A model for merging black holes

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Abstract

We present a model in which *SING*, a singularity-inspired notion (viXra:1812.0480 [v1]), plays some role in black hole merging to suggest that evaporation of black hole leaves something behind.

For a model is a disappearing schema.

—— Gadamer, H.-G., "Truth and Method," Crossroad Publishing 1982 p. 128.

1 Glossary

BH: black hole .

GIMP: GNU Image Manipulation Program .

iff: if and only if .

LHS: left-hand side .

MM: magnetic monopole .

O: the origin $(0, 0)$.

QE: quadratic equation .

RHS: right-hand side .

ST: string theory .

wrt: with respect to .

$\cup$: set union .

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2 Introduction

Inspired by physical application of differential form [1], we treat BH [2,3] merging from the viewpoint of *SING*, a singularity-inspired notion [4].

3 Modeling

While tackling the problem of merger of BH's, we consider the following steps.

Step 1: *Approach*. Two BH's get closer;

Step 2: *Contact*. They touch;

Step 3: *Merger*. They fuse.

We will elaborate on each step in what follows.

3.1 Approach

Spherical symmetry being a convenient simplification [5], we suppose that merging BH's are two circles that are separate in a Cartesian coordinate plane as shown below.

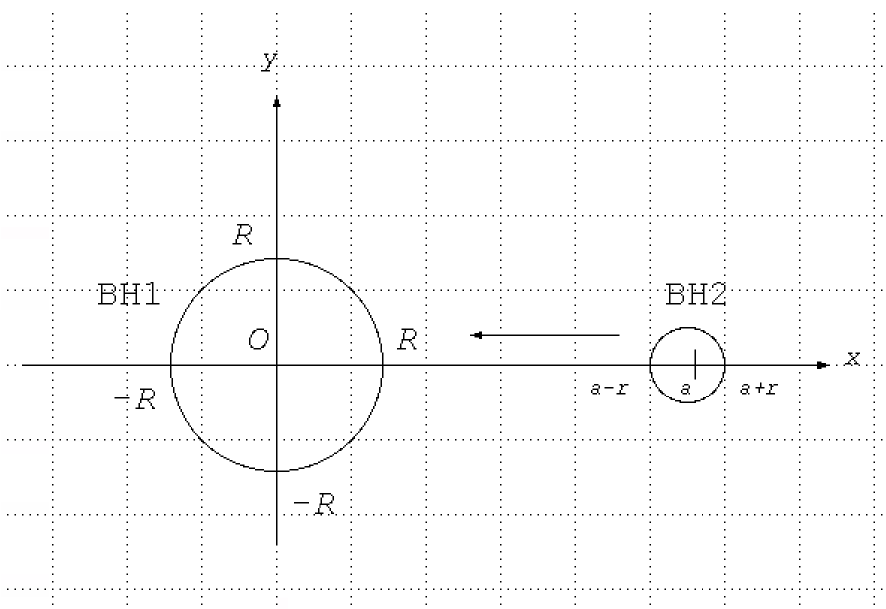

Fig. 1. BH2 approaches BH1 [1].

Note that in the above figure, BH2, whose center and radius are a and r, respectively, moves toward BH1, whose center and radius are O and R, respectively. We mathematically describe these BH's as

[1] Tgif ver. 4.2 is used for the preparation of figures unless otherwise specified.

$$
\begin{cases}
\text{BH1} : x^2 + y^2 = R^2, & (1) \\
\text{and} \\
\text{BH2} : (x-a)^2 + y^2 = r^2, & (2) \\
\text{with} \\
R > r > 0. & (3)
\end{cases}
$$

3.2 Contact

After a while, these BH's contact:

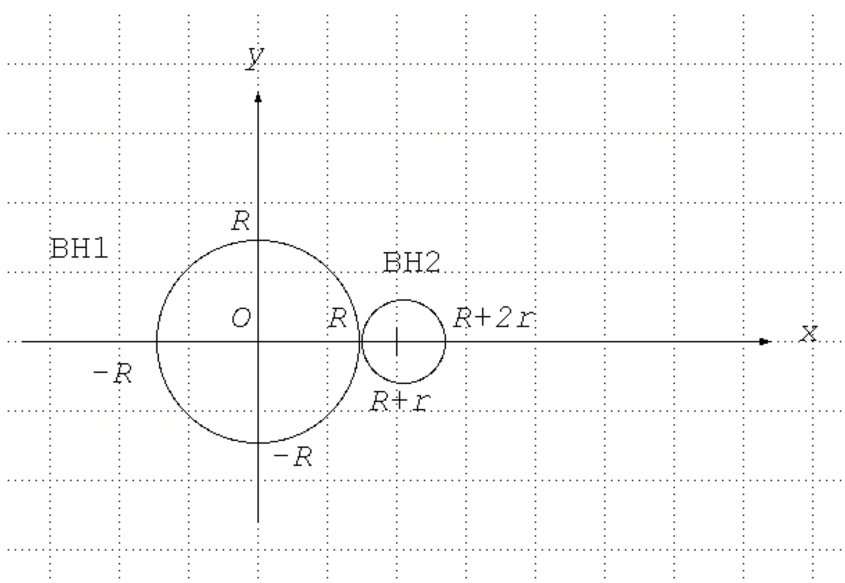

Fig. 2. BH2 comes into contact with BH1.

