Abstract

We studied the Robin defect associated with the inequality $\sigma(n) < e^{\gamma} n \log\log n$, express its Laplace transform through Ramanujan's transformation for the divisor Lambert series and isolate the precise difference between smoothed positivity and coefficientwise positivity. The main results give us an exact Ramanujan-transformed identity for the Robin defect, an equivalent coefficientwise formulation of the Riemann Hypothesis showing why transform-level positivity cannot by itself prove the hypothesis and an extremal reduction to colossally abundant and highest abundant numbers. Numerical data for early extremal integers illustrate how the normalized defect behaves past the exceptional value 5040. Ramanujan's identities provide powerful global control, but the Riemann Hypothesis requires pointwise positivity at the extremal divisor-rich integers.

Keywords: Ramanujan identities; Riemann Hypothesis; Robin inequality

1. Introduction

Riemann Hypothesis (RH) explains the distribution of prime numbers, where the zeros of the analytically continued zeta function entered prime number theory in a decisive way [1]. A different but closely related path comes from extremal divisor sums. Grönwall proved in 1913 that the maximal order of the sum-of-divisors function satisfies

$$\underset{n\to\infty}{lim\,sup} \frac{\sigma(n)}{n \log\log n} = e^{\gamma} \qquad (1)$$

where γ denotes Euler's constant [2]. Ramanujan then studied highly composite and related extremal integers in 1915, including results whose later publication made clear how deeply his work anticipated the theory of superabundant and colossally abundant numbers [3]. Alaoglu and Erdős systematized the study of highly abundant, superabundant and colossally abundant numbers in 1944 [4]. Ingham later used a Ramanujan divisor identity to give a proof of the non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$, thereby connecting Ramanujan's arithmetic identities to the analytic mechanism behind the prime number theorem [5]. Robin's 1984 theorem then gave one of the most striking elementary reformulations of the RH which is true precisely when

$$\sigma(n) < e^{\gamma} n \log\log n (n > 5040) \qquad (2)$$

This criterion is especially powerful because any failure of (2) must occur among highly structured divisor-rich integers [6]. Lagarias later produced an elementary inequality equivalent to RH involving harmonic numbers and $\sigma(n)$, strengthening the role of divisor sums in the subject [7]. Also Roger's survey on Ramanujan and the zeta function emphasizes that Ramanujan's contributions touch both special zeta values and prime distribution, including the use of Ramanujan identities to prove non-vanishing results for zeta, Dirichlet and Artin L-functions [8]. Musin's work on Ramanujan's theorem and highest abundant numbers develops a modern extremal viewpoint on Robin-type inequalities and shows that the geometry of such numbers changes sharply according to the truth value of RH [9]. Recent Robin–Lagarias analogues further confirm that divisor-sum inequalities remain an active and precise way to probe RH-equivalent statements [10].

The purpose of this paper is to give a clean Ramanujan–Robin framework. We prove exact identities and reductions showing where the real difficulty lies. The central object is the Robin defect

$$\Delta(n) := e^{\gamma} n \log\log n - \sigma(n) \qquad (3)$$

defined for $n \geq 3$. Robin's theorem says that RH is equivalent to $\Delta(n) > 0$ for every $n > 5040$. Ramanujan's divisor Lambert series gives exact analytic control of the smoothed sequence $\sigma(n)$. The main question is whether such smoothed control can force the pointwise positivity in Robin's criterion. Ramanujan transformations give exact global identities, but the missing step is coefficientwise positivity at the extremal integers.

2. Notation and preliminary facts

For a positive integer n

$$\sigma_{\alpha}(n) := \textstyle\sum_{d|n} d^{\alpha} \qquad (4)$$

where $\alpha \in \mathbb{C}$. The usual sum-of-divisors function is $\sigma(n) = \sigma_1(n)$. The Riemann zeta function is denoted by $\zeta(s)$. All logarithms are natural logarithms.

We shall use the following divisor Lambert series:

$$S(x) := \textstyle\sum_{n=1}^{\infty} \sigma(n) e^{-nx}, x > 0 \qquad (5)$$

The series converges absolutely for every positive x, since $\sigma(n) = O_{\varepsilon}(n^{1+\varepsilon})$ for every fixed $\varepsilon > 0$.

