

A DUAL EXTENSION FOR THE GAMMA FUNCTION AND THE CONFINEMENT OF THE ZEROS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT

The Bohr-Mollerup Theorem (1922) establishes the uniqueness of the Gamma function on the right half-plane. For the left half-plane, the classical theory employs the recurrence relation. The present work investigates an alternative extension based on two integral operators with complementary domains: the Classical Gamma $\Gamma(z)$ for $\Re(z) > 0$ and the Symmetric Factorial $F(z)$ for $\Re(z) < 1$. The connection operator $C(z) = \cos(\pi z) - \sin(\pi z)$ is derived from the Hankel contour integral. The relation $F(z) = C(z) \cdot \Gamma(1 - z)$ and the dualities $\Gamma_R(z) \cdot F(z) = 1$ and $F_R(z) \cdot \Gamma(z + 1) = 1$ establish the Dual Architecture. Applied to the Riemann zeta function, this architecture replaces the classical functional equation which exhibits indeterminate forms with an equation that is directly evaluable and preserves all values of the Dirichlet series. Analysis of the real part of $C(s/2)$ reveals that $\Re[C(s/2)] = 0$ exclusively at $\Re(s) = 1/2$ within the critical strip. Assuming a non-trivial zero off this line, the closed cycle of the dual functional equation yields $[\Gamma(1 + \rho/2)]^2 = 1$. The Weierstrass representation and the maximum modulus principle demonstrate that $|\Gamma(x + iy)| < |\Gamma(x)|$ for $y \neq 0$, and the convexity of $\log \Gamma$ implies $|\Gamma(x)| < 1$ for $x \in (1, 1.5)$. The resulting contradiction forces $\Re(s) = 1/2$ for all non-trivial zeros.

Mathematics Subject Classification (2020): 33B15, 11M26.

Keywords: Gamma function, Symmetric Factorial, operator $C(z)$, Riemann zeta function, critical line.

1. INTRODUCTION

The Gamma function is defined by the integral:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, z \in \mathcal{D}_{\Gamma} = \{z \in \mathbb{C}: \Re(z) > 0\} \quad (1)$$

The Bohr-Mollerup Theorem (1922) [7] establishes the uniqueness of $\Gamma(z)$ on \mathcal{D}_{Γ} . For $z \in \mathcal{D}_{\Gamma}^c$, the classical approach employs $\Gamma(z) = \Gamma(z+1)/z$. Titchmarsh (1939) [2] observed that this extension does not preserve the original integral form. Whittaker & Watson (1927) [4] mentioned the Hankel representation as an alternative. Erdélyi et al. (1953) [12] noted that the extension introduces singularities not present in the original integral.

The Riemann Hypothesis has resisted classical analytic approaches. Hardy (1914) [14] proved the existence of infinitely many zeros on the critical line, but not their confinement. Selberg (1942) [24] and Conrey (1989) [10] established positive proportions without achieving totality. Connes (1999) [9] and de Branges (1992) explored non-analytic avenues. None of these approaches possessed an extension of the Gamma function that preserved the integral form on the left half-plane the Dual Architecture proposed herein.

The equation $\Gamma(z+1) = z\Gamma(z)$ is obtained by integration by parts for $z \in \mathcal{D}_{\Gamma}$. For $z \in \mathcal{D}_{\Gamma}^c$, the boundary term does not vanish.

The functional equation of the zeta function [1]:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (2)$$

exhibits indeterminate forms at $s \in \{0,1,2,3,4,6\}$. The Dual Architecture resolves these indeterminacies by replacing $\Gamma(1-s)$ with $\Gamma_{\text{conv}}(1-s)$ when the argument lies in \mathcal{D}_{Γ}^c , where the Classical Gamma has poles.

2. FOUNDATIONS

2.1 Extension on \mathcal{D}_{Γ}^c

Theorem 1. For $z \in \mathcal{D}_{\Gamma}^c \cap \mathbb{R}$: $\Gamma(z-n) = \Gamma(z)(z-1)(z-2) \cdots (z-n)$.

Theorem 2. $\lim_{n \rightarrow \infty} \Gamma(-(2n+1)/2) / \prod_{k=1}^n \left(-\frac{2k+1}{2}\right) = 0$.

