Accurate circle configurations and numerical conformal mapping in polynomial time

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Abstract — **According to a remarkable reinterpretation of a theorem of E.M. Andreev (1970) by W.P. Thurston (≈1982), there is a unique (up to inversive transformations) packing of interior-disjoint circles in the plane, whose contact graph is any given polyhedral graph G, and such that an analagous "dual" circle packing simultaneously exists, whose contact graph is the planar dual graph G^*, and such that the primal and dual circle packin1qDgs have the same set of tangency points and the primal circles are orthogonal to the dual ones at these tangency points.**

This note shows that relatively and absolutely accurate coordinates for the primal and dual circles may be obtained in time polynomial in N, the number of vertices of the polyhedral graph, and D, the number of decimals of accuracy desired.

Consequently one may also accurately "midscribe" a polyhedron – and simultaneously its dual – in polynomial time.

Also consequently, one may implement Riemann's conformal mapping theorem numerically, in polynomial time with provable accuracy.

Our result is obtained by generalizing and reformulating ideas found in the doctoral thesis of Walter Brägger [Math. Institut, Rheinsprung 21, CH-4051 Basel, Feb. 1991] to reduce our problem to maximizing a smooth convex function. This maximization problem is then solved by using Khachian's "ellipsoid method" or Vaidya's algorithm.

Keywords — Conformal Mapping, midscribed polyhedra, circle packings, ellipsoid algorithm, Clausen's integral, hyperbolic geometry, inversion.

1 Background

IT WILL BE ASSUMED that the reader is familiar with the basics of graph theory, hyperbolic geometry, and convex programming, and with the notion of "inversions" about Euclidean spheres. The uninitiated reader may consult [5, 18, 8, 9, 1], or, should he be lucky enough to obtain a copy, most of the relevant ideas and terminology in these areas are discussed in [13].

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2 Introduction

The version of the Andreev-Thurston theorem which we are interested in states that[1]

Theorem 1 [Andreev-Thurston in plane] *There is a unique (up to inversive transformations) packing of interior-disjoint circles in the plane, whose contact graph is any given polyhedral graph G. [If G is maximal planar, this circle configuration is unique up to inversive transformations of the plane.] Furthermore an analagous "dual" circle packing simultaneously exists, whose contact graph is the planar dual graph G^*, and such that the primal and dual circles have the* same *set of tangency points and the primal circles are orthogonal to the dual ones at these tangency points.*

A graph is "polyhedral" iff it is 3-connected and planar (a famous theorem of Steinitz [9]). Any planar graph is a subgraph of a polyhedral graph, if it isn't polyhedral itself, so theorem 1 of course shows the existence of circle packings whose contact graph is any planar graph. Indeed, we see (after a decomposition into 3-connected components, and consideration of the inversive freedom of the circle packing corresponding to each component) that there is *nonuniqueness* of the combined circle packing corresponding to any non-polyhedral planar graph.

Upon transforming the plane onto the surface of a sphere by a stereographic projection (which is *conformal* and maps circles to circles), theorem 1 becomes:

Corollary 2 [Andreev-Thurston on sphere] *A packing of interior-disjoint spherical caps on a sphere exists whose contact-graph is any polyhedral graph. If the graph is maximal planar, then this packing is unique (once the graph is specified) up to inversive transformations of 3-space.*

[1] This result has not yet been published in the open literature, but it was stated in 1982 lecture notes by Thurston and was presented by him in an address entitled "The finite Riemann mapping theorem" at the International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University March 1985. It also appears in the manuscript of a book-in-progress [19] by Thurston, who is at the Mathematics department of Princeton University, Princeton NJ, and is exposited in a preprint by A.Marden and B.Rodin. A different existence proof, arising from a considerably more general theory, may be found in [13].

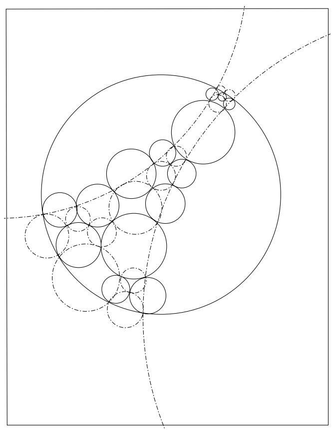

Figure 1: A circle packing (drawn dot-dashed) and its dual circle packing (solid), as in theorem 1.