Lemma 1. For every $x > 0$,

$$S(x) = \textstyle\sum_{m=1}^{\infty} \frac{m}{e^{mx}-1} \qquad (6)$$

Proof. Absolute convergence allows a rearrangement of the double sum:

$$\sum_{n=1}^{\infty} \sigma(n) e^{-nx} = \sum_{n=1}^{\infty} \sum_{d|n} d\, e^{-nx}$$

Writing $n = dm$ gives

$$\sum_{d=1}^{\infty} d \sum_{m=1}^{\infty} e^{-dmx} = \sum_{d=1}^{\infty} \frac{d\, e^{-dx}}{1 - e^{-dx}},$$

which is exactly (6).

The following Mellin identity records the zeta-theoretic content of $S(x)$.

Lemma 2. For $\Re(s) > 2$,

$$\textstyle\int_0^{\infty} S(x) x^{s-1}\, dx = \Gamma(s)\zeta(s)\zeta(s-1) \qquad (7)$$

Proof. Since the defining series for $S(x)$ is absolutely convergent in the required range, termwise integration gives

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Gamma(s)}{n^s}$$

Thus the integral equals

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$$

The Dirichlet series for $\sigma(n)$ is $\zeta(s)\zeta(s-1)$ for $\Re(s) > 2$.

3. Ramanujanś divisor identity

Ramanujan identity is one of the cleanest places where divisor sums and products of zeta functions meet. It is best stated in terms of σ_α.

Theorem 3. Let $a, b \in \mathbb{C}$. In every half-plane where the following Dirichlet series and Euler products converge absolutely,

$$\textstyle\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} \qquad (8)$$

Proof of Theorem 3. The function $n \mapsto \sigma_a(n)\sigma_b(n)$ is multiplicative. Therefore the Dirichlet series on the left has an Euler product. For a prime p, put $X = p^{-s}$, $A = p^a$, and $B = p^b$. The local factor is

$$\sum_{k=0}^{\infty} (1 + A + \cdots + A^k)(1 + B + \cdots + B^k)X^k.$$

Using the finite geometric identity for each bracket and summing the resulting four geometric series gives the local expression

$$\frac{1 - ABX^2}{(1-X)(1-AX)(1-BX)(1-ABX)}$$

Multiplication over all primes gives exactly (8).

A classical consequence is Ingham's non-vanishing mechanism. It explains why Ramanujan's identity belongs naturally beside the prime number theorem.

Proposition 4. The identity (8) implies that $\zeta(1+it) \neq 0$ for every real t.

Proof. The case $t = 0$ is immediate because $\zeta(s)$ has a pole at $s = 1$. Assume $t \neq 0$ and suppose, for contradiction, that $\zeta(1+it) = 0$. Taking $a = it$ and $b = -it$ in (8) gives

$$\textstyle\sum_{n=1}^{\infty} \frac{|\sigma_{it}(n)|^2}{n^s} = \frac{\zeta(s)^2\zeta(s-it)\zeta(s+it)}{\zeta(2s)} \qquad (9)$$

All coefficients on the left are non-negative and the first coefficient is 1. By the assumed zero and its conjugate, the possible pole at $s = 1$ is removed. The possible poles at s=1+it and s=1-it are also removed because the factor $\zeta(s)^2$ goes to zero at those points. So, the right side is holomorphic past the real line segment $\Re(s)$>1/2, except that the denominator has a pole at 2s=1. As real s approaches 1/2 from the right, the right side approaches 0. Landau's theorem for Dirichlet series with non-negative coefficients then requires the abscissa of convergence of the left side to be at most 1/2. However, the value of the left side cannot approach 0 as real s goes to $1/2^+$ from the right since every partial sum is at least as large as its first term. This contradiction proves the claim.

Remark 1. Proposition 4 is included to locate Ramanujan's identity correctly. It gives a proof of the non-vanishing needed for the prime number theorem. The RH problem addressed by Robin's criterion is coefficientwise and strictly sharper.

3.1 Ramanujanś transformation

The next result is the modular transformation that supplies the analytic engine of this paper.

Theorem 5. For every $x > 0$,

$$S(x) = \tfrac{\pi^2}{6x^2} - \tfrac{1}{2x} + \tfrac{1}{24} - \tfrac{4\pi^2}{x^2} S\left(\tfrac{4\pi^2}{x}\right) \qquad (10)$$

Proof of Theorem 5. We let

$$E_2(z) := 1 - 24\textstyle\sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}, \Im z > 0 \qquad (11)$$

The quasimodular transformation law for the Eisenstein series E_2 is

$$E_2(-1/z) = z^2 E_2(z) + \tfrac{6z}{\pi i} \qquad (12)$$

With setting $z = ix/(2\pi)$. Then $e^{2\pi i z} = e^{-x}$ and

$$e^{2\pi i(-1/z)} = e^{-4\pi^2/x}$$

Substituting these into (11) and (12) gives

$$1 - 24S\left(\frac{4\pi^2}{x}\right) = -\frac{x^2}{4\pi^2}\{1 - 24S(x)\} + \frac{3x}{\pi^2}$$

Solving for $S(x)$ yields (10).