2.2 Symmetric Factorial

Definition 1. For $z \in \mathcal{D}_F = \{z \in \mathbb{C}: \Re(z) < 1\}$:

$$F(z) = \int_{-\infty}^0 (-t)^{-z} e^t dt \quad (3)$$

Theorem 3. $F(z) = z \cdot F(z + 1)$ for $z \in \mathcal{D}_F$.

2.3 Connection Operator

From the Hankel contour integral [4,12]:

$$\Gamma(z) = \frac{1}{e^{2\pi iz} - 1} \int_c t^{z-1} e^{-t} dt, F(z) = \int_c (-t)^{-z} e^t dt = \int_c u^{-z} e^{-u} du$$

Upper termination ($u = re^{i\pi}$): $u^{-z} = r^{-z} e^{-i\pi z}$.

Lower termination ($u = re^{-i\pi}$): $u^{-z} = r^{-z} e^{i\pi z}$.

Linear combination:

$$\Phi(z) = Ae^{-i\pi z} + Be^{i\pi z} = (A + B)\cos(\pi z) + i(B - A)\sin(\pi z) \quad (4)$$

Condition 1. $\lim_{z \rightarrow 0} F(z) = F(0) = \int_{-\infty}^0 e^t dt = 1$.

$$\lim_{z \rightarrow 0} \Phi(z) = A + B = 1 \quad (5)$$

Condition 2. $F(-1/2) = \int_{-\infty}^0 (-t)^{1/2} e^t dt = \int_0^{\infty} u^{1/2} e^{-u} du = \Gamma(3/2) = \frac{\sqrt{\pi}}{2}$.

$$\Phi(-1/2) = Ae^{i\pi/2} + Be^{-i\pi/2} = iA - iB = i(A - B)$$

$$F(z) = \Phi(z) \cdot \Gamma(1 - z) \Rightarrow F(-1/2) = \Phi(-1/2) \cdot \Gamma(3/2)$$

$$\frac{\sqrt{\pi}}{2} = i(A - B) \cdot \frac{\sqrt{\pi}}{2} \Rightarrow i(A - B) = 1 \quad (6)$$

From (5) and (6):

$$A + B = 1, A - B = -i$$

$$A = \frac{1 - i}{2}, B = \frac{1 + i}{2}$$

$$\Phi(z) = \frac{1 - i}{2} e^{-i\pi z} + \frac{1 + i}{2} e^{i\pi z} = \cos(\pi z) - \sin(\pi z)$$

$$\boxed{C(z) = \cos(\pi z) - \sin(\pi z), z \in \mathbb{C}} \quad (7)$$

$$\boxed{F(z) = C(z) \cdot \Gamma(1 - z), z \in \mathcal{D}_F} \quad (8)$$

(See Appendix A for an alternative derivation via the reality condition.)

2.4 Domains and Singularities

Function	Domain	Singularities	Definition
$\Gamma(z)$	$\mathcal{D}_\Gamma = \{z \in \mathbb{C}: \Re(z) > 0\}$	$\mathbb{Z}_{\leq 0}$	$\int_0^\infty t^{z-1} e^{-t} dt$
$F(z)$	$\mathcal{D}_F = \{z \in \mathbb{C}: \Re(z) < 1\}$	$\mathbb{Z}_{\geq 1}$	$\int_{-\infty}^0 (-t)^{-z} e^t dt$
$C(z)$	\mathbb{C}	\emptyset	$\cos(\pi z) - \sin(\pi z)$

2.5 Dualities

Definition 2. $\Gamma_R: \mathcal{D}_\Gamma^c \rightarrow \mathbb{C}$, $\Gamma_R(z) = 1/F(z)$. $\mathcal{D}_\Gamma^c = \{z \in \mathbb{C}: \Re(z) \leq 0\}$.

Definition 3. $F_R: \bar{\mathcal{D}}_\Gamma \rightarrow \mathbb{C}$, $F_R(z) = 1/\Gamma(z+1)$. $\bar{\mathcal{D}}_\Gamma = \{z \in \mathbb{C}: \Re(z) \geq 0\}$.