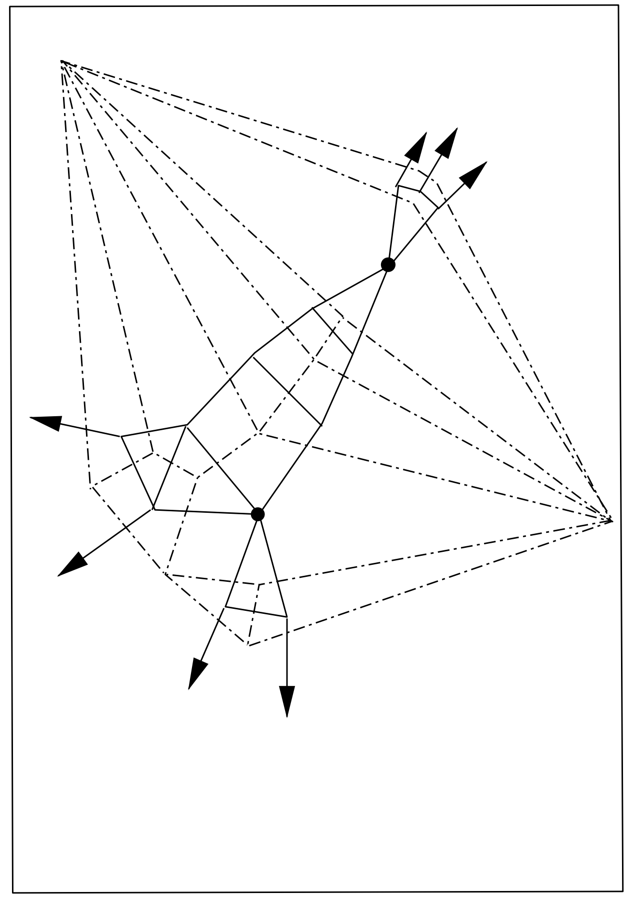

Figure 2: Contact graph of circle packing in previous figure (drawn dot-dashed) and its planar dual graph (solid). The edges with arrows are supposed to go to a special vertex "at ∞," corresponding to the outer dual circle. Note that in fact these graphs are *not* polyhedral, since they are only 2-connected, as may be seen by considering the removal of the vertex at ∞ and either one of the highlit vertices.

There is also a circle packing, whose contact graph is any polyhedral graph, such that its circle-tangency points are identical to the circle tangency points of a circle packing with the dual *contact graph, and the primal and dual circles are perpendicular at these points, and this combined realization is unique up to inversive transformations preserving the sphere.*

By considering a polyhedron which is the convex hull of the apices of cones tangent to the sphere at each circle, we obtain a third version:

Corollary 3 [Andreev-Thurston midscribing theorem] *Every polyhedral type is realizable by a convex polyhedron with a "midscribed" sphere, i.e. with each edge tangent to a sphere. If the polyhedron is simplicial, then this realization is unique up to projective transformations of 3-space that preserve the sphere.*

Also, every polyhedral type is realizable in this midscribed way, in such a way that its dual polyhedron is simultaneously midscribed with its set of edge-tangency points being identical to the set of edge-tangency points of the primal polyhedron, and the primal and dual edges are perpendicular at these points where they coincide. This combined realization is unique, up to projective transformations of 3-space that preserve the sphere.

An iterative algorithm (due to Thurston) converging to this embedding, for a *simplicial* polyhedral graph (synonym: maximal planar graph), is described in appendix 2 of [17], and there is a convergence proof due to Thurston which should appear in [19]. That algorithm, although simple, convergent, and practical for small graphs, has not been shown to run to completion in polynomial time.

The purpose of the present paper is to give a new algorithm, accompanied by a proof of polynomial running time, and which works for *general* polyhedral graphs, not just simplicial ones.