Lemma 6. As $x \to 0^+$,

$$S(x) = \tfrac{\pi^2}{6x^2} - \tfrac{1}{2x} + \tfrac{1}{24} + O\left(\tfrac{e^{-4\pi^2/x}}{x^2}\right) \qquad (13)$$

Proof. By Putting $y = 4\pi^2/x$. From (6) we have,

$$S(y) \le \frac{1}{1 - e^{-y}} \sum_{m=1}^{\infty} m e^{-my} = \frac{e^{-y}}{(1 - e^{-y})^3}$$

Substituting this bound into (10) gives (13).

Remark 2. Formula (10) is exact; formula (13) is its small-x consequence. This distinction matters because exact transformed identities preserve arithmetic information, while asymptotic smoothing can erase coefficientwise sign changes.

4. Robin defect

For $n \ge 3$, define the Robin defect by (3). Robin's theorem states that RH is equivalent to strict positivity of this defect for every $n > 5040$.

Theorem 7. The RH is true if and only if

$$\Delta(n) > 0 (n \ge 5041). \qquad (14)$$

Proof of Theorem 7. This is Robin's criterion, written in the notation of (3).

We now form the Laplace transform of the Robin defect over the Robin range:

$$D(x) := \sum_{n=5041}^{\infty} \Delta(n)e^{-nx}, A(x) := \sum_{n=5041}^{\infty} n \log\log n \, e^{-nx}, P(x) := \sum_{n=1}^{5040} \sigma(n)e^{-nx}. \qquad (15)$$

Theorem 8. For every $x > 0$,

$$D(x) = e^{\gamma} A(x) - \tfrac{\pi^2}{6x^2} + \tfrac{1}{2x} - \tfrac{1}{24} + \tfrac{4\pi^2}{x^2} S\left(\tfrac{4\pi^2}{x}\right) + P(x). \qquad (16)$$

Proof of Theorem 8. By the definition of $\Delta(n)$,

$$D(x) = e^{\gamma} A(x) - \sum_{n=5041}^{\infty} \sigma(n)e^{-nx}$$

The last sum is $S(x) - P(x)$. Substituting the transformation (10) for $S(x)$ gives (16).

Equation (16) is the central Ramanujan–Robin identity of this paper. It separates the three essential components: the elementary Robin weight $n \log\log n$, the modular main terms from $S(x)$ and the exponentially transformed remainder.

4.1 Coefficientwise positivity

The exact transform (16) is useful only if its information can be pushed back to individual coefficients. The following result makes the issue precise.

Proposition 9. We let

$$\Phi(q) := \sum_{n=5041}^{\infty} \Delta(n)q^n, \mid q \mid < 1 \qquad (17)$$

Then RH is equivalent to every coefficient of $\Phi(q)$ being strictly positive.

Proof. The coefficient of q^n in (17) is $\Delta(n)$ for $n \geq 5041$. The conclusion is exactly Theorem 7.

The positivity of the transform $D(x) = \Phi(e^{-x})$ is weaker than the coefficientwise positivity in Proposition 9.

Lemma 10. As $x \to 0^+$,

$$A(x) \sim \tfrac{\log\log(1/x)}{x^2} \qquad (18)$$

Proof. Let $L(u) = \log\log u$. Since L is slowly varying, $L(t/x)/L(1/x) \to 1$ for each fixed $t > 0$ as $x \to 0^+$. Write

$$x^2 A(x) = \sum_{n=5041}^{\infty} (nx)e^{-nx} L(n)x$$

This is a Riemann sum for $\int_0^{\infty} te^{-t} L(t/x)\,dt$, with the missing finite initial interval contributing $o(L(1/x))$. Uniform convergence on compact subintervals of $(0, \infty)$, together with the exponential decay of te^{-t}, gives

$$x^2 A(x) \sim L(1/x) \int_0^{\infty} te^{-t}\,dt$$

The integral equals 1.

Corollary 11. There exists $x_0 > 0$ such that $D(x) > 0$ for every $0 < x < x_0$, independently of RH.

Proof. Combining (13), (15), and (18) gives

$$x^2 D(x) = e^\gamma \log\log(1/x) - \frac{\pi^2}{6} + o(\log\log(1/x)) \qquad (19)$$

The right side tends to $+\infty$. Hence $D(x)$ is positive for all sufficiently small positive x.