Letting the above 'contact state' reflect (1) and (2), we write

$$\text{BH1/2} : x^2 + y^2 = R^2 \cup \{x - (R+r)\}^2 + y^2 = r^2. \tag{4}$$

In order to compute *SING* [4] of this, we set

$$\phi = (x^2 + y^2 - R^2)[\{x - (R+r)\}^2 + y^2 - r^2] = 0 \tag{5}$$

and compute the following after expansion of the middle term of (5).

$$\begin{aligned}\frac{d\phi}{dx} &= \frac{d}{dx}[(x^2+y^2-R^2)\{x^2+y^2-2(R+r)x+R^2+2Rr\}]\\ &= \frac{d}{dx}(x^4+y^4+2x^2y^2-2Rx^3-2rx^3+2Rrx^2-2Rxy^2-2rxy^2+2Rry^2+2R^3x+2R^2rx\\ &\quad -R^4-2R^3r)\\ &= 4x^3+4y^3\frac{dy}{dx}+4xy^2+4x^2y\frac{dy}{dx}-6Rx^2-6rx^2+4Rrx-2Ry^2-4Rxy\frac{dy}{dx}-2ry^2-4rxy\frac{dy}{dx}\\ &\quad +4Rry\frac{dy}{dx}+2R^3+2R^2r,\end{aligned}$$

from which we obtain the $1-$form

$$\begin{aligned}\omega = d\phi &= (4x^3+4xy^2-6Rx^2-6rx^2+4Rrx-2Ry^2-2ry^2+2R^3+2R^2r)dx\\ &\quad +(4y^3+4x^2y-4Rxy-4rxy+4Rry)dy\\ &= 2(2x^3+2xy^2-3Rx^2-3rx^2+2Rrx-Ry^2-ry^2+R^3+R^2r)dx\\ &\quad +4y(y^2+x^2-Rx-rx+Rr)dy. \end{aligned} \tag{6}$$

To get *SING* from (6), we solve

$$\begin{cases} 2x^3+2xy^2-3Rx^2-3rx^2+2Rrx-Ry^2-ry^2+R^3+R^2r=0, & (7)\\ y(y^2+x^2-Rx-rx+Rr)=0, & (8)\end{cases}$$

and (8) leads us to consider the following three cases.

Case 1. If $y=0$ and $y^2+x^2-Rx-rx+Rr=0$, substituting $y=0$ into the latter condition, we get $x^2-Rx-rx+Rr=0$. Factoring the LHS of this yields $(x-R)(x-r)=0$, that is, $x=R,\ r$. Thus, $(x,\ y)=(R,\ 0)$ and $(r,\ 0)$, which leads us to further consider the following subcases.

Subcase 1.1. If $(x,\ y)=(R,\ 0)$, substituting $(x,\ y)=(R,\ 0)$ into the LHS of (7) yields $2R^3+2\cdot R\cdot 0^2-3R\cdot R^2-3r\cdot R^2+2Rr\cdot R-R\cdot 0^2-r\cdot 0^2+R^3+R^2r=0$, which amounts to its RHS. So $(x,\ y)=(R,\ 0)$ is a solution to (7) and (8).

Subcase 1.2. If $(x,\ y)=(r,\ 0)$, likewise, the LHS of (7) becomes $-r^3-Rr^2+R^3+R^2r$. Factoring this yields $(R+r)^2(R-r)$. Since $R+r>0$ [2] , for $(x,\ y)=(r,\ 0)$ to be a solution to (7), $R=r$ needs to hold, which contradicts (3), though [3] .

Case 2. If $y=0$ and $y^2+x^2-Rx-rx+Rr\neq 0$, substituting $y=0$ into the LHS of the latter condition yields $x^2-Rx-rx+Rr\neq 0$. Factoring the LHS of this, one gets $(x-R)(x-r)\neq 0$. So $x\neq R,\ r$. And substituting $y=0$ into the LHS of (7) yields $2x^3-3Rx^2-3rx^2+2Rrx+R^3+R^2r=0$. Factoring the LHS of this, one gets $(x-R)\{2x^2-(R+3r)x-R^2-Rr\}=0$. Solving this wrt x yields $x=R,\ \frac{R+3r\pm\sqrt{9R^2+14Rr+9r^2}}{4}$. However, since $x\neq R$, we drop $x=R$ to get $(x,\ y)=(\frac{R+3r\pm\sqrt{9R^2+14Rr+9r^2}}{4},\ 0)$.

[2] See (3).

[3] Curiously, we will try to extract something from this contradiction later. See **4**.

Case 3. If $y \neq 0$ and $y^2 + x^2 - Rx - rx + Rr = 0$, we obtain $y^2 = -x^2 + Rx + rx - Rr$ from the latter condition and substitute this into the LHS of (7). Then, we get $2x^3 + 2x(-x^2 + Rx + rx - Rr) - 3Rx^2 - 3rx^2 + 2Rrx - R(-x^2 + Rx + rx - Rr) - r(-x^2 + Rx + rx - Rr) + R^3 + R^2 r$. After some computation, this becomes $-(R^2 + 2Rr + r^2)x + R(R^2 + 2Rr + r^2)$, which amounts to $(R - x)(R + r)^2$. Equating this with 0, we get $x = R$, since $R + r > 0$ [4]. Substituting $x = R$ into the LHS of (8) yields $y^3 = 0$. Hence, $y = 0$, but this contradicts $y \neq 0$, the former condition in this case [5].

3.2.1 Showing where *SING*'s are

Taken together, we have got

$$(x, y) = (R, 0),\ (\frac{R + 3r \pm \sqrt{9R^2 + 14Rr + 9r^2}}{4}, 0). \tag{9}$$

These are the *SING*'s of 'contact state', and the RHS of (9) are each labeled as the following.

$$S1(R, 0), \qquad S2(\tfrac{R+3r-\sqrt{9R^2+14Rr+9r^2}}{4}, 0), \qquad S3(\tfrac{R+3r+\sqrt{9R^2+14Rr+9r^2}}{4}, 0).$$

Putting $S1$ aside, we try to know the whereabouts of $S2$ and $S3$ relative to the centers of BH1 and BH2.