3 Statement and proof of main result.

Given the graph P of an arbitrary polyhedron. (3-connected planar. *Not* necessarily simplicial or simple.). We wish to find the unique (up to inversions) circle packing with this contact graph such that the circle tangency points are exactly the circle tangency points of a dual, orthogonal circle packing with the dual graph.

Theorem 4 [Main result] *We describe an algorithm whose run time is bounded by a polynomial in N, D, and $\log \kappa$, where N is the number of vertices of P, D is the number of decimal places of accuracy to which we find the circles's centers and radii (say these output numbers are binary rationals), and κ is the "condition number" of a certain minimization problem.*

Remarks. Here D is a lower bound on the number of decimals in (i.e. the size of) the output. We will show that κ is bounded by a $N^{O(N)}$ times the maximum (or reciprocal minimum) ratio of a primal radius to the radius of an incident dual circle. If the circle radii are wanted to some *relative* accuracy, such as 10%, and it is desired that the positions of each centers be off by no more than 10% of the corresponding radius, then the number of decimals D in the output will necessarily obey $10^D > \kappa$. The point of this remark is that $\log \kappa$, and hence the running time of our algorithm, will be bounded by polynomial function of D and N alone.

Secondly we remark that the true radii of the circles are generally algebraic numbers of high degree, so no "exact" algorithm can be hoped for.

Finally, we remark that the algorithm in our theorem and its proof, would seem to lead to an alternate proof of the uniqueness statements in the Andreev-Thurston theorem, although we will not harp on the matter.

Proof of theorem 4. We will give a top-level argument now; details are deferred to lemmas which will be proven in the next section.

Any polyhedral graph P, or its dual $P*$, must contain some triangular face (lemma 8) and inversive freedom allows us to choose the 3 vertices of this triangular face to form any desired triangle in the plane. Thus it is allowable, and we will find it convenient, to in fact regard $P*$ as P if necessary, and to demand that such a facial triangle actually be embedded in the plane as some specific pleasant triangle (such an an equilateral one) and be the exterior face of P. (Later, of course, we can find an inversive transformation mapping the resulting circle packing into any other.)

To every polyhedral graph P there corresponds a "quadrangulation" Q. Each quadrangle q of Q is bilaterally symmetric (see figure 3) and the 2 antipodal vertices on the line of symmetry, having angles β and α ($\alpha + \beta = \pi$) corresponding to a P-vertex and a centerpoint of an incident P-face, respectively. The other 2 vertices of q, which have right angles, correspond to points on the edges of the graph of P. The total number of quadrangles $q \in Q$ is $2E$, where E is the number of edges of P.

One will observe that an actual tiling by such quadrangles would lead to a drawing of P in the plane, using line segments for edges, such that each face is a convex polygon.[2] In fact, such a drawing always exists (this fact was known before, without recourse to the Andreev-Thurston theorem [20]) corresponding to the primal-dual simultaneous circle-packing of theorem 1, since our quadrangulations correspond to these circle packings as follows: The primal circle radii correspond to the B's and the dual circle radii to the A's; the circle centers are the α and β angles of the quadrangles.

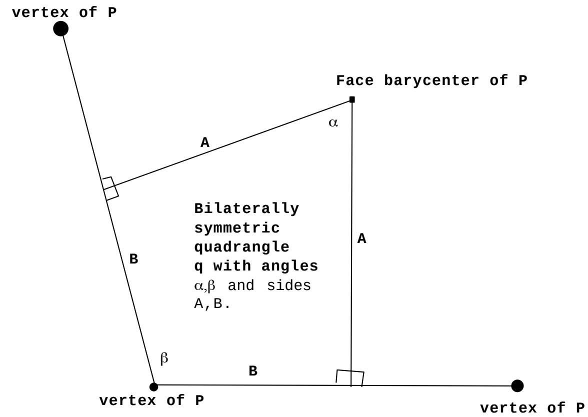

Figure 3: A quadrangle in the quadrangulation corresponding to polyhedral graph P.