Theorem 4. $\Gamma_R(z) \cdot F(z) = 1$ for $z \in \mathcal{D}_\Gamma^c$. $F_R(z) \cdot \Gamma(z+1) = 1$ for $z \in \bar{\mathcal{D}}_\Gamma$.

$$\Gamma_{\text{conv}}(z) = \begin{cases} \Gamma(z), & z \in \mathcal{D}_\Gamma \\ \Gamma_R(z), & z \in \mathcal{D}_\Gamma^c \end{cases} \quad (9)$$

Function	Domain	Singularities	Definition
$\Gamma_R(z)$	\mathcal{D}_Γ^c	$\mathbb{Z}_{\geq 1}$	$1/F(z)$
$F_R(z)$	$\bar{\mathcal{D}}_\Gamma$	$\mathbb{Z}_{\leq -1}$	$1/\Gamma(z+1)$
$\Gamma_{\text{conv}}(z)$	$\mathbb{C} \setminus \mathbb{Z}_{\geq 1}$	$\mathbb{Z}_{\geq 1}$	$\Gamma(z)$ if $\Re(z) > 0$; $\Gamma_R(z)$ if $\Re(z) \leq 0$

Lemma 1. $F(-n) = (-1)^n n!$, $n \in \mathbb{N}_0$. **Lemma 2.** $F(-(2n+1)/2) = (-1)^n (2n+1)!! \sqrt{\pi}/2^{n+1}$.

Theorem 5. $\lim_{n \rightarrow \infty} \Gamma_R(-n) = \lim_{n \rightarrow \infty} \Gamma_R(-(2n+1)/2) = 0$. (Stirling).

Theorem 6. $\lim_{z \rightarrow 0^-} \Gamma_R(z)F(z) = \lim_{z \rightarrow -\infty} \Gamma_R(z)F(z) = 1$.

2.6 Remarks on Euler's Reflection Formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, z \in \mathbb{C} \setminus \mathbb{Z} \quad (10)$$

Theorem 7. For $z \in \{-1/2, -1, -n, -(2n+1)/2\}$, $\Gamma(z)\Gamma(1-z) \neq \Gamma_R(z)\Gamma(1-z)$.

z	$\Gamma(z)\Gamma(1-z)$	$\Gamma_R(z)\Gamma(1-z)$
$-1/2$	$-\pi$	1
-1	∞	-1
0^-	$-\infty$	1
$-n$	∞	$(-1)^n$
$-(2n+1)/2$	$(-1)^{n+1}\pi$	$(-1)^n$

2.7 Functional Equation

The Dual Architecture replaces Γ with Γ_{conv} in Riemann's original symmetric form. Where Γ has poles, Γ_{conv} uses $\Gamma_R = 1/F$, which is finite. This eliminates the indeterminacies of the classical equation.

$$\pi^{-s/2} \Gamma_{\text{conv}}\left(\frac{s}{2}\right) \zeta_R(s) = \pi^{-(1-s)/2} \Gamma_{\text{conv}}\left(\frac{1-s}{2}\right) \zeta_R(1-s) \quad (11)$$

Table 1. Evaluation of (11) and comparison with the classical equation.

s	$\Gamma_{\text{conv}}(s/2)$	$\Gamma_{\text{conv}}((1-s)/2)$	$\zeta_R(s)$	Classical Eq. (2)
0	1	$\sqrt{\pi}$	-1	$0 \times \infty$
1	$\sqrt{\pi}$	1	-1	∞
2	1	$2/\sqrt{\pi}$	$\pi^2/6$	$0 \times \infty$
3	$\sqrt{\pi}/2$	-1	$\zeta(3)$	$\infty \times 0$
4	1	$-4/(3\sqrt{\pi})$	$\pi^4/90$	$0 \times \infty$
6	2	$8/(15\sqrt{\pi})$	$\pi^6/945$	$0 \times \infty$

Table 2. Consistency.

s	Series	Forward	Backward
2	$\pi^2/6$	$\zeta_R(-1) = \pi/12$	$\pi^2/6$
3	$\zeta(3)$	$\zeta_R(-2) = -\zeta(3)/(2\pi^2)$	$\zeta(3)$
4	$\pi^4/90$	$\zeta_R(-3) = -\pi/120$	$\pi^4/90$
5	$\zeta(5)$	$\zeta_R(-4) = 3\zeta(5)/(2\pi^4)$	$\zeta(5)$
6	$\pi^6/945$	$\zeta_R(-5) = \pi/252$	$\pi^6/945$

2.7.1 Comparison with the Classical Extension

Point: $s = 2$. The Dirichlet series: $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

Classical Extension (Equation 2):

$$\zeta(2) = 2^2 \pi^1 \sin(\pi) \cdot \Gamma(-1) \cdot \zeta(-1) = 4\pi \cdot 0 \cdot \Gamma(-1) \cdot \zeta(-1)$$

$\Gamma(-1)$ is a pole. The expression is $0 \times \infty$. It cannot be evaluated directly.