<u>*S*2</u>

We suppose

$$-R < \frac{R + 3r - \sqrt{9R^2 + 14Rr + 9r^2}}{4} < 0 \tag{10}$$

and check whether (10) holds.

First, we compute $\frac{R+3r-\sqrt{9R^2+14Rr+9r^2}}{4} - (-R) = \frac{5R+3r-\sqrt{9R^2+14Rr+9r^2}}{4}$. So if the inequality

$$5R + 3r > \sqrt{9R^2 + 14Rr + 9r^2} \tag{11}$$

holds,

$$-R < \frac{R + 3r - \sqrt{9R^2 + 14Rr + 9r^2}}{4} \tag{12}$$

also holds. Here we note that we have

$$0 \leq a \leq b \text{ iff } 0 \leq a^n \leq b^n, \text{ for } n > 0.$$ [6]

[4] See (3).

[5] Unfortunately, we find it difficult to deal with this contradiction at this point. *Cf.* footnote 3.

[6] We will implicitly use this in the following arguments from time to time.

Since it follows from (3) that both sides of (11) are positive, one can say that if

$$(5R+3r)^2 > 9R^2+14Rr+9r^2, \tag{13}$$

then (12) holds. To know whether (13) holds, we compute

$$(5R+3r)^2-(9R^2+14Rr+9r^2) = 25R^2+30Rr+9r^2-9R^2-14Rr-9r^2 = 16R(R+r).$$

Since $R > r > 0$ [7] , the rightmost term is positive, which means that (13) holds. Hence, (12) holds.

Next, we check whether

$$\frac{R+3r-\sqrt{9R^2+14Rr+9r^2}}{4} < 0 \tag{14}$$

holds. Using similar reasoning, we compute

$$9R^2+14Rr+9r^2-(R+3r)^2 = 9R^2+14Rr+9r^2-R^2-6Rr-9r^2 = 8R(R+r).$$

Again, the rightmost term is positive [8] , which means that (14) holds. Because (12) and (14) hold, (10) also holds. So we can say that $S2$ lies between $(-R,0)$ and O.

<u>$S3$</u>

Likewise, we suppose

$$R+r < \frac{R+3r+\sqrt{9R^2+14Rr+9r^2}}{4} < R+2r \tag{15}$$

and check whether (15) holds.

First, we compute $\frac{R+3r+\sqrt{9R^2+14Rr+9r^2}}{4}-(R+r) = \frac{-3R-r+\sqrt{9R^2+14Rr+9r^2}}{4}$. So if the inequality

$$\sqrt{9R^2+14Rr+9r^2} > 3R+r \tag{16}$$

holds,

$$R+r < \frac{R+3r+\sqrt{9R^2+14Rr+9r^2}}{4} \tag{17}$$

also holds. We compute

$$9R^2+14Rr+9r^2-(3R+r)^2 = 9R^2+14Rr+9r^2-9R^2-6Rr-r^2 = 8r(R+r).$$

Since $R > r > 0$ [9] , the rightmost term is positive. So (16) holds. Hence, (17) holds.

[7] See (3).
[8] Ditto.
[9] Ditto.

Next, we check whether

$$\frac{R+3r+\sqrt{9R^2+14Rr+9r^2}}{4} < R+2r \tag{18}$$

holds. Using similar reasoning, we compute

$$\{4(R+2r)-(R+3r)\}^2-(9R^2+14Rr+9r^2)=(3R+5r)^2-9R^2-14Rr-9r^2=16r(R+r).$$

Again, the rightmost term is positive [10], which means that (18) holds. Because (17) and (18) hold, (15) also holds. Hence, $S3$ lies between $(R+r,0)$ and $(R+2r,0)$.

We visualize the results we have obtained:

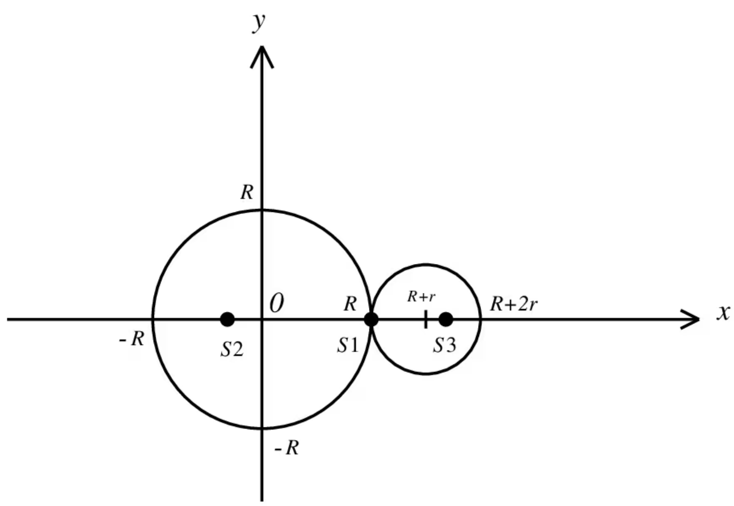

Fig. 3. Whereabouts of $S1$, $S2$, and S3

Remark 3.2.1.1. We classify $S1$ into the category **ON** and $S2$ and $S3$ into the category **IN** in terms of *SING* [11] .