Remark 3. Corollary 11 is the smoothing barrier. It proves that eventual positivity of the Ramanujan-smoothed Robin defect is automatic and therefore cannot settle RH. Robin's criterion requires the stronger assertion in Proposition 9.

5. Extremal reduction

The obstruction to RH through Robin's criterion is not spread evenly over the integers. It is concentrated among integers where $\sigma(n)/n$ is exceptionally large.

A positive integer N is colossally abundant if there exists $\varepsilon > 0$ such that

$$\frac{\sigma(k)}{k^{1+\varepsilon}} \leq \frac{\sigma(N)}{N^{1+\varepsilon}} (k \geq 1) \qquad (20)$$

We define

$$G(n) := \frac{\sigma(n)}{n \log\log n} \qquad (21)$$

The inequality (2) is simply $G(n) < e^\gamma$ for $n > 5040$.

Theorem 12. RH is true if and only if:

$$G(N) < e^\gamma$$

for every colossally abundant integer $N > 5040$.

Proof of Theorem 12. If RH is true, the conclusion follows directly from Theorem 7. Conversely, Robin proved that if (2) fails, then a counterexample can be chosen among colossally abundant numbers. Therefore positivity on the colossally abundant subsequence is sufficient.

Musin's highest abundant framework gives a second geometric way to organize the same obstruction. For real s, we set

$$R_s(n) := \Delta(n)(\log n)^s, n \geq 5041 \qquad (22)$$

An integer $m \geq 5041$ is called highest abundant of order s if there exists a real number a such that $R_s(n) - an$ attains its minimum at $n = m$ over the prescribed domain.

Proposition 13. On every finite interval $5041 \leq n \leq X$, the highest abundant numbers of order s are exactly the vertices of the lower convex envelope of the point set

$$\{(n, R_s(n)) : 5041 \leq n \leq X\} \qquad (23)$$

Proof. A point $(m, R_s(m))$ lies on the lower convex envelope exactly when there is a supporting line of slope a passing through it with all other points on or above that line. This condition is precisely

$$R_s(m) - am \leq R_s(n) - an$$

for every integer n in the interval.

Remark 4. Theorem 12 reduces Robin's criterion to colossally abundant integers. Proposition 13 supplies a computable convex-envelope method for locating the integers where normalized Robin defects are extremal. These are complementary reductions which the first is arithmetic and the second is geometric.

The numerical values in Figure 1 were computed exactly from the factorization formula

$$\sigma\left(\prod_{j=1}^{r} p_j^{\alpha_j}\right) = \prod_{j=1}^{r} \frac{p_j^{\alpha_j+1}-1}{p_j-1}. \qquad (24)$$

For each colossally abundant integer $N > 5040$, it gives $G(N)$, $\Delta(N)$ and the normalized defect

$$T(N) := \frac{\Delta(N)\sqrt{\log N}}{N} \qquad (25)$$

The data show that the first colossally abundant integers after 5040 satisfy Robin's inequality. They also indicate that the normalized defect is not consistent at the level of $G(N)$, since $G(367567200)$ increases relative to the previous value. Finally, the data explain why extremal subsequences are essential: ordinary integers are too many, while colossally abundant and highest abundant integers focus on the area where Robin's inequality is most at risk.