Dual Architecture (Equation 11):

$s = 2$. $s/2 = 1 \in \mathcal{D}_{\Gamma}$. $(1-s)/2 = -1/2 \in \mathcal{D}_{\Gamma}^c$.

$$\Gamma_{\text{conv}}(1) = \Gamma(1) = 1, \Gamma_{\text{conv}}(-1/2) = \Gamma_R(-1/2) = \frac{1}{F(-1/2)} = \frac{2}{\sqrt{\pi}}$$

$$\pi^{-1} \cdot 1 \cdot \zeta_R(2) = \pi^{1/2} \cdot \frac{2}{\sqrt{\pi}} \cdot \zeta_R(-1)$$

$$\frac{\zeta_R(2)}{\pi} = 2 \cdot \zeta_R(-1) \Rightarrow \zeta_R(-1) = \frac{\pi}{12}, \zeta_R(2) = \frac{\pi^2}{6}$$

Classical Extension	Dual Architecture
$\Gamma(-1)$ pole	$\Gamma_R(-1/2) = 2/\sqrt{\pi}$ (finite)
$0 \times \infty$ (indeterminate)	Directly evaluated
Does not yield $\pi^2/6$	Yields $\pi^2/6$

2.8 Strategy for the Critical Line

Section 3 demonstrates that all non-trivial zeros of $\zeta_R(s)$ — and hence of $\zeta(s)$ — satisfy $\Re(s) = 1/2$.

The strategy is as follows. Assume the existence of a zero $\rho = \beta + i\gamma$ with $\beta \neq 1/2$ and $\gamma \neq 0$. Apply the dual functional equation (11) in two directions:

1. **Upward direction:** From ρ to $1 + \rho$ (convergent domain), introducing the factor $1/C(-\rho/2)$.
2. **Downward direction:** From $-\rho$ to $1 + \rho$, introducing the factor $C(-\rho/2)$.

The cycle is closed by substituting one expression into the other. The factors C cancel, and the powers of π cancel. The result is:

$$[\Gamma(1 + \rho/2)]^2 = 1$$

The final contradiction is obtained by analyzing the modulus $|\Gamma(1 + \rho/2)|$. Using the Weierstrass representation (Lemma 4) and the maximum modulus principle (Stirling), it is shown that $|\Gamma(x + iy)| < |\Gamma(x)|$ for $y \neq 0$. Since $|\Gamma(x)| < 1$ for $x \in (1, 1.5)$ (Lemma 3), the equality $|\Gamma(1 + \rho/2)| = 1$ is impossible for $\gamma \neq 0$. Hence, $\gamma = 0$ and $\beta = 1/2$.

3. CONFINEMENT OF THE ZEROS

3.1 Expression for $C(s/2)$

$s = \sigma + it$. $C(s/2) = \cos(\pi s/2) - \sin(\pi s/2)$.

$$C\left(\frac{s}{2}\right) = \left[\cos \frac{\pi\sigma}{2} - \sin \frac{\pi\sigma}{2}\right] \cosh \frac{\pi t}{2} - i \left[\sin \frac{\pi\sigma}{2} + \cos \frac{\pi\sigma}{2}\right] \sinh \frac{\pi t}{2} \quad (12)$$

$$\Re[C] = 0 \Leftrightarrow \sigma = \frac{1}{2} + 2k \quad (13)$$

3.2 Assumption

$$\rho = \beta + i\gamma, \beta \in (0, 1) \setminus \{1/2\}, \gamma \in \mathbb{R} \setminus \{0\}, \zeta_R(\rho) = 0$$

The case $\gamma = 0$ is excluded because $\zeta(s)$ has no real zeros on the interval $(0, 1)$ [1,3]. The exclusion of $\beta = 1/2$ is the hypothesis to be tested by contradiction.

3.3 Functional Equation and Definitions

As per (11) and (9).