Remark 3.2.1.2. Since $(\frac{R+3r-\sqrt{9R^2+14Rr+9r^2}}{4}+\frac{R+3r+\sqrt{9R^2+14Rr+9r^2}}{4})/2=\frac{R+3r}{4}$, $M(\frac{R+3r}{4}, 0)$ is the midpoint of $S2$ and $S3$.

Remark 3.2.1.3. A straightforward computation shows $(\frac{3\cdot r+1\cdot R}{1+3}, \frac{3\cdot 0+1\cdot 0}{1+3})=(\frac{R+3r}{4}, 0)$. So the point M can be regarded as the point dividing the line segment that joins the points $(r,0)$ and $(R,0)$ in the ratio 1 : 3 [12] .

We visualize *Remarks* 3.2.1.2 and 3.2.1.3:

[10] Ditto.
[11] See *e.g.*, [4, **Table**].
[12] *Cf.* here .

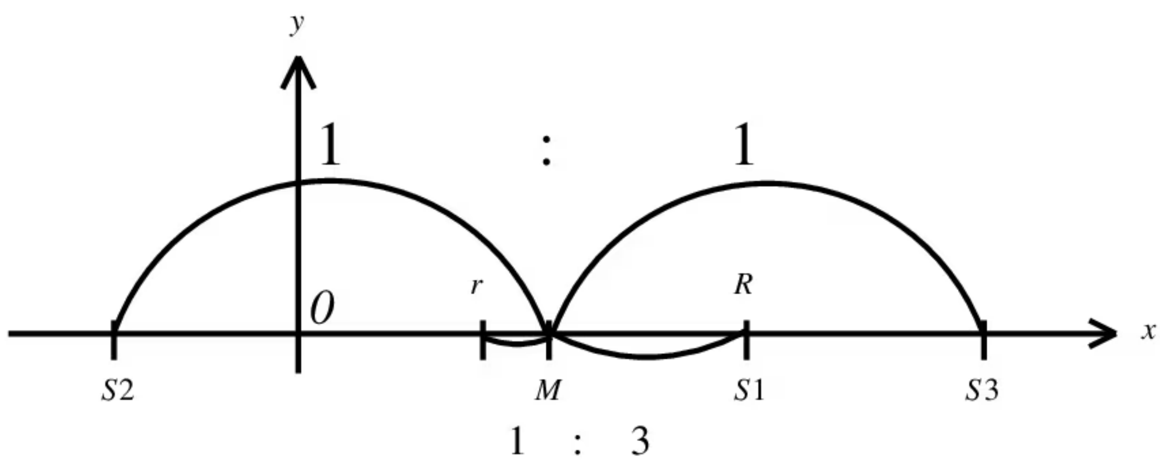

Fig. 4. Some remarks visualized [13]

3.3 Merger

In the figure below

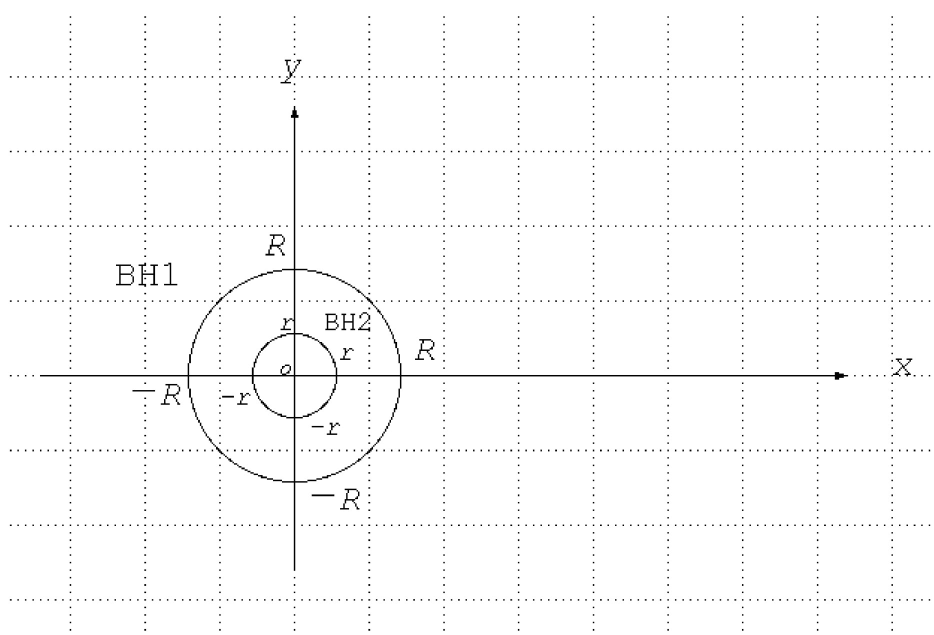

Fig. 5. Merger of BH1 and BH2

we consider

[13] PostScript was used for the preparation of Fig.'s 3 and 4.

$$\begin{cases} \text{BH1} : x^2 + y^2 = R^2, \\ \text{BH2} : x^2 + y^2 = r^2 \end{cases}$$

and write

$$\text{BH1/2} : x^2 + y^2 = R^2 \cup x^2 + y^2 = r^2 \text{ [14]}.$$

Like (5), we set

$$\phi = (x^2 + y^2 - R^2)(x^2 + y^2 - r^2) = 0$$

and compute the following after expansion of the middle term.