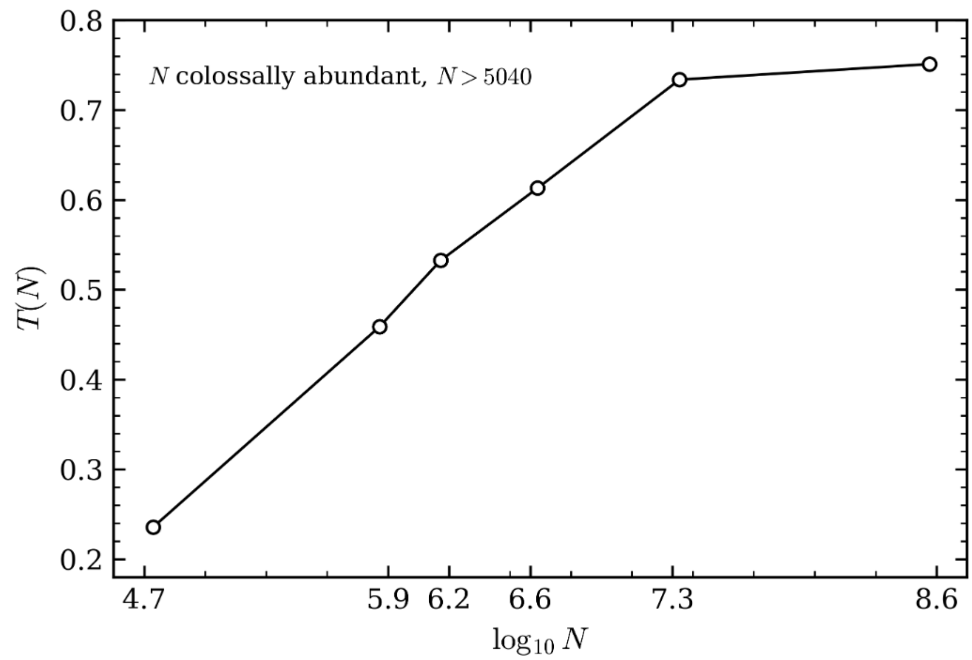

Figure 1: Normalized Robin defect

6. Conclusion

Ramanujan's divisor identities and Lambert-series transformations provide exact control over smoothed divisor sums. Robin's theorem changes the Riemann Hypothesis into a pointwise inequality for the same arithmetic function. This paper's main contribution is to combine these two ideas into one clear framework. The exact identity (16) shows the Laplace transform of the Robin defect using Ramanujan's transformation. Proposition 9 identifies the necessary statement related to the Riemann Hypothesis: every coefficient of the Robin-defect series must be positive. Corollary 11 proves that smoothed positivity near x=0 is unconditional. Therefore, no proof of the Riemann Hypothesis can come solely from the asymptotic positivity of the transformed Lambert series.

The remaining obstruction is arithmetic and extremal, captured by Theorem 12 and the convex-envelope formulation in Proposition 13. In this form, Ramanujan supplies the transformation; Robin supplies the coefficientwise target; the extremal theory of abundant numbers identifies where a proof must act. A future proof along this route would need a uniform lower bound for $\Delta(N)$ on the colossally abundant subsequence or an equivalent convex-envelope bound for the normalized defects $R_s(N)$. The present article supplies the rigorous reduction and separates what is already forced by Ramanujan's identities from what remains genuinely equivalent to RH.

Dedicated to. . .

To the most brilliant mathematician, dear ***Srinivasa Ramanujan***, who devoted his life to mathematics with purest imagination and faith in hidden order of numbers. And to my beautiful daughter ***Artemisia***, whose lovely eyes gave me courage in hard hours and reminded me why I must keep moving forward.

References

[1] B. Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", November 1859. 136-144.

[2] T. H. Gronwall, "Some Asymptotic Expressions in the Theory of Numbers," *Trans. Am. Math. Soc.*, vol. 14, no. 1, p. 113, Jan. 1913, doi: 10.2307/1988773.

[3] S. Ramanujan, "Highly Composite Numbers," *Proceedings of the London Mathematical Society*, vol. s2_14, no. 1, pp. 347–409, Jan. 1915, doi: 10.1112/PLMS/S2_14.1.347.

[4] L. Alaoglu and P. Erdos, "On Highly Composite and Similar Numbers," *Trans. Am. Math. Soc.*, vol. 56, no. 3, p. 448, Nov. 1944, doi: 10.2307/1990319.

[5] A. E. Ingham, "Some Tauberian Theorems Connected with the Prime Number Theorem," *J. Lond. Math. Soc.*, vol. s1-20, no. 3, pp. 171–180, Jun. 1945, doi: 10.1112/JLMS/S1-20.3.171.

[6] "Grandes valeurs de la fonction somme des diviseurs | PDF | Théorie des nombres | Mathématiques." Accessed: Jul. 01, 2026. `https://www.scribd.com/document/812768752/Robin-1984-noBackground#google_vignette`.

[7] J. C. Lagarias, "An Elementary Problem Equivalent to the Riemann Hypothesis," *American Mathematical Monthly*, vol. 109, no. 6, pp. 534–543, Aug. 2000, doi: 10.2307/2695443.

[8] M. Rogers, "IDENTITIES FOR THE RAMANUJAN ZETA FUNCTION," 2013.

[9] O. R. Musin, "Ramanujan's theorem and highest abundant numbers," *Arnold Mathematical Journal*, vol. 6, no. 1, pp. 119–130, Feb. 2020, doi: 10.1007/s40598-020-00136-w.

[10] L. C. Washington and A. Yang, "Analogues of the Robin-Lagarias Criteria for the Riemann Hypothesis," Aug. 2020, Accessed: Jul. 01, 2026. [Online]. Available: `https://arxiv.org/pdf/2008.04787`

In memory of Srinivasa Ramanujan (1887–1920)