The quadrangles of Q in such a tiling would be disjoint except on their boundaries, *except* for the quadrangles which subdivide the "outer" face f of P, which, of course, actually subdivide the interior of f and not its exterior. (The dual circle whose center is the center of f, is special since it surrounds the other dual circles.)

We will not have to worry about the special quadrangles subdividing f since we have demanded that f is a fixed triangle in the plane; thus the appropriate special dual circle is easily found (after all the other circles are known) as the circle enclosing all the other dual circles and tangent to the appropriate three of them. The ruler and compass construction of the circles tangent to 3 given circles is a hoary geometrical problem known as the "problem of Apollonius." The slickest solution is due to Gergonne [7].

To each quadrangle $q \in Q$ there corresponds a convex polyhedron in *hyperbolic geometry* $\mathbf{H}^3$ which we now specify by describing its face planes. It has 6 faces: erect 4 faces as the sides of an infinite Euclidean right prism with base q where q lies in the boundary plane of the conformal halfspace model [3] of $\mathbf{H}^3$, (these sides are Euclideanly orthogonal to q) and the other 2 faces are Eu-

[2] Incidentally, here are two related observations: (1) observe that each face of the midscribed polyhedron P is a convex plane polygon (since the entire polyhedron is convex in 3-space). Thus (2) the drawing of the contact graph of the circles on the sphere, obtained by projecting P radially onto the sphere's surface, is a subdivision of this spherical surface into spherically convex polygons whose sides are geodesic arcs.

[3] The reader will recall that in the conformal halfspace model, $\mathbf{H}^3$ is represented by a halfspace of $\mathbf{R}^3$, and the hyperbolic geodesics correspond to Euclidean semicircles and halflines perpendicular to the bounding plane. Euclidean and hyperbolic angles are equal since the mapping is "conformal." The isometries to $\mathbf{H}^3$ are Euclidean rescalings, translations, and rotations preserving the bounding plane, and inversions in spheres centered in the bounding halfplane.

clidean hemispheres with centers at the vertices of q with angles α and β respectively, having Euclidean radii A and B respectively. This polytope is divisible into 2 hyperbolic tetrahedra by cutting along a Euclidean plane perpendicular to q and containing the two right-angled vertices of q. The first tetrahedron is as shown in figure 4.

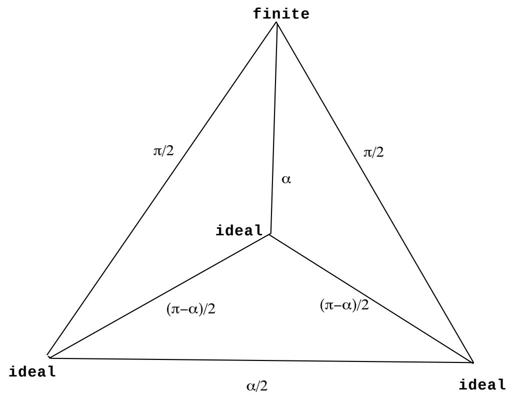

Figure 4: A hyperbolic tetrahedron. Edges are labeled with their dihedral angles. All vertices are ideal except for the one labeled "finite."

The other tetrahedron is same but having β's instead of α's.

Our **plan** is to *maximize* the sum of the hyperbolic volumes of all the quadrangle polyhedrons by choice of the *angles* α_i and $\beta_i = \pi - \alpha_i$ determining their shapes. Observe that since Euclidean rescaling of the conformal halfspace model is a hyperbolic isometry, the hyperbolic volumes of the quadrangle polyhedra indeed do depend purely on the angles α, β and not on the edge lengths.

We argue (lemma 5) that at any maxima (or indeed, at any stationary point) of the sum-of-volumes function in angle-space, subject to the linear constraints that

1. All angles lie in $(0, \pi)$,
2. The angles ringing an interior vertex of the quadrangulation sum to 2π,
3. The sum of the interior angles at an exterior vertex of P equals the total angle there – which is a *fixed constant* since we are demanding that the exterior face of P be a fixed triangle.

(1)

it will magically happen that the quadrangle shapes that arise will in fact fit together perfectly to form a quadrangulation.