$$\begin{aligned} \frac{d\phi}{dx} &= \frac{d}{dx}(x^4 + y^4 + 2x^2y^2 - R^2x^2 - R^2y^2 - r^2x^2 - r^2y^2 + R^2r^2) \\ &= 4x^3 + 4y^3\frac{dy}{dx} + 4xy^2 + 4x^2y\frac{dy}{dx} - 2R^2x - 2R^2y\frac{dy}{dx} - 2r^2x - 2r^2y\frac{dy}{dx}, \end{aligned}$$

from which we obtain the $1-$form

$$\begin{aligned} \omega = d\phi &= (4x^3 + 4xy^2 - 2R^2x - 2r^2x)dx + (4y^3 + 4x^2y - 2R^2y - 2r^2y)dy \\ &= 2x(2x^2 + 2y^2 - R^2 - r^2)dx + 2y(2y^2 + 2x^2 - R^2 - r^2)dy. \end{aligned}$$

So we need to solve

$$\begin{cases} x(2x^2 + 2y^2 - R^2 - r^2) = 0, & (19) \\ y(2x^2 + 2y^2 - R^2 - r^2) = 0, & (20) \end{cases}$$

and (19) leads us to consider the following three cases.

Case 1. If $x = 0$ and $2x^2 + 2y^2 - R^2 - r^2 = 0$, the latter condition satisfies (20). Solving both conditions, one gets

$$(x, y) = (0, \pm\sqrt{\frac{R^2 + r^2}{2}}), \tag{21}$$

solution of this case.

Case 2. If $x = 0$ and $2x^2 + 2y^2 - R^2 - r^2 \neq 0$, it follows from the latter condition and (20) that $y = 0$. Since $x = 0$, we get the solution $(x, y) = (0, 0)$.

Case 3. If $x \neq 0$ and $2x^2 + 2y^2 - R^2 - r^2 = 0$, these conditions satisfy (19) and (20). So it suffices to consider them. That is, since $x \neq 0$, we remove the points in (21) from $2x^2 + 2y^2 = R^2 + r^2$, the circle $x^2 + y^2 = \frac{R^2+r^2}{2}$.

We visualize each case:

[14] *Cf.* (4).

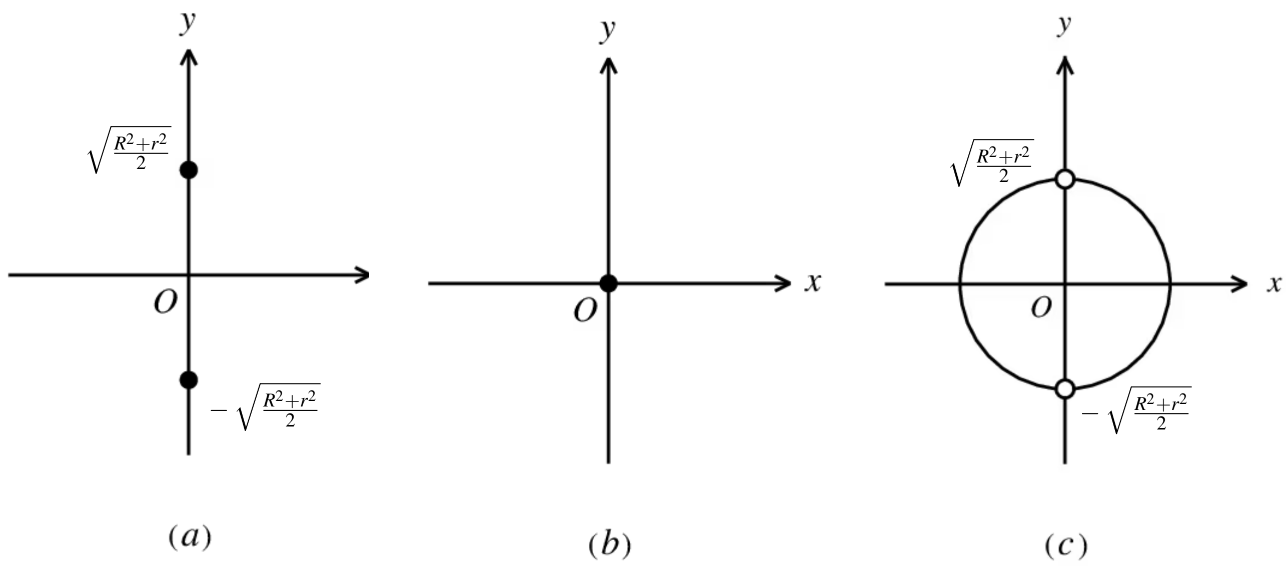

Fig. 6. Visualization of each case [15] . (a), (b), and (c) correspond to *Cases* $1-3$, respectively.

Putting together Fig.'s $6(a)-(c)$, we show the whereabouts of *SING*'s as follows.

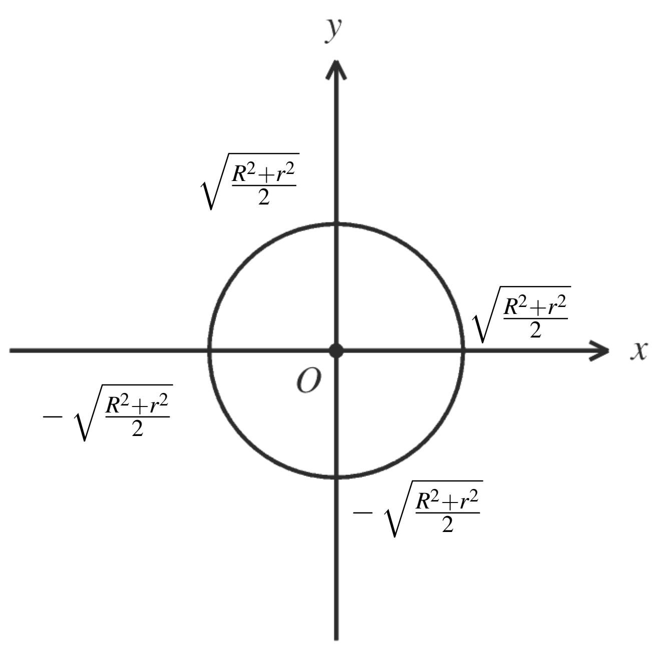

Fig. 7. *SING*'s of merged BH's [16]

[15]PostScript was used for the preparation of these figures.
[16]This figure was prepared like the above way.