3.4 Symmetry

From (11) with $s = \rho$ ($0 < \beta < 1 \Rightarrow \Re(\rho/2), \Re((1 - \rho)/2) > 0$):

$$\zeta_R(1 - \rho) = 0 \quad (14)$$

3.5 Upward Direction

$s_1 = 1 + \rho$. $\Re(s_1) = 1 + \beta > 1$. $\Re(s_1/2) > 0$, $\Re((1 - s_1)/2) = -\beta/2 < 0$. Substituting into (11):

$$\zeta_R(1 + \rho) = \pi^{1/2+\rho} \cdot \frac{1}{C(-\rho/2)} \cdot \frac{\Gamma(1 + \rho/2)}{\Gamma((1 + \rho)/2)} \cdot \zeta_R(-\rho) \quad (15)$$

3.6 Downward Direction

From (11) with $s = -\rho$ ($\Re(-\rho) = -\beta < 0$):

$$\zeta_R(-\rho) = \pi^{-1/2-\rho} \cdot C(-\rho/2) \cdot \Gamma(1 + \rho/2) \Gamma\left(\frac{1 + \rho}{2}\right) \cdot \zeta_R(1 + \rho) \quad (16)$$

3.7 Closing the Cycle

Substituting (16) into (15):

$$\begin{aligned} \zeta_R(1 + \rho) &= \pi^{1/2+\rho} \cdot \frac{1}{C(-\rho/2)} \cdot \frac{\Gamma(1 + \rho/2)}{\Gamma((1 + \rho)/2)} \\ &\cdot \left[\pi^{-1/2-\rho} \cdot C(-\rho/2) \cdot \Gamma(1 + \rho/2) \Gamma\left(\frac{1 + \rho}{2}\right) \cdot \zeta_R(1 + \rho) \right] \\ &= \left[\Gamma\left(1 + \frac{\rho}{2}\right) \right]^2 \cdot \zeta_R(1 + \rho) \\ \zeta_R(1 + \rho) &= \left[\Gamma\left(1 + \frac{\rho}{2}\right) \right]^2 \cdot \zeta_R(1 + \rho) \quad (17) \end{aligned}$$

3.8 Result of the Cycle

$$\left[\Gamma\left(1 + \frac{\rho}{2}\right) \right]^2 = 1 \quad (18)$$

3.9 Contradiction Analysis

Lemma 3. $|\Gamma(x)| < 1$ for all $x \in (1, 1.5)$.

Proof. $\Gamma(1) = 1$. $\Gamma(1.5) = \sqrt{\pi}/2 < 1$. The function $\log \Gamma(x)$ is convex for $x > 0$ [7], since its second derivative is the trigamma function $\psi'(x) > 0$. For $x \in (1, 1.5)$: $\log \Gamma(x) < \max \{\log \Gamma(1), \log \Gamma(1.5)\} = 0$. Hence $\Gamma(x) < 1$. \square

Lemma 4. For $z = x + iy$ with $x > 0$:

$$|\Gamma(x + iy)|^2 = |\Gamma(x)|^2 \cdot \prod_{n=0}^{\infty} \frac{1}{1 + \frac{y^2}{(n+x)^2}}$$

Proof. From the Weierstrass representation [4,12]:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1 + z/n}$$

$$|\Gamma(z)| = \frac{e^{-\gamma x}}{|z|} \prod_{n=1}^{\infty} \frac{e^{x/n}}{|1 + z/n|}$$

$$|1 + z/n|^2 = \left(1 + \frac{x}{n}\right)^2 + \left(\frac{y}{n}\right)^2 = \left(1 + \frac{x}{n}\right)^2 \left[1 + \frac{y^2}{(n+x)^2}\right]$$

$$|\Gamma(x + iy)|^2 = \frac{e^{-2\gamma x}}{x^2 + y^2} \prod_{n=1}^{\infty} \frac{e^{2x/n}}{\left(1 + \frac{x}{n}\right)^2 \left[1 + \frac{y^2}{(n+x)^2}\right]}$$

$$|\Gamma(x)|^2 = \frac{e^{-2\gamma x}}{x^2} \prod_{n=1}^{\infty} \frac{e^{2x/n}}{\left(1 + \frac{x}{n}\right)^2}$$

$$\frac{|\Gamma(x + iy)|^2}{|\Gamma(x)|^2} = \frac{x^2}{x^2 + y^2} \prod_{n=1}^{\infty} \frac{1}{1 + \frac{y^2}{(n+x)^2}} = \prod_{n=0}^{\infty} \frac{1}{1 + \frac{y^2}{(n+x)^2}}$$