We then argue (lemma 6) that in fact the sum-of-volumes function is a smooth, strictly concave-down function within a simple convex polytopal domain (defined by the linear constraints we just mentioned) in angle space $\mathbf{R}^{2E}$.

Further (lemma 7), it is easy to evaluate this function and its gradient to any number of decimal places.

We now use either Khachian's ellipsoid algorithm [8] or Vaidya's algorithm [21] for maximizing concave-down functions within convex regions of $\mathbf{R}^n$. This finds all the angles α_i, β_i at the unique maximum to D decimal places in time polynomial in N, D, and $\log \kappa$, where κ is the condition number at the maximum, i.e. the ratio of the lengths of the principle axes of the (near-ellipsoidal) contours of the sum-of-volumes function near this minimum.[4]

One may then, of course, paste the quadrangles together to find a suitable circle packing.

A quick look at the formula in lemma 6 reveals that difficulties (large condition numbers κ) can arise only when some of the angles α_i or β_i are near 0. This can happen (referring to figure 3) only if the ratio of the two side lengths of a quadrilateral is large, or correspondingly the ratio of the radius of some primal circle to the radius of an incident dual circle is very large or very small. This leads to the claimed bounds on κ. (The $N^{O(N)}$ safety factor in the first remark was to account for the possible effect of the integer constraint matrix, cf. lemma 9.) □

4 Lemmas.

Lemma 5 *If four bilaterally symmetric Euclidean plane quadrangles ring a common right angle as in figure 5 and the sum of their "hyperbolic volumes" (as discussed above) is maximal over possible choices of the angles* α_i*, subject to the linear constraints 1, it will automatically happen that*

$$A_1 A_2 A_3 A_4 = B_1 B_2 B_3 B_4 \qquad (2)$$

so that if the quadrilaterals are scaled so that edge B_1 *matches* A_2*, edge* B_2 *matches* A_3*, and edge* B_3 *matches* A_4*, then it will automatically happen that edge* B_4 *matches* A_1*.*

Proof. From Schläfli's formula for the derivative of hyperbolic volume (suitably modified to handle ideal vertices [10]) we see that the derivative of the sum of the hyperbolic-volume functions of the 4 quadrangles as the α_i's and β_i's change according to a infinitesimal perturbation of the type described by the $\pm$'s in figure 5 (where each angle with a $+$ sign is increased by ϵ, each angle with a $-$ is decreased by ϵ, where the ϵ's are all equal infinitesimals), is proportional to

$$\sum_i (\log A_i \ \partial\alpha_i \ - \ \log B_i \ \partial\beta_i) \ = \ \epsilon \log \prod_i \frac{A_i}{B_i}. \qquad (3)$$

Observe that this perturbation leaves all linear constraints of form 1 unaffected. Hence any maximal-volume choice of the α's must have volume invariant under infinitesimal perturbations of this form; hence the right hand side must be *zero.* □

[4] In fact, re lemma 7, we do not need the function value itself, only its gradient, for use in Vaidya's algorithm. Other minimization algorithms might be preferred in practice, which do require this information, however.

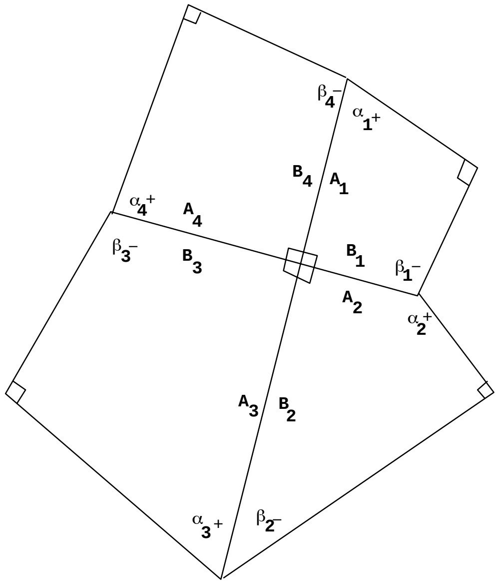

Figure 5: Four quadrangles. (Warning: The present angle and side labeling differs from the rest of the paper.)