We denote the circle and the point at O in Fig. 7 by $S4$ and $S5$, respectively. Using $R > r > 0$ [17] , we get $R^2 > r^2$. Adding R^2 to both sides of this gives $2R^2 > R^2 + r^2$, that is, $R^2 > \frac{R^2+r^2}{2}$. So

$$R > \sqrt{\frac{R^2+r^2}{2}}. \tag{22}$$

Likewise, we get

$$\sqrt{\frac{R^2+r^2}{2}} > r. \tag{23}$$

And it follows from (22) and (23) that

$$R > \sqrt{\frac{R^2+r^2}{2}} > r. \tag{24}$$

Taking (24) into consideration, we merge Fig.'s 5 and 7 to get the following.

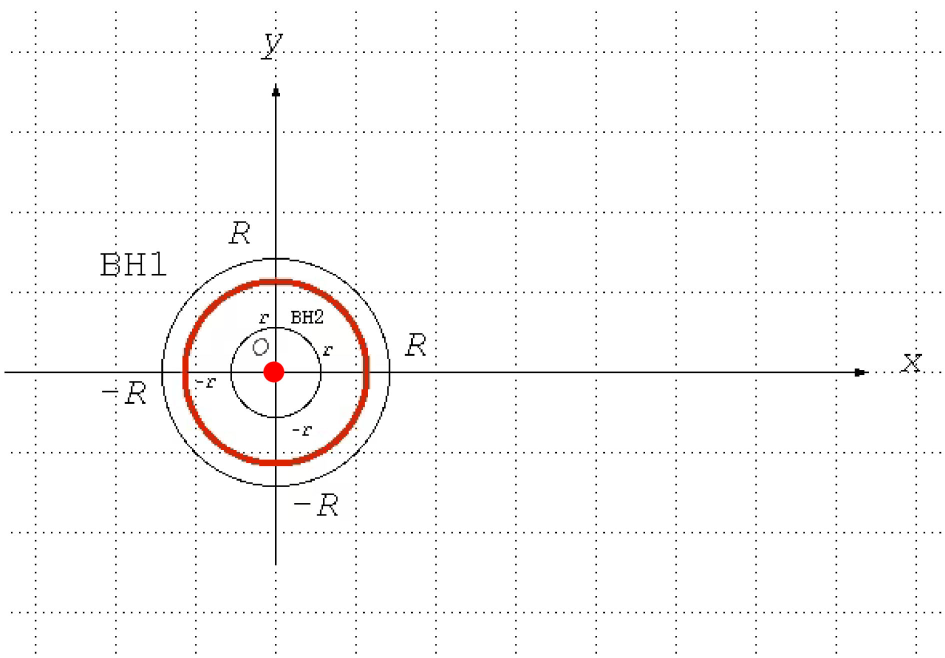

Fig. 8. Showing where *SING*'s are in merged BH's. $S4$ and $S5$ are indicated by a red circle and a red dot, respectively [18] .

[17]See (3).

[18]Original Tgif image was slightly retouched by GIMP ver. 3.2.4.

4 What if two BH's are of a size?

In **3.2**, the relation $R = r$ in *Subcase* 1.2 gave rise to contradiction unfortunately. However, we try computing *SING* of such a case mainly out of curiosity. We skip *Step* 1 and start directly from *Step* 2.

4.1 Contact

Replacing r in (9) by R, one gets

$$(x,y) = (R,0),\ (\tfrac{R+3R\pm\sqrt{9R^2+14R\cdot R+9R^2}}{4},0).$$

So in this case, *SING*'s are

$$(x,y) = (R,0),\ ((1\pm\sqrt{2})R,0). \tag{25}$$

We label them as the following.

$$S6((1-\sqrt{2})R,\,0), \qquad S7(R,\,0), \qquad S8((1+\sqrt{2})R,\,0).$$

These are visualized with two BH's:

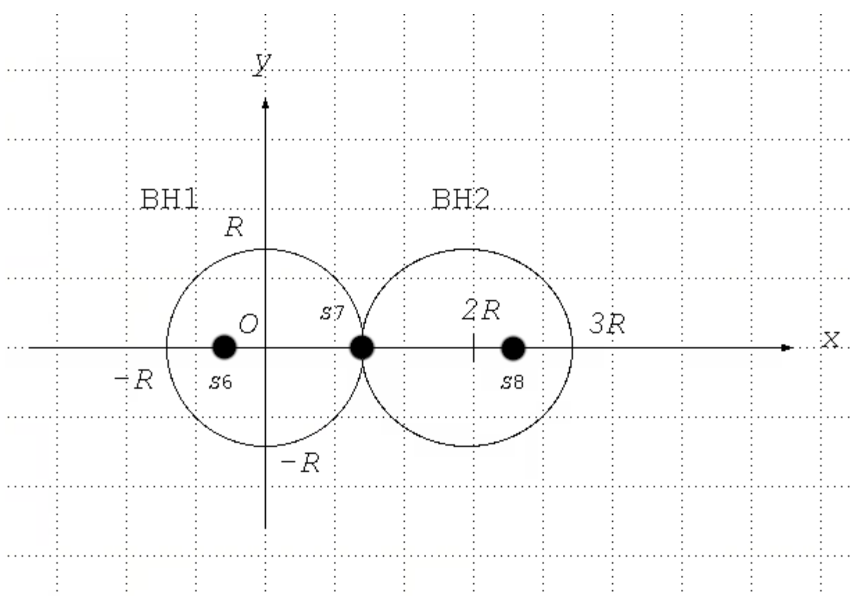

Fig. 9. Whereabouts of $S6$, $S7$, and $S8$ [19]

Remark 4.1.1. We classify $S7$ into the category **ON** and $S6$ and $S8$ into the category **IN** in terms of *SING* [20] .