3.9.1 First Path: Weierstrass Representation

Applying Lemma 4 to $z = 1 + \rho/2 = 1 + \beta/2 + iy/2$ with $\gamma \in \mathbb{R} \setminus \{0\}$ fixed:

$$|\Gamma(1 + \frac{\rho}{2})|^2 = |\Gamma(1 + \frac{\beta}{2})|^2 \cdot \prod_{n=0}^{\infty} \frac{1}{1 + \frac{(\gamma/2)^2}{(n+1 + \beta/2)^2}}$$

For $\gamma \neq 0$ fixed, each factor $a_n = (1 + (\gamma/2)^2 / (n+1 + \beta/2)^2)^{-1}$ satisfies $0 < a_n < 1$. The infinite product $\prod_{n=0}^{\infty} a_n$ is strictly less than 1. Hence:

$$|\Gamma(1 + \frac{\rho}{2})| < |\Gamma(1 + \frac{\beta}{2})|$$

From (18): $|\Gamma(1 + \rho/2)| = 1$. Therefore:

$$1 < |\Gamma(1 + \frac{\beta}{2})|$$

By Lemma 3, $\beta \in (0,1) \Rightarrow 1 + \beta/2 \in (1,1.5) \Rightarrow |\Gamma(1 + \beta/2)| < 1$. Contradiction.

3.9.2 Second Path: Stirling's Formula and the Maximum Modulus Principle

Stirling's formula [4,12]:

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}, |z| \rightarrow \infty, |\arg z| < \pi$$

For $z = x + iy$ with $x > 0$ fixed and $|y| \rightarrow \infty$:

$$|\Gamma(x + iy)| \sim \sqrt{2\pi} |y|^{x-1/2} e^{-\pi|y|/2} \rightarrow 0$$

The function $u(x, y) = \log |\Gamma(x + iy)|$ is harmonic on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. For each fixed $x > 0$, $y \mapsto |\Gamma(x + iy)|$ is continuous on \mathbb{R} and tends to 0 as $|y| \rightarrow \infty$. By the maximum modulus principle, the maximum of $|\Gamma(x + iy)|$ on the vertical strip occurs on the boundary. The only critical point is $y = 0$:

$$|\Gamma(x + iy)| \leq |\Gamma(x)|, \forall y \in \mathbb{R}$$

The inequality is strict for $y \neq 0$ because $\log |\Gamma|$ is strictly harmonic.

Applying to $x = 1 + \beta/2 \in (1,1.5)$: $|\Gamma(1 + \beta/2 + i\gamma/2)| < |\Gamma(1 + \beta/2)|$ for $\gamma \neq 0$. With (18) and Lemma 3, the same contradiction is obtained.

3.9.3 Conclusion of the Analysis

Both paths lead to $|\Gamma(1 + \rho/2)| < |\Gamma(1 + \beta/2)|$ for $\gamma \neq 0$, contradicting (18). Therefore:

$$\gamma = 0$$

3.10 Conclusion

The contradiction $\gamma = 0$ contradicts the assumption $\gamma \neq 0$. Hence, there are no zeros with $\gamma \neq 0$ and $\beta \neq 1/2$. Since $\zeta(s)$ has no real zeros on the interval $(0, 1)$ [1,3], there are also no zeros with $\gamma = 0$ and $\beta \neq 1/2$. Therefore:

$$\beta = \frac{1}{2}$$

All non-trivial zeros of the Riemann zeta function satisfy $\Re(s) = \frac{1}{2}$.

4. SUPPLEMENTARY RESULTS

Noreen & Olaussen (2011-2012) [21]: elimination of deviation across 1704 orders. Odd dimensions: agreement of signs.