Lemma 6 *The hyperbolic volume of a quadrangle-polyhedron with angles α and $\beta = \pi - \alpha$ is (up to certain additive and multiplicative constants, which we may and will ignore)*

$$\int_{\pi/4}^{\alpha} \log\cot\frac{\alpha}{2}\ \partial\alpha. \tag{4}$$

This is a strictly concave down function of α for $0 < \alpha < \pi$.

Proof. Let $C^2 = A^2 + B^2$. Then $A = C\sin\frac{\beta}{2}$ and $B = C\sin\frac{\alpha}{2}$. Now Schläfli's differential of volume is proportional to $\log A\partial\alpha + \log B\partial\beta$. Since $\alpha + \beta = \pi$ we see that $\sin\frac{\beta}{2} = \cos\frac{\alpha}{2}$ and $\partial\alpha + \partial\beta = 0$, causing the $\log C$ terms to cancel, and the final result is that Schläfli's differential of volume is proportional to $\log\cot\frac{\alpha}{2}\partial\alpha$. Observe that this is strictly decreasing in $\alpha \in [0, \pi]$, proving the concavity claim. □

Lemma 7 *The function (EQ 4) and its derivative may be computed to D decimal places in time polynomial in D.*

Proof. An algorithm is described in chapter 7 of [2] which will compute n decimals of $\exp z$ from n decimals of z, z complex, in $O(M(n)\log n)$ bit-operations, where $M(n)$ is the time it takes to multiply two n-bit numbers. This enables fast computation of π, all trigonometric functions, their inverses, and $\log x$. Thus certainly the derivative of (EQ 4) is calculable.

The integral, however, is somewhat more daunting. Integrals like

$$\int \log\cot x\ dx \tag{5}$$

may be expressed as differences of *Clausen integrals*

$$\mathrm{Cl}_2(u) = -\int_0^u \log|2\sin\frac{x}{2}|\ dx \tag{6}$$

with appropriate linearly transformed arguments. These Clausen integrals are discussed at length in [12] although not from a particularly algorithmic point of view. For example the Fourier series

$$\mathrm{Cl}_2(u) = \sum_{n\geq 1}\frac{\sin nu}{n^2}, \tag{7}$$

given there, while convergent everywhere, is not of great utility. However, $\mathrm{Cl}_2(u)$ obeys various identities

$$\mathrm{Cl}_2(u \pm 2n\pi) = \mathrm{Cl}_2(u), \tag{8}$$

$$\mathrm{Cl}_2(\pi + u) + \mathrm{Cl}_2(\pi - u) = 0, \tag{9}$$

$$\mathrm{Cl}_2(n\pi) = 0, \tag{10}$$

$$\frac{1}{2}\mathrm{Cl}_2(2u) = \mathrm{Cl}_2(u) - \mathrm{Cl}_2(\pi - u) \tag{11}$$

permitting its evaluation on $\mathbf{R}$ to be reduced to the problem of evaluating it for $u \in [0, 2\pi/3]$. Then, n terms of the series

$$\mathrm{Cl}_2(u) = u\left(1 - \log|u| + \sum_{n\geq 1}\frac{B_n u^{2n}}{(2n)(2n+1)(2n)!}\right), \tag{12}$$

(which converges for $|u| < 2\pi$) where the B_n are Bernoulli numbers $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, etc. suffice to obtain $\mathrm{Cl}_2(u)$ to error of order 3^{-n} – that is, to $\Theta(n)$-bits of accuracy – if $u \leq 2\pi/3$, and since the nth Bernoulli number may be evaluated in polynomial time using standard recurrences, we conclude that polynomial time suffices to find n decimals of these integrals for arbitrary argument. □

Lemma 8 *If P is a polyhedral graph and $P*$ is its dual, then P or $P*$ between them contain a total of at least 8 triangular faces.*

Proof. Let V, E, and F be the number of vertices, faces, and edges of P, respectively. $V + F = E + 2$ is Euler's formula. If k is the total number of triangle faces in P and $P*$ combined, all other faces being of order ≥ 4, then $4V + 4F - k = 4E$ by summing the valencies of the vertices in P and $P*$. Hence, $k \geq 8$. □