Remark 4.1.2. $S7$ is the midpoint of $S6$ and $S8$, since $\{(1-\sqrt{2})R+(1+\sqrt{2})R\}/2 = R$.

[19]Ditto.

[20]*Cf.* footnote 11.

4.2 Merger

Letting r in (24) tend to R gives

$$R = R = R,$$

which means that the radii of BH1, BH2, and $S4$ in Fig. 8 get identical as shown below.

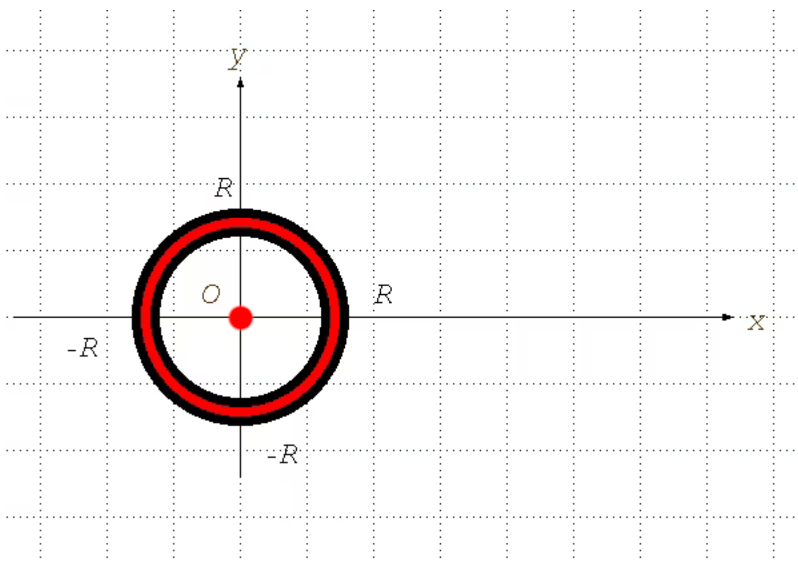

Fig. 10. Outcome of merger [21]

In the above figure, we label the red circle and the red point as $S9$ and $S10$, respectively.

Remark 4.2.1. Since ex-BH1, ex-BH2, and $S9$ are all $x^2 + y^2 = R^2$, they are indistinguishable mathematically. $S9$ is thus classified into the category of **IT** [22] .

Remark 4.2.2. Just for the sake of visual explanation, they are differentiated as a red circle and two adjacent black circles [23] .

Remark 4.2.3. $S10$ is classified into the category **IN** [24] .

5 Discussion

In order to understand the mechanism of black hole merging, we made a model and got '*SING* solutions' $x = R, \frac{R+3r\pm\sqrt{9R^2+14Rr+9r^2}}{4}$ [25] by solving $(x-R)\{2x^2-(R+3r)x-R^2-Rr\} = 0$ [26].

[21] Original Tgif image was slightly retouched by GIMP ver. 3.2.4.
[22] See *e.g.*, [4, **Table**].
[23] *Cf.* [4, Fig. 3].
[24] See *e.g.*, [4, **Table**].
[25] *Cf.* (9).
[26] See *Case* 2 in **3.2**.

Here we notice that setting $R = r = 1$ in the QE $2x^2 - (R+3r)x - R^2 - Rr = 0$ yields

$$2x^2 - 4x - 2 = 0. \tag{26}$$

Solving this yields the values $1 \pm \sqrt{2}$ we have already seen in (25). So although not purely mathematical, we may have made a 'slightly deep' answer to the 'deep' question raised in [6, **3**], which was

What does solving QE mean?

By the way, (26) is essentially the same as the QE $x^2 - 2x - 1 = 0$, its positive solution, $1+\sqrt{2}$, being the silver ratio . As for its negative solution, or $1-\sqrt{2}$, by thinking of its absolute value, we compute

$$|1-\sqrt{2}| = \sqrt{2} - 1 = 0.4142135\ldots.$$

Geometrically, this concerns circular coverage [7]. Physically, it is known to be related to ferromagnetism [8], which leads us to propose the following.

CONJECTURE 5.1. MM is a constituent of BH (as suggested by ST). Prove or disprove this experimentally.

With regard to Fig. 8, BH1 encloses $S4$, BH2, and $S5$, whereas BH2 does $S5$, which might be useful for probing into the internal structure of BH. As to *SING*'s in Fig. 9, we notice they lie in tandem on the x-axis like Fig. 3, the point $(R,0)$ being between two other *SING*'s. As for Fig. 10, $S4$ in Fig. 8 seems to have 'surfaced' to become a part of the contour of the merged BH as shown by the red circle between black circles, whereas $S5$ in that figure remains at O. If (conventional) BH singularity [9] is regarded as harmful and *SING* is relevant to it at all, this is a rather dangerous situation in which one *SING* is 'exposed without shield' [27]. However, it seems unlikely that such 'twin' BH's actually exist somewhere and will further merge as desired. Moreover, this exposure might be just a 'mirage' that was inadvertently obtained out of mathematical curiosity.