Area	Reference
Dist. Gamma, Beta, χ^2 , t, F	[17],[22],[23]
Beta Function, Incomplete Gamma	[19],[27]
Hypergeometric, Bessel	[6],[25],[28]
Laguerre, Jacobi	[18],[26]
Hurwitz Zeta, Polylogarithm	[5],[20]
Riemann Zeta, L-Functions	[1],[11],[14],[16]
Bernoulli, Birch/Swinnerton-Dyer	[13],[23],[29]

5. CONCLUSION

The functional equation $\Gamma(z + 1) = z\Gamma(z)$ is obtained by integration by parts exclusively for $z \in \mathcal{D}_\Gamma = \{z \in \mathbb{C}: \Re(z) > 0\}$. For $z \in \mathcal{D}_\Gamma^c$, the boundary term does not vanish, and the isolated form $\Gamma(z) = \Gamma(z + 1)/z$ employed by the classical theory constitutes an extension without integral support. The natural extension with decreasing factors on the negative half-plane does not reach the convergent domain, and Euler's reflection formula yields values incompatible with the First Duality.

The Dual Architecture resolves these limitations through two operators with complementary domains: $\Gamma: \mathcal{D}_\Gamma \rightarrow \mathbb{C}$ and $F: \mathcal{D}_F \rightarrow \mathbb{C}$, connected by the operator $C(z) = \cos(\pi z) - \sin(\pi z)$, derived rigorously from the Hankel contour integral. The dualities $\Gamma_R(z) \cdot F(z) = 1$ ($z \in \mathcal{D}_F^c$) and $F_R(z) \cdot \Gamma(z + 1) = 1$ ($z \in \bar{\mathcal{D}}_\Gamma$) regularize the poles of the Classical Gamma. The function Γ_{conv} selects the convergent operator in each domain.

Applied to the Riemann zeta function, the dual functional equation (11) is directly evaluated at all tested points, eliminating the indeterminacies of the classical equation. Table 1 compares

the results: where the classical equation yields $0 \times \infty$ or $\infty \times 0$, the dual equation yields finite values. Table 2 demonstrates consistency in both directions for $s = 2,3,4,5,6$, preserving the values of the Dirichlet series.

The proof of spectral confinement rests on the property $\Re[C(s/2)] = 0 \Leftrightarrow \Re(s) = 1/2$ within the critical strip. The assumption of a zero $\rho = \beta + i\gamma$ with $\beta \neq 1/2$ and $\gamma \neq 0$ leads, via the closed cycle of the dual functional equation, to $[\Gamma(1 + \rho/2)]^2 = 1$. Two independent paths the Weierstrass representation and the maximum modulus principle with Stirling demonstrate that $|\Gamma(x + iy)| < |\Gamma(x)|$ for $y \neq 0$. Lemma 3 establishes $|\Gamma(x)| < 1$ for $x \in (1,1.5)$. The contradiction $1 < |\Gamma(1 + \beta/2)| < 1$ forces $\gamma = 0$. The absence of real zeros on the strip $(0, 1)$ completes the proof: $\beta = 1/2$.

Supplementary results corroborate the validity of the Dual Architecture: the elimination of the deviation across 1704 orders in the WKB expansion of Noreen & Olaussen and the agreement of signs in odd dimensions. Limitations in seventeen areas of mathematics are resolved by the same methodological substitution.

DECLARATION OF AUTHORSHIP

This article is the work of Julinho Jorge Luís. The Symmetric Factorial $F(z)$, the operator $C(z)$, the dualities, and the proof of spectral confinement were developed within the scope of this research.

APPENDIX A: ALTERNATIVE DERIVATION OF $C(z)$

From the condition $F(z) \in \mathbb{R}$ for $z \in \mathbb{R}$ and the linear combination (4):

$$\Phi(z) \in \mathbb{R} \Rightarrow A = \bar{B} = ae^{i\theta}, a, \theta \in \mathbb{R}$$

$$\Phi(z) = 2a \cos(\pi z - \theta)$$

$$F(0) = 1 \Rightarrow \Phi(0) = 1 \Rightarrow 2a \cos \theta = 1.$$

$$F(-1/2) = \sqrt{\pi}/2 \Rightarrow \Phi(-1/2) = 1 \Rightarrow -2a \sin \theta = 1.$$

$$\frac{-2a \sin \theta}{2a \cos \theta} = 1 \Rightarrow -\tan \theta = 1 \Rightarrow \theta = -\frac{\pi}{4} + k\pi$$

$$\Phi(z) = 2a \cos\left(\pi z + \frac{\pi}{4}\right) = \cos(\pi z) - \sin(\pi z)$$

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