Lemma 9 *Let M be a nonsingular $N \times N$ integer matrix, each of whose entries obeys $|M_{ij}| \leq A$. Let the determinant of M be D and let its eigenvalues be λ_i, $i = 1 \ldots N$. Then $1 \leq |D| \leq A^N N^{N/2}$, $\max_i |\lambda_i| \leq AN$, and $\min_i |\lambda_i| \geq (AN)^{1-N}$.*

Proof. The bound on D is Hadamard's bound, and arises from the interpretation of the determinant as the Euclidean N-volume of the parallelipiped generated by the rows of M. The bound on $\lambda_{\max}$ is obvious, and the bound on $\lambda_{\min}$ arises from the fact that $D = \prod_i \lambda_i$. □

5 Conformal mapping

Theorem 1 leads to a very amusing new way to view the Riemann mapping theorem [17, 15, 16]. These references show that a way to find an approximate conformal mapping between a disk in the complex plane and any other simply connected region R, is to pack the interior of R with a hexagonal grid of equal circles and then to form their contact graph P. Modify this graph by adding a maximal number of additional contacts forcing circles at the boundary of R to contact a special, extra, exterior circle, obtaining a maximal planar graph P', and then use our theorem 4 to find a circle packing with this modified contact graph. Voila – R is mapped to the disk in the sense that the circles packed inside R are mapped to known corresponding circles packed inside the disk, having the same combinatorial structure, and in some sense this map is "conformal" since small circles are mapped to small circles. (A *picture* of this procedure may be found in [17].) Indeed Rodin and Sullivan proved that in the limit as the circles in R approach zero radius, this map tends to a Riemann map.

How does this algorithm for conformal mapping compare with the competition – the usual numerical methods? A survey of the latter, which is somewhat out of date, but will at least serve to get a novice pointed in the right directions, may be found in [11].

Our method has the theoretical advantage that it comes with a proof of polynomial running time and a proof of accuracy, and also it has the psychological advantage that there are all these highly visible and reassuring circles lying about. I have not found any proof of polynomial running time for any competing algorithm. Indeed, many of the competing methods depend on iterations whose convergence in *any* amount of time, is not proven – unless your initial guess is luckily close enough to the limit point of the iteration, in which case they all converge like lightning (e.g. quadratically).

Several strong statements may be made in defense of the best of the competing numerical methods, however. First, these methods usually only worry about mapping the *boundary* or R to the circle (interior points are then easily mapped by some simpler scheme) – saving a large amount of time. Second, they often have excellent convergence properties in practice, even if no proof is available. Third, they take advantage of fast Fourier transform tricks, saving even more time. Fourth, the discretization of the continuous mapping problem that the Andreev-Thurston method corresponds to, is not a particularly good one near the boundary of R. Most of the competing methods would be better in this respect.

In view of these practical advantages, which cannot be matched by the algorithm of the present paper (although our algorithm *can* achieve ultimate quadratic convergence by switching from the ellipsoid algorithm to a conjugate gradient method [6], say, at an apposite moment), we doubt our result will have much direct practical impact.

6 Acknowledgements

Peter G. Doyle pointed out Brägger's work to me and helped explain it. The present paper is really only a slight expansion of Brägger's ideas, which in turn (to some extent) depend on ideas found in a paper of Y. Colin de Verdiere's [4]. Specifically, the two new ideas found here are 1. the quadrangulation approach, which permits extending Brägger's results to non-simplicial polyhedra, and 2. the idea of applying the ellipsoid algorithm, which is rather old.

Doyle now has informed me that he too thought of the quadrangulation idea, in fact before I did, so perhaps this should be a joint paper? Doyle didn't seem keen on writing a joint paper, or even having any paper, but I suppose I am willing.

A forthcoming paper by Rivin and Smith [14] will treat "generalized circle packings," and thus, in some sense, will render the present paper obsolete. But the present paper is nevertheless of interest because it is simpler and because the special case that it treats, is the one of the greatest interest.

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