Instead of discussing such 'impossible twin' BH's, what if we interpret Fig. 10 as follows?

Interpretation 5.2. Fig. 10 intimates the endgame of a *single* BH.

This interpretation leads us to wonder whether something remains after the so-called BH evaporation . In Fig.'s 5 and 8, BH's merge, but don't disappear. On the other hand, if we interpret $S9$ in Fig. 10 as 'takeover' of aboriginal BH's, one can say BH merging has given rise to evaporation. However, for better or worse, such evaporation doesn't seem tantamount to negating something, since $S9$ and $S10$ remain. That is, BH's have undergone evaporation to leave *SING*'s behind in our model. Therefore, we hope *SING*'s will shed some light on the problem of BH evaporation .

As to mathematical contradictions we have encountered in **3.2** [28], recalling the (celebrated) BH information paradox , some might ironically find them to be intriguing, though we refrain from claiming that they parallel it.

[27] Compare Fig. 10 with Fig. 8 showing that $S4$ is 'shielded' by BH1 and that $S5$ is 'shielded doubly' by BH1 and BH2.

[28] See *Subcase* 1.2 and *Case* 3.

Actually, since real bodies are seldom spherically symmetric [5], we may have run the risk of oversimplification in our modeling, where BH's have been assumed to be sheer circles. And again, '*SING* emergence' might be just a mathematical mirage. Furthermore, frankly speaking, we are embarrassed at our course of arguments through which we have somehow arrived at Fig. 10 in the face of the aforementioned contradictions, for it seems wise to abandon further argument, when one encounters such contradiction(s).

Nonetheless and finally, from an experimental point of view, we also wonder if our model helps to explain a flash of light that seems related to GW190521 and/or expected merging in Markarian 501 at least qualitatively.

Acknowledgment. We would like to thank the developers of Atril , GIMP, Okular , PostScript, and Tgif for their indirect help, which enabled us to prepare figures for submission.

References

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[2] Poisson, E. and Will, C. M., "Gravity," Cambridge University Press 2014 p. 267.

[3] Needham, T., "Visual Differential Geometry and Forms," Princeton University Press 2021 pp. 329-331.

[4] Suzuki, K., "Classifying conic sections in terms of differential forms," viXra:1812.0480 [v1] .

[5] Poisson, E. and Will, C. M., *ibid* p. 30.

[6] Suzuki, K., "Cubic equation revisited: part 1," viXra:2412.0187 [v1] .

[7] Finch, S. R., "Mathematical Constants," Cambridge University Press 2003 pp. 484-488.

[8] *Idem*, *ibid* p. 396.

[9] Poisson, E. and Will, C. M., *ibid* p. 271.

[10] Needham, T., *ibid* p. 322.

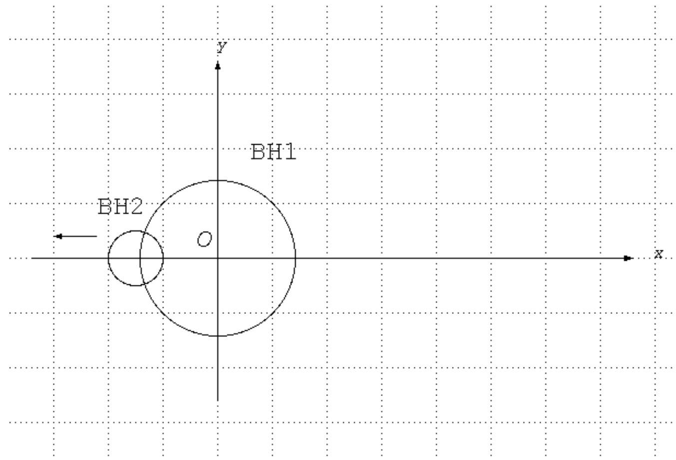
6 Appendix

6.1 Completion of merger: is that all?

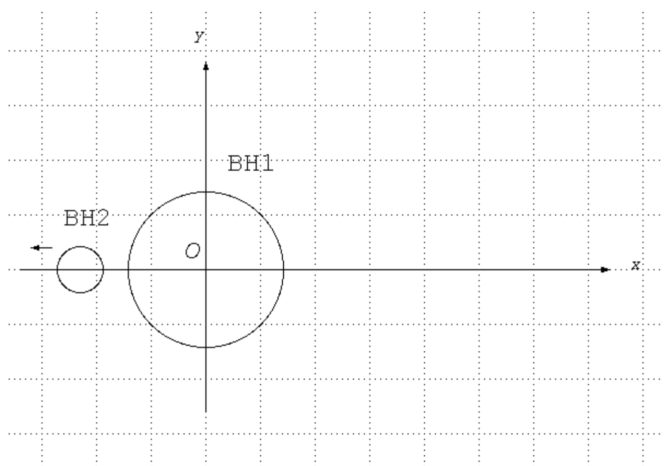
In other words, our question is

Nothing ensues after BH's merge?

To address this question, we imagine a counterpart of quantum tunnelling by which BH2 somehow crosses BH1 via the process of merger as if nothing had happened, like the figures below.



BH2 is ‘crossing’ BH1 [29] .



BH2 has ‘crossed’ BH1.

Of course, this might be a kind of *Gedankenexperiment* at best, but if such ‘BH crossing’ should happen….

6.2 What about rotating BH?

Though we have treated BH’s as if they didn’t rotate, $S4$ and $S9$ remind us of ring singularity of a rotating BH . So we might discuss rapidly rotating BH’s [10, footnote 10] elsewhere.

[29] Some might recall